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A GENERAL FIXED POINT THEOREM ON G-METRIC SPACES*

Abstract. In this paper, we prove a general fixed point theorem on G-metric spaces by an implicit relation. This result unifies many fixed point theorems in [3], [6], [9], [12].

Key words: fixed point theorem, G-metric space.

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1. Introduction

In [8], Z. Mustafa and B. Sims introduced the concept of G-metric spaces as follows.

Definition 1 ([8], Definition 3)**.** *Let X be a nonempty set and the function $G : X \times X \times X \longrightarrow \mathbb{R}_+$ satisfy the following.*

($G1$) $G(x, y, z) = 0$ *if* $x = y = z$.

($G2$) $0 < G(x, x, y)$ *for all* $x \neq y \in X$.

($G3$) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y \neq z \in X$.

($G4$) *The symmetry on three variables:*

$$\begin{aligned} G(x,y,z) &= G(x,z,y) = G(y,x,z) = G(y,z,x) \\ &= G(z,x,y) = G(z,y,x) \end{aligned}$$

for all $x, y, z \in X$.

($G5$) *The rectangle inequality:*

$$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$$

for all $x, y, z, a \in X$.

*Then G is called a G-*metric *on X and the pair (X, G) is called a G-*metric *space.*

Many authors have been interested in the fixed point problem on G-metric spaces and many results have been obtained in [1], [2], [3], [4], [5], [6], [7],

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[9], [10], [11], [12]. In this paper, we prove a general fixed point theorem on G-metric spaces by an implicit relation. This result unifies many fixed point theorems in [3], [6], [9], [12].

2. Main results

First we recall some notions and lemmas.

Definition 2 ([8]). *Let (X, G) be a G-metric space and $x_0 \in X$, $r > 0$. The set*

$$B_G(x_0, r) = \{x \in X : G(x_0, x, x) < r\}$$

*is called a G-*ball with center x_0 and radius r. *The family of all G-balls forms a base of a topology $\tau(G)$ on X, and $\tau(G)$ is called a G-*metric topology. *The sequence $\{x_n\}$ is called to be G-*convergent *to x in X if $x_n \to x$ in the G-metric topology $\tau(G)$. The sequence $\{x_n\}$ is called to be G-*Cauchy *in X if $G(x_n, x_m, x_l) \to 0$ as $m, n, l \to \infty$. X is called a* complete G-metric *space if every G-Cauchy sequence is G-convergent.*

Lemma 1 ([8], Proposition 6). *Let (X, G) be a G-metric space. Then the following statements are equivalent.*

(i) *x_n is G-convergent to x in X.*
(ii) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
(iii) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
(iv) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Lemma 2 ([8], Proposition 9). *Let (X, G) be a G-metric space. Then the following statements are equivalent.*

(i) *$\{x_n\}$ is a G-Cauchy sequence.*
(ii) *$G(x_n, x_m, x_m) \to 0$ as $m, n \to \infty$.*

Lemma 3 ([8], Proposition 8). *Let (X, G) be a G-metric space. Then G is jointly continuous in all three of its variables.*

Now we introduce an implicit relation to state the main result. Let $\mathcal{M}$ be the set of all continuous ten-variables functions $M : \mathbb{R}_+^{10} \longrightarrow \mathbb{R}_+$. We consider following conditions for all $x, y, z, x_i, y_i, z_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, and some $k \in [0, 1)$.

$(C1)$ If $y \leq M(x, x, z, z, y, 0, y, y, 0, y)$ and $z \leq x + y$, then $y \leq kx$.

$(C2)$ If $y \leq M(0, 0, y, y, 0, x, 0, 0, x, 0)$, then $y \leq kx$.

$(C3)$ $M(x, x, x, x, x, x, x, 0, 0, 0) \leq kx$
$M(0, x, x, 0, x, x, 0, 0, x, x) \leq kx$

and if $x_i \leq y_i + z_i$, then $M(x_0, \ldots, x_9) \leq M(y_0, \ldots, y_9) + M(z_0, \ldots, z_9)$.

Next we state the main result of the paper with respect to the above implicit relation.

Theorem 1. *Let T be a self-map on a complete G-metric space (X,G) and*

$$G(Tx,Ty,Tz) \leq M\Big(G(x,y,z), G(x,T(x),T(x)), G(x,T(y),T(y)), G(x,T(z),T(z)), G(y,T(y),T(y)), G(y,T(x),T(x)), G(y,T(z),T(z)), G(z,T(z),T(z)), G(z,T(x),T(x)), G(z,T(y),T(y))\Big) \tag{1}$$

for some $M \in \mathcal{M}$ and all $x,y,z \in X$. Then we have

(i) If M satisfies the condition (C1), then T has a fixed point.

(ii) If M satisfies the condition (C2) and T has a fixed point, then the fixed point is unique.

(iii) If M satisfies the condition (C3) and T has a fixed point, then T is continuous at the fixed point.

Proof. (i) For any $x_0 \in X$ and all $n \in \mathbb{N}$, put $x_n = Tx_{n-1}$. It follows from (1) that

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq M\Big(G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(x_{n-1}, Tx_n, Tx_n), G(x_{n-1}, Tx_n, Tx_n), \\ &\quad G(x_n, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1}), \\ &\quad G(x_n, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n), \\ &\quad G(x_n, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n)\Big) \\ &= M\Big(G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), \\ &\quad G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\Big) \\ &= M\Big(G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), \\ &\quad G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), 0, G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), 0, G(x_n, x_{n+1}, x_{n+1})\Big). \end{aligned}$$

Since

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})$$

by $(G5)$ and M satisfies the condition $(C1)$, then there exists $k \in [0,1)$ such that

$$G(x_n, x_{n+1}, x_{n+1}) \le kG(x_{n-1}, x_n, x_n) \le k^n G(x_0, x_1, x_1). \tag{2}$$

For all $n < m$, by using $(G5)$ and (2) we have

$$\begin{aligned} 0 \le G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m) \\ &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \ldots + G(x_{m-1}, x_m, x_m) \\ &\le (k^n + k^{n+1} + \ldots + k^{m-1})G(x_0, x_1, x_1) \\ &\le \frac{k^n}{1-k} G(x_0, x_1, x_1). \end{aligned}$$

Taking the limit as $m, n \to \infty$ we get $G(x_n, x_m, x_m) \to 0$. By Lemma 2, $\{x_n\}$ is a G-Cauchy sequence. Since X is complete, then $x_n \to u$. Now we prove that u is a fixed point of T. By using (1) again we get

$$\begin{aligned} G(x_{n+1}, Tu, Tu) &= G(Tx_n, Tu, Tu) \\ &\le M\Big(G(x_n, u, u), \\ &\quad G(x_n, Tx_n, Tx_n), G(x_n, Tu, Tu), G(x_n, Tu, Tu), \\ &\quad G(u, Tu, Tu), G(u, Tx_n, Tx_n), G(u, Tu, Tu), \\ &\quad G(u, Tu, Tu), G(u, Tx_n, Tx_n), G(u, Tu, Tu)\Big) \\ &= M\Big(G(x_n, u, u), \\ &\quad G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tu, Tu), G(x_n, Tu, Tu), \\ &\quad G(u, Tu, Tu), G(u, x_{n+1}, x_{n+1}), G(u, Tu, Tu), \\ &\quad G(u, Tu, Tu), G(u, x_{n+1}, x_{n+1}), G(u, Tu, Tu)\Big). \end{aligned}$$

By Lemma 3 and $M \in \mathcal{M}$, taking the limit as $n \to \infty$ we have

$$\begin{aligned} G(u, Tu, Tu) &\le M\Big(G(u, u, u), \\ &\quad G(u, u, u), G(u, Tu, Tu), G(u, Tu, Tu), \\ &\quad G(u, Tu, Tu), G(u, u, u), G(u, Tu, Tu), \\ &\quad G(u, Tu, Tu), G(u, u, u), G(u, Tu, Tu)\Big) \\ &= M\Big(0, 0, G(u, Tu, Tu), G(u, Tu, Tu), \\ &\quad G(u, Tu, Tu), 0, G(u, Tu, Tu), \\ &\quad G(u, Tu, Tu), 0, G(u, Tu, Tu)\Big). \end{aligned}$$

Since M satisfies the condition $(C1)$, then $G(u,Tu,Tu) \leq k.0$. This proves that $G(u,Tu,Tu)=0$ or $Tu=u$.

(ii) Let $u,v \in X$ and

$$Tu = u, \qquad Tv = v.$$

We shall prove that $u=v$. By using (1) again we get

$$\begin{aligned} G(v,u,u) &= G(Tv,Tu,Tu) \\ &\leq M\Big(G(v,Tv,Tv), \\ &\quad G(v,Tv,Tv), G(v,Tu,Tu), G(v,Tu,Tu), \\ &\quad G(u,Tu,Tu), G(u,Tv,Tv), G(u,Tu,Tu) \\ &\quad G(u,Tu,Tu), G(u,Tv,Tv), G(u,Tu,Tu)\Big) \\ &= M\Big(G(v,v,v) \\ &\quad G(v,v,v), G(v,u,u), G(v,u,u), \\ &\quad G(u,u,u), G(u,v,v), G(u,u,u) \\ &\quad G(u,u,u), G(u,v,v), G(u,u,u)\Big) \\ &= M\Big(0,0,G(v,u,u),G(v,u,u), \\ &\quad 0, G(u,v,v), 0, 0, G(u,v,v), 0\Big). \end{aligned}$$

Since M satisfies the condition $(C2)$, then

(3) $$G(v,u,u) \leq kG(u,v,v).$$

By a similar argument we get

(4) $$G(u,v,v) \leq kG(v,u,u).$$

It follows from (3) and (4) that

$$G(v,u,u) \leq k^2 G(v,u,u).$$

Then $G(v,u,u)=0$. This proves that $u=v$. (1). Let $u=Tu$ and $x_n \to u$ in X. To prove T is continuous at u, we shall prove that $Tx_n \to Tu$. By using (1) again we get

$$\begin{aligned} G(Tx_n,Tx_n,Tu) &\leq M\Big(G(x_n,x_n,u), \\ &\quad G(x_n,Tx_n,Tx_n), G(x_n,Tx_n,Tx_n), G(x_n,Tu,Tu), \\ &\quad G(x_n,Tx_n,Tx_n), G(x_n,Tx_n,Tx_n), G(x_n,Tu,Tu), \\ &\quad G(u,Tu,Tu), G(u,Tx_n,Tx_n), G(u,Tx_n,Tx_n)\Big) \end{aligned}$$

$$
\begin{aligned}
&= M\Big(G(x_n,x_n,u),\\
&\quad G(x_n,Tx_n,Tx_n),G(x_n,Tx_n,Tx_n),G(x_n,u,u),\\
&\quad G(x_n,Tx_n,Tx_n),G(x_n,Tx_n,Tx_n),G(x_n,u,u),\\
&\quad 0,G(u,Tx_n,Tx_n),G(u,Tx_n,Tx_n)\Big).
\end{aligned}
$$

It follows from $(G5)$ that

$$
\begin{aligned}
G(x_n,x_n,u) &\leq G(x_n,x_n,u)+0\\
G(x_n,Tx_n,Tx_n) &\leq G(x_n,u,u)+G(u,Tx_n,Tx_n)\\
G(x_n,u,u) &\leq G(x_n,u,u)+0\\
0 &\leq 0+0\\
G(u,Tx_n,Tx_n) &\leq 0+G(u,Tx_n,Tx_n).
\end{aligned}
$$

Since M satisfies the condition $(C3)$, then

$$
\begin{aligned}
G(Tx_n,Tx_n,Tu) &\leq M\Big(G(x_n,x_n,u),G(x_n,u,u),\\
&\quad G(x_n,u,u),G(x_n,u,u),G(x_n,u,u),\\
&\quad G(x_n,u,u),G(x_n,u,u),0,0,0\Big)\\
&+ M\Big(0,G(u,Tx_n,Tx_n),G(u,Tx_n,Tx_n),0,\\
&\quad G(u,Tx_n,Tx_n),G(u,Tx_n,Tx_n),0,\\
&\quad 0,G(u,Tx_n,Tx_n),G(u,Tx_n,Tx_n)\Big)\\
&\leq k.G(x_n,x_n,u)+k.G(u,Tx_n,Tx_n).
\end{aligned}
$$

It implies that

$$0 \leq G(Tx_n,Tx_n,u) \leq \frac{k}{1-k}G(x_n,x_n,u).$$

Taking the limit as $n \to \infty$ we get $G(Tx_n,Tx_n,u) \to 0$. Then by Lemma 1 we have

$$Tx_n \to u = Tu.$$

This proves that T is continuous at u. ■

Next we show that many known fixed point theorems are particular cases of Theorem 1.

Remark 1. By choosing

$$M(t_0,t_1,\ldots,t_9) = k\max\{0,t_1,\ldots,t_9\}$$

with $k \in \left[0, \frac{1}{2}\right)$ and $t_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [12, Theorem 1] and [12, Corollary 1] are particular cases of Theorem 1.

Remark 2. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\{0, t_1 + t_4 + t_7, t_2 + t_5 + t_9, t_3 + t_6 + t_8\}$$

with $k \in \left[0, \frac{1}{4}\right)$ and $t_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [12, Theorem 2] and [12, Corollary 2] are particular cases of Theorem 1.

Remark 3. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\{t_0, t_1, t_4, t_7, t_2, t_6, t_8\}$$

with $k \in \left[0, \dfrac{1}{2}\right)$ and $t_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, then $M \in \mathcal{M}$ and M satisfies the conditions (C1), (C2) and (C3). So [9, Theorem 2.1] and [9, Corollary 2.3] are particular cases of Theorem 1.

Remark 4. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\{t_2 + t_5, t_6 + t_9, t_3 + t_8\}$$

with $k \in \left[0, \frac{1}{2}\right)$ and $t_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [9, Theorem 2.4] and [9, Corollary 2.5] are particular cases of Theorem 1.

Remark 5. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\{t_4 + t_2, 2t_5\}$$

with $k \in \left[0, \frac{1}{3}\right)$ and $t_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [9, Theorem 2.6] and [9, Corollary 2.7] are particular cases of Theorem 1.

Remark 6. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\{t_8 + t_5, t_6 + t_3, t_2 + t_9\}$$

with $k \in \left[0, \frac{1}{3}\right)$ and $t_i \in \mathbb{R}_+$, $0 \leq i \leq 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [9, Theorem 2.8] is a particular case of Theorem 1.

Remark 7. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\left\{t_0, t_1, t_4, t_7, \frac{t_2 + t_8}{2}, \frac{t_2 + t_5}{2}, \frac{t_6 + t_9}{2}, \frac{t_3 + t_8}{2}\right\}$$

with $k \in [0, 1)$ and $t_i \in \mathbb{R}_+$, $0 \le i \le 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [3, Theorem 2.1] is a particular case of Theorem 1.

Remark 8. By choosing

$$M(t_0, t_1, \ldots, t_9) = k \max\{t_0, t_1, t_4, t_2, t_5, t_7\}$$

with $k \in [0, 1)$ and $t_i \in \mathbb{R}_+$, $0 \le i \le 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$, then the first part of [3, Theorem 2.2] is a particular case of Theorem 1.

Remark 9. By choosing

$$M(t_0, t_1, \ldots, t_9) = k(t_1 + t_4 + t_7)$$

with $k \in \left[0, \frac{1}{3}\right)$ and $t_i \in \mathbb{R}_+$, $0 \le i \le 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [6, Theorem 2.1] and [6, Corollary 1] are particular cases of Theorem 1.

Remark 10. By choosing

$$M(t_0, t_1, \ldots, t_9) = \alpha t_0 + \beta(t_1 + t_4 + t_7)$$

with $\alpha + 3\beta \in [0, 1)$ and $t_i \in \mathbb{R}_+$, $0 \le i \le 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [6, Theorem 2.2] and [6, Corollary 2] are particular cases of Theorem 1.

Remark 11. By choosing

$$M(t_0, t_1, \ldots, t_9) = \alpha t_0 + \beta \max\{t_1, t_4, t_7\}$$

with $\alpha + \beta \in [0, 1)$ and $t_i \in \mathbb{R}_+$, $0 \le i \le 9$, then $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$. So [6, Theorem 2.3] and [6, Corollary 3] are particular cases of Theorem 1.

Remark 12. We can have more new fixed point theorems if we combine Theorem 1 with some examples of $M \in \mathcal{M}$ and M satisfies the conditions $(C1)$, $(C2)$ and $(C3)$.

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