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Arkojyoti Biswas

ON RELATIVE ORDER OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Abstract. In this paper we obtain some relationship between relative order, relative lower order,Gol'dberg order and lower Gol'dberg order of an entire functions of several complex veriables which improves some earlier results.

Key words: Gol'dberg order, entire functions, relative order.

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1. Introduction, definitions and notations

Let f be a non constant entire function of a single complex variable in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Then $M_f(r)$ is strictly increasing, its inverse

$$M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists and is such that

$$\lim_{r\to\infty} M_f^{-1}(r) = \infty.$$

The order and lower order of an entire function are defined in the following way:

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as follows:*

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} \quad \textit{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

The function f is said to be of regular growth if $\rho_f = \lambda_f$.

For two entire functions f and g, Bernal [2] introduced the definition of relative order $\rho_g(f)$ of f with respect to g as follows:

Definition 2 ([2]). *Let f and g be two entire functions.The relative order $\rho_g(f)$ of f with respect to g is defined as:*

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \\ &\qquad\qquad \textit{for all sufficiently large values of } r\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

It is easy to see that if $g(z) = \exp z$ then $\rho_g(f) = \rho_f$.

Similarly the relative lower order $\lambda_g(f)$ of f with respect to g is defined by

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

The function f is said to be of regular relative growth with respect to g if $\rho_g(f) = \lambda_g(f)$.

The notion of relative order of two complex variables was introduced by Banerjee and Dutta [1]. They defined the relative order for two complex variables as follows:

Definition 3 ([1]). *Let $f(z_1, z_2)$ and $g(z_1, z_2)$ be two non constant entire functions of two complex variables z_1 and z_2 holomorphic in the closed polydisc*

$$\{(z_1, z_2) : |z_j| \le r_j;\ j = 1, 2\}$$

and let

$$\begin{aligned}F(r_1, r_2) &= \max\{|f(z_1, z_2)| : |z_j| \le r_j;\ j = 1, 2\}, \\ G(r_1, r_2) &= \max\{|g(z_1, z_2)| : |z_j| \le r_j;\ j = 1, 2\}.\end{aligned}$$

The relative order of f with respect to g denoted by $\rho_g(f)$, is defined as

$$\begin{aligned}\rho_g(f) = \inf\{\mu > 0 : F(r_1, r_2) < G(r_1^{\mu}, r_2^{\mu}) \\ \textit{for } \ r_1 \ge R(\mu),\ r_2 \ge R(\mu)\}.\end{aligned}$$

To study the notion of the relative order of entire functions of n complex variables we first recall the following notations and definitions.

Let $\mathbb{C}^n$ be the n-dimentional complex space.We denote the point $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $(m_1, m_2, \ldots, m_n) \in I^n$ by z and m respectively where I denotes the set of all non negetive integers. The modulus of z, denoted by $|z|$ is defined as

$$|z| = \left(|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2\right)^{\frac{1}{2}}.$$

Also we write $\|m\| = m_1 + m_2 + \ldots + m_n$.

Let $D \subset \mathbb{C}^n$ be an arbitrary bounded complete $n-$circular domain with centre at origin.Let

$$M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$$

where f is an entire function of n complex variables and for $R > 0$ a point $z \in D_R$ iff $\frac{z}{R} \in D$.

Let g be a non constant entire function. Then $M_{g,D}(R)$ is strictly increasing, continuous and its inverse

$$M_{g,D}^{-1} : (|g(0)|, \infty) \to (0, \infty)$$

exists and is such that

$$\lim_{R\to\infty} M_{g,D}^{-1}(R) = \infty.$$

The Gol'dberg order (briefly G-order) of an entire function of n complex variables is defined in the following way.

Definition 4 ([3])**.** *The Gol'dberg order* (*briefly G-order*) $\rho_{f,D}$ *of* f *with respect to domain* D *is defined as follows:*

$$\rho_{f,D} = \limsup_{R\to\infty} \frac{\log\log M_{f,D}(R)}{\log R}.$$

The lower Gol'dberg order $\lambda_{f,D}$ *of* f *with respect to domain* D *is defined as*

$$\lambda_{f,D} = \liminf_{R\to\infty} \frac{\log\log M_{f,D}(R)}{\log R}.$$

We say that f is of regular growth if $\lambda_{f,D} = \rho_{f,D}$.

It is known that {cf.[3]} the order $\rho_{f,D}$ is independent of the choice of the domain D and therefore we denote the order of f as ρ_f.

In a recent paper Mondal and Roy [4] introduced the concept of relative order of entire functions of n complex variables.In this regard they gave the following definition.

Definition 5 ([4])**.** *Let* f *and* g *be entire functions of* n *complex variables and* D *be a bounded complete* $n-$*circular domain with centre at the origin in* $\mathbb{C}^n$*. Then the relative order* $\rho_{g,D}(f)$ *of* f *with respect to* g *in the domain* D *is defined by*

$$\begin{aligned}\rho_{g,D}(f) &= \inf\{\mu > 0 : M_{f,D}(R) < M_{g,D}(R^{\mu}) \;\; for \;\; R \geq R(\mu)\} \\ &= \limsup_{R\to\infty} \frac{\log M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.\end{aligned}$$

If we take $g(z) = e^z = e^{(z_1, z_2, \ldots, z_n)}$ *then the relative order* $\rho_{g,D}(f)$ *of* f *with respect to* g *in the domain* D *coincides with the Goldberg order* $\rho_{f,D}$ *of* f *with respect to domain* D.

We define the relative lower order $\lambda_{g,D}(f)$ of f with respect to g in the domain D as

$$\lambda_{g,D}(f) = \lim \inf_{R\to\infty} \frac{\log M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.$$

We say that f is of regular relative growth in the domain D if $\lambda_{g,D}(f) = \rho_{g,D}(f)$.

Mondal and Roy [4] proved that the relative order of f with respect to g is independent of the choice of the domain D and therefore we denote the relative order of f with respect to g as

$$\rho_{g,D}(f) = \rho_g(f) = \lim \sup_{R\to\infty} \frac{\log M_g^{-1} M_f(R)}{\log R}.$$

Similarly we denote the relative lower order as

$$\lambda_{g,D}(f) = \lambda_g(f) = \lim \sup_{R\to\infty} \frac{\log M_g^{-1}(M_f(R))}{\log R}.$$

Throughout the paper, unless otherwise mentioned, we consider transcendental entire functions of n complex variables and D will represent a bounded complete $n-$circular domain.

In this paper we obtain some relationship between relative order, relative lower order,Gol'dberg order and lower Gol'dberg order of an entire functions of several complex veriables which improves some earlier results of [4] and [5].

2. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be entire functions of n complex variables such that $0 < \lambda_g \le \rho_g$ and $0 < \lambda_f \le \rho_f$ .Then*

$$\frac{\lambda_f}{\rho_g} \le \lambda_g(f) \le \min\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \max\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \rho_g(f) \le \frac{\rho_f}{\lambda_g}.$$

Proof. From the definition of Gol'dberg order and lower Gol'dberg order we get for arbitrary $\varepsilon(>0)$ and all large values of R that

$$M_f(R) < \exp(R^{\rho_f+\varepsilon}), \tag{1}$$

$$M_g(R) < \exp(R^{\rho_g+\varepsilon}), \tag{2}$$

$$M_f(R) > \exp(R^{\lambda_f-\varepsilon}) \tag{3}$$

and

$$M_g(R) > \exp(R^{\lambda_g - \varepsilon}). \tag{4}$$

Also for a sequence $\{R_n\}$ tending to infinity we get that

$$M_f(R_n) > \exp(R_n^{\rho_f - \varepsilon}), \tag{5}$$

$$M_g(R_n) > \exp(R_n^{\rho_g - \varepsilon}), \tag{6}$$

$$M_f(R_n) < \exp(R_n^{\lambda_f + \varepsilon}) \tag{7}$$

and

$$M_f(R_n) < \exp(R_n^{\lambda_g + \varepsilon}). \tag{8}$$

Now from the definition of relative order we get for arbitrary $\varepsilon_1(> 0)$ and all large values of R that

$$\rho_g(f) + \varepsilon_1 > \frac{\log M_g^{-1} M_f(R)}{\log R}.$$

Now from (5) we get for a sequence $\{R_n\}$ tending to infinity that

$$\begin{aligned}\rho_g(f) + \varepsilon_1 &> \frac{\log M_g^{-1}\left[\exp(R_n^{\rho_f - \varepsilon})\right]}{\log R_n} = \frac{\log M_g^{-1}\left[\exp\left(R_n^{\frac{\rho_f - \varepsilon}{\rho_g + \varepsilon}}\right)^{\rho_g + \varepsilon}\right]}{\log R_n} \\ &> \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\rho_f - \varepsilon}{\rho_g + \varepsilon}}\right)}{\log R_n} = \frac{\rho_f - \varepsilon}{\rho_g + \varepsilon}.\end{aligned}$$

As $\varepsilon_1(> 0)$ and $\varepsilon(> 0)$ are arbitrary we obtain that

$$\rho_g(f) \geq \frac{\rho_f}{\rho_g}. \tag{9}$$

Also from (1) we get for arbitrary $\varepsilon(> 0)$ and for all large values of R that

$$\begin{aligned}\frac{\log M_g^{-1} M_f(R)}{\log R} &< \frac{\log M_g^{-1}\left[\exp(R^{\rho_f + \varepsilon})\right]}{\log R} \\ &= \frac{\log M_g^{-1}\left[\exp\left(R^{\frac{\rho_f + \varepsilon}{\rho_g - \varepsilon}}\right)^{\rho_g - \varepsilon}\right]}{\log R}.\end{aligned}$$

Now from (6) we get for a sequence $\{R_n\}$ tending to infinity that

$$\frac{\log M_g^{-1} M_f(R_n)}{\log R_n} < \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\rho_f+\varepsilon}{\rho_g-\varepsilon}}\right)}{\log R_n}$$
$$i.e., \liminf_{R_n\to\infty} \frac{\log M_g^{-1} M_f(R_n)}{\log R_n} \le \frac{\rho_f+\varepsilon}{\rho_g-\varepsilon}.$$

As ε (>0) is arbitrary we have

$$\lambda_g(f) \le \frac{\rho_f}{\rho_g}. \tag{10}$$

Now from the definition of relative lower order we get for arbitrary ε_2 (>0) and for all large values of R that

$$\lambda_g(f) - \varepsilon_2 < \frac{\log M_g^{-1} M_f(R)}{\log R}.$$

Now from (7) we get for a sequence $\{R_n\}$ tending to infinity that

$$\lambda_g(f) - \varepsilon_2 < \frac{\log M_g^{-1}\left[\exp(R_n^{\lambda_f+\varepsilon})\right]}{\log R_n} = \frac{\log M_g^{-1}\left[\exp\left(R_n^{\frac{\lambda_f+\varepsilon}{\lambda_g-\varepsilon}}\right)^{\lambda_g-\varepsilon}\right]}{\log r_n}$$
$$< \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\lambda_f+\varepsilon}{\lambda_g-\varepsilon}}\right)}{\log R_n} = \frac{\lambda_f+\varepsilon}{\lambda_g-\varepsilon}.$$

As ε_2 (>0) and ε (>0) are arbitrary we obtain that

$$\lambda_g(f) \le \frac{\lambda_f}{\lambda_g}. \tag{11}$$

Now from (3) we get for arbitrary $\varepsilon(>0)$ and for all large values of R that

$$\frac{\log M_g^{-1} M_f(R)}{\log R} > \frac{\log M_g^{-1}\left[\exp(R^{\lambda_f-\varepsilon})\right]}{\log R}$$
$$= \frac{\log M_g^{-1}\left[\exp\left(R^{\frac{\lambda_f-\varepsilon}{\lambda_g+\varepsilon}}\right)^{\lambda_g+\varepsilon}\right]}{\log R}.$$

Now from (8) we obtain for a sequence $\{R_n\}$ tending to infinity that

$$\frac{\log M_g^{-1} M_f(R_n)}{\log R_n} > \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\lambda_f-\varepsilon}{\lambda_g+\varepsilon}}\right)}{\log R_n}$$

$$i.e., \ \limsup_{R_n\to\infty} \frac{\log M_g^{-1} M_f(R_n)}{\log R_n} \geq \frac{\lambda_f-\varepsilon}{\lambda_g-\varepsilon}.$$

As ε (>0) is arbitrary we get that

$$\rho_g(f) \geq \frac{\lambda_f}{\lambda_g}. \tag{12}$$

Now we get for arbitrary ε_3 (>0) and for a sequence $\{R_n\}$ tending to infinity that

$$\begin{aligned}\rho_g(f)-\varepsilon_3 &< \frac{\log M_g^{-1} M_f(R_n)}{\log R_n} < \frac{\log M_g^{-1}\left[\exp(R_n^{\rho_f+\varepsilon})\right]}{\log R_n} \\ &= \frac{\log M_g^{-1}\left[\exp\left(R_n^{\frac{\rho_f+\varepsilon}{\lambda_g-\varepsilon}}\right)^{\lambda_g-\varepsilon}\right]}{\log R_n} \\ &< \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\rho_f+\varepsilon}{\lambda_g-\varepsilon}}\right)}{\log R_n} = \frac{\rho_f+\varepsilon}{\lambda_g-\varepsilon}.\end{aligned}$$

As ε_3 (>0) and ε (>0) are arbitrary we have

$$\rho_g(f) \leq \frac{\rho_f}{\lambda_g}. \tag{13}$$

Now we get for arbitrary ε_4 (>0) and for a sequence $\{R_n\}$ tending to infinity that

$$\begin{aligned}\lambda_g(f)+\varepsilon_4 &> \frac{\log M_g^{-1} M_f(R_n)}{\log R_n} > \frac{\log M_g^{-1}\left[\exp(R_n^{\lambda_f-\varepsilon})\right]}{\log R_n} \\ &= \frac{\log M_g^{-1}\left[\exp\left(R_n^{\frac{\lambda_f-\varepsilon}{\rho_g+\varepsilon}}\right)^{\rho_g+\varepsilon}\right]}{\log R_n} \\ &> \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\lambda_f-\varepsilon}{\rho_g+\varepsilon}}\right)}{\log R_n} = \frac{\lambda_f-\varepsilon}{\rho_g+\varepsilon}.\end{aligned}$$

As ε_4 (>0) and ε (>0) are arbitrary we obtain that

$$\lambda_g(f) \geq \frac{\lambda_f}{\rho_g}. \tag{14}$$

The theorem follows from (9), (10), (11), (12), (13) and (14). ■

Corollary 1. *If g is of regular growth with nonzero order then*

$$\rho_g(f) = \frac{\rho_f}{\rho_g}.$$

Corollary 2. *If both f and g are of regular growth with nonzero order then*

$$\rho_g(f) = \rho_f(g) \quad iff \quad \rho_f = \rho_g.$$

Corollary 3. *If both f and g are of regular growth with nonzero order then* $\lambda_g(f) = \rho_g(f)$.

Remark 1. Theorem 1 improves Theorem 2.5 of Mondal and Roy [4].

Remark 2. The $\leq$ sign in Theorem 1 cannot be replaced by $<$ which is evident from the following example.

Example 1. Let $f(z) = e^{2z_1 z_2 \ldots z_n}$ and $g(z) = e^{z_1 z_2 \ldots z_n}$.Then $\lambda_g = \rho_g = \lambda_f = \rho_f = 1$.Also $\rho_g(f) = \lambda_g(f) = 1$ and therefore

$$\frac{\lambda_f}{\rho_g} = \lambda_g(f) = \min\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} = \max\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} = \rho_g(f) = \frac{\rho_f}{\lambda_g}.$$

Remark 3. As both f and g are of regular growth with nonzero order and $\rho_g(f) = \rho_f(g) = 1$ the Example1 also verifies the above three corollaries.

Theorem 2. *Let f and g be entire functions of n complex variables such that* $0 < \rho_f < \infty$ *and* $\rho_g = 0$. *Then*

$$\rho_g(f) = \infty.$$

Proof. From the definition of relative order we get for arbitrary $\varepsilon_1(>0)$ and all large values of R that

$$\rho_g(f) + \varepsilon_1 > \frac{\log M_g^{-1} M_f(R)}{\log R}.$$

Now from (5) we get for a sequence $\{R_n\}$ tending to infinity that

$$\rho_g(f) + \varepsilon_1 > \frac{\log M_g^{-1}\left[\exp(R_n^{\rho_f - \varepsilon})\right]}{\log R_n} = \frac{\log M_g^{-1}\left[\exp\left(R_n^{\frac{\rho_f-\varepsilon}{\varepsilon}}\right)^{\varepsilon}\right]}{\log R_n}$$

$$> \frac{\log M_g^{-1} M_g\left(R_n^{\frac{\rho_f-\varepsilon}{\varepsilon}}\right)}{\log R_n} = \frac{\rho_f - \varepsilon}{\varepsilon}.$$

As $\varepsilon_1(>0)$ and ε (>0) are arbitrary it follows that $\rho_g(f)=\infty$. ■

Theorem 3. *Let f and g be entire functions of n complex variables such that $0<\rho_g<\infty$ and $\rho_f=0$. Then*

$$\lambda_g(f)=0.$$

Proof. From the definition of Gol'dberg order we have for arbitrary ε (>0) and for all large values of R that

$$M_f(R)<\exp(R^{\varepsilon})$$

$$\begin{aligned} i.e., \ \frac{\log M_g^{-1}M_f(R)}{\log R} &< \frac{\log M_g^{-1}\left[\exp(R^{\varepsilon})\right]}{\log R} \\ &= \frac{\log M_g^{-1}\left[\exp\left(R^{\frac{\varepsilon}{\rho_g-\varepsilon}}\right)^{\rho_g-\varepsilon}\right]}{\log R}. \end{aligned}$$

Now from (6) we get for a sequence $\{R_n\}$ tending to infinity that

$$\begin{aligned} \frac{\log M_g^{-1}M_f(R_n)}{\log R_n} &< \frac{\log M_g^{-1}M_g\left(R_n^{\frac{\varepsilon}{\rho_g-\varepsilon}}\right)}{\log R_n} \\ i.e., \ \liminf_{r_n\to\infty}\frac{\log M_g^{-1}M_f(R_n)}{\log R_n} &\le \frac{\varepsilon}{\rho_g-\varepsilon}. \end{aligned}$$

As ε (>0) is arbitrary it follows that $\lambda_g(f)=0$. ■

Remark 4. If $\rho_f=\rho_g=0$ then Theorem 2 and Theorem 3 are not true which is evident from the following example.

Example 2. Let $f(z)\equiv z$ and $g(z)=z^2$.Then $\rho_f=\rho_g=0$, but

$$\rho_g(f) = \limsup_{r\to\infty}\frac{\log M_g^{-1}M_f(R)}{\log R}=\limsup_{r\to\infty}\frac{\log\sqrt{R}}{\log R}=\frac{1}{2}.$$

Also

$$\lambda_f(g) = \liminf_{r\to\infty}\frac{\log M_f^{-1}M_g(R)}{\log R}=\liminf_{r\to\infty}\frac{\log R^2}{\log R}=2.$$

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ARKOJYOTI BISWAS
RANAGHAT YUSUF INSTITUTION
P.O.-RANAGHAT, DIST-NADIA, PIN-741201
WEST BENGAL, INDIA
e-mail: arkojyotibiswas@gmail.com

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