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SOME CLASSES OF IDEAL CONVERGENT DIFFERENCE SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY ORLICZ FUNCTION

Abstract. An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In [25], Kostyrko et. al introduced the concept of ideal convergence as a sequence (x_k) of real numbers is said to be I-convergent to a real number ℓ, if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}$ belongs to I. In this article we introduce the concept of ideal convergent sequence of fuzzy numbers using difference operator and Orlicz functions and study their basic facts. Also we investigate the different algebraic and topological properties of these classes of sequences.

Key words: ideal, I-convergence, fuzzy number, normal space, symmetric space, difference space.

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1. Introduction

The concept of fuzzy set and fuzzy set operations were first introduced by Zadeh [45] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarly relations and fuzzy orderings, fuzzy measures of fuzzy events. In fact the fuzzy set theory has become an area of active research for the last 40 years. To overcome the limitations induced by vagueness and uncertainty of real life data, neoclassical analysis [3] has been developed. It extends the scope and results of classical mathematical analysis objects; such as functions, sequences and series. The fuzzy set theory has been used widely not only in many engineering applications, such as, in bifurcation of non-linear dynamical systems [22], in the computer programming [13], in the non-linear operator [32], in population dynamics [2], but also in various branches of mathematics, such as, in the theory of linear systems [35], in approximations theory ([1], [6]).

On the other hand the concept of ordinary convergence of sequences of fuzzy numbers was firstly introduced by Matloka [30], where he proved some basic theorems for sequences of fuzzy numbers. Nanda [33] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers from a complete metric space. Recently Kumar and Kumar [27] introduced the ideal convergence of sequences of fuzzy numbers.

Throughout the article $w^F, \ell_\infty^F, c^F, c_0^F$ denote the classes of all, bounded, convergent and null sequence spaces of fuzzy real numbers, respectively.

The notions of I-convergence was introduced and studied at the initial stage by Kostyrko et al. [25], it is generalized form of statistical convergence, which was introduced by Fast [12]. Later on it was further investigated by Šalát et al. ([36],[37]), Tripathy and Hazarika ([40], [41], [42], [43]), Tripathy and Mahanta [39] , Esi and Hazarika ([7], [8]), Hazarika ([15],[16], [17], [18], [19], [20], [21]), Hazarika and Savas [14] and many others. Also I-convergence has been discussed in more general abstract spaces such as the fuzzy numbers spaces [35], 2-normed linear spaces [38].

Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S if I is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e.$A \in I, B \subseteq A \Rightarrow B \in I$). An ideal $I \subseteq P(S)$ is said to be non trivial if $I \subseteq P(S)$. A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S if $\phi \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and for each $A \in F$ and $B \supset A$, implies $B \in F$. For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$. A non-trivial ideal $I \subseteq P(S)$ is called an *admissible ideal* on S if and only if it contains all singletons, i.e., if it contains $\{\{x\} : x \in S\}$. A non-trivial ideal I is said to be *maximal*, if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

The difference sequence space was introduced by Kizmaz [24] as follows:

$$Z(\Delta) = \{(x_k) \in w : \Delta x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ and $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The idea of difference sequences was generalized by Colak and Et [4], Colak et al. [5], Et and Basarir [9], Et and Colak [10], Et et al. [11]. The operator $\Delta^m : w^F \to w^F$ is defined by

$$(\Delta^0 X_k) = X_k, (\Delta^1 X_k) = \Delta^1 X_k = X_k - X_{k+1},$$
$$(\Delta^m X_k) = \Delta^1(\Delta^{m-1} X_k), m \geq 2 \text{ for all } k \in \mathbb{N},$$

which is equivalent to the following binomial representation,

$$\Delta^m X_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} X_{k+\nu} \text{ for all } k \in \mathbb{N}.$$

Tripathy, et al [44] further generalized this notion and introduced the following. For $m \geq 1$ and $n \geq 1$,

$$Z(\Delta_n^m) = \{(x_k) \in w : \Delta_n^m x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$.

This generalized difference has the following binomial representation,

$$\Delta_n^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+\nu n} \text{ for all } k \in \mathbb{N}. \tag{1}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(0) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

At the initial stage Lindberg [28] was studied Orlicz space in connection with Banach space with symmetric. Nung and Lee [34] were studied different classes of sequence spaces defined by Orlicz function. Later on the notion was studied by Mursaleen et al. [31] and many others.

Remark 1. It is well known if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to be satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$(see Krasnoselski and Rutitsky [26]).

Lindenstrauss and Tzafriri [29] used the idea of Orlicz function to construct the sequence space,

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

2. Definitions and preliminaries

A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$ i.e. a mapping $X : \mathbb{R} \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is convex if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$. If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$, then the fuzzy number X is called *normal*. The α -level set of a fuzzy real number $X, 0 < \alpha \leq 1$ denoted by X^α is defined as $X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$.

A fuzzy number X is said to be *upper-semi continuous* if for each $\varepsilon 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $\mathbb{R}(J)$.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ in D, we define

$$X \leq Y \text{ if and only if] } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Define a metric d on D by

$$d(X,Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation $\leq$ is a partial order on D.

The absolute value $|X|$ of $X \in \mathbb{R}(J)$ is defined as

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}$ be defined by

$$\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $(\mathbb{R}(J), \bar{d})$ is a complete metric space.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$, for all $\alpha \in J$. The additive identity and multiplicative identity in $\mathbb{R}(J)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

A sequence $X = (X_k)$ of fuzzy numbers is said to bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

A sequence (X_k) of fuzzy real numbers is said to be *convergent* to a fuzzy real number X_0 if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq n_0$.

A subset A of $\mathbb{N}$ is said to have *asymptotic density* (or *density*) $\delta(A)$ if

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k)$$

exists, where χ_A is the characteristic function of A.

A sequence (X_k) of fuzzy numbers is said to be *statistical convergent* if there exists a fuzzy number X_0 such that for each $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : \overline{d}(X_k, X_0) \geq \varepsilon\}) = 0$. In this case we write $st - \lim X_k = X_0$.

A sequence (X_k) of fuzzy real numbers is said to be *I-convergent* if there exists a fuzzy real number X_0 such that for each $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon\} \in I.$$

We write $I - \lim X_k = X_0$.

A sequence space E_F of fuzzy numbers is said to be *normal* (or *solid*) if $(\alpha_k X_k) \in E_F$ whenever $(X_k) \in E_F$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ and $|Y_k| \leq |X_k|$, for all $k \in \mathbb{N}$.

A sequence space E_F of fuzzy numbers is said to be *symmetric* if $(X_{\pi(k)}) \in E_F$, whenever $(X_k) \in E_F$, where π is a permutation of $\mathbb{N}$.

Let $K = \{k_1 < k_2 < \ldots\} \subseteq \mathbb{N}$ and E be a sequence space. A *K-step space* of E is a sequence space

$$\lambda_K^{E_F} = \{(X_{k_n}) \in w^F : (k_n) \in E_F\}.$$

A canonical preimage of a sequence $\{(X_{k_n})\} \in \lambda_K^{E_F}$ is a sequence $\{Y_k\} \in w^F$ defined as

$$Y_k = \begin{cases} X_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda_K^{E_F}$ is a set of canonical preimages of all elements in $\lambda_K^{E_F}$, i.e. Y is in canonical preimage of $\lambda_K^{E_F}$ if and only if Y is canonical preimage of some $X \in \lambda_K^{E_F}$.

A sequence space E_F of fuzzy numbers is said to be *monotone* if E_F contains the cannical pre-image of all its step spaces.

A sequence space E_F is said to be a *sequence algebra* if $(X_k \otimes Y_k) \in E_F$, whenever $(X_k), (Y_k) \in E_F$

A sequence space E_F is said to be *convergence free* if $(Y_k) \in E_F$ whenever $(X_k) \in E_F$ and $X_k = 0$ implies $Y_k = 0$.

A subset A of $\mathbb{N}$ is said to have *logarithmic density* $d(A)$ if $d(A) = \lim_{n\to\infty} d_n(A) = \frac{1}{s_n} \sum\limits_{k=1}^{n} \frac{\chi_A(k)}{k}$ exists, for all $n \in \mathbb{N}$, where $s_n = \sum\limits_{k=1}^{n} \frac{1}{k}$.

Example 1. Let $I_\delta\{A \subset \mathbb{N} : \delta(A) = 0\}$. Then I_δ is an ideal of $\mathbb{N}$. Also all finite subsets of $2^{\mathbb{N}}$ have zero asymptotic density and $\delta(A^c) = \delta(\mathbb{N} - A) = 1 - \delta(A)$.

Example 2. Let $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then I_d is an ideal of $\mathbb{N}$. Also all finite subsets of $2^{\mathbb{N}}$ have zero logarithmic density and $d(A^c) =$

$d(\mathbb{N}-A)=1-d(A)$. Since $\sum\limits_{k=1}^{n}\frac{1}{k}=\log n+\gamma+O(\frac{1}{n})$, where γ is the Euler constant, so if $d(A)$ exists, then it is equal to $\lim_{n\to\infty}\frac{1}{\log n}\sum\limits_{k=1}^{n}\frac{\chi_A(k)}{k}$.

If $I=I_f$ (class of all finite subsets of $\mathbb{N}$), then I_f-convergence coincides with the usual convergence of fuzzy numbers. If $I=I_\delta$, then I_δ - convergence coincides with the statistical convergence of fuzzy numbers.

The following result will be used for establishing some results in this article.

Lemma 1. *Every normal space is monotone.(Please refer to Kamthan and Gupta [23], page 53).*

3. Main results

In this section we introduce some sequence spaces using the difference operator and Orlicz functions. Let M be an Orlicz function and $p=(p_k)$ be a sequence of strictly positive real numbers. For some $\rho>0$, we define the following sequence spaces:

$$\begin{aligned}
c_0^{IF}(M,\Delta_n^r,p) &= \left\{(X_k)\in w^F:\left\{k\in\mathbb{N}:\left[M\left(\frac{\overline{d}(\Delta_n^r X_k,\overline{0})}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I\right\}\\
c^{IF}(M,\Delta_n^r,p) &= \left\{(X_k)\in w^F:\left\{k\in\mathbb{N}:\left[M\left(\frac{\overline{d}(\Delta_n^r X_k,X_0)}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I\right\}\\
\ell_\infty^{F}(M,\Delta_n^r,p) &= \left\{(X_k)\in w^F:\sup_k\left[M\left(\frac{\overline{d}(\Delta_n^r X_k,\overline{0})}{\rho}\right)\right]^{p_k}<\infty\right\}.
\end{aligned}$$

for $X_0\in\mathbb{R}(J)$. By using these spaces, we can construct the sequence spaces

$$m^{IF}(M,\Delta_n^r,p)=c^{IF}(M,\Delta_n^r,p)\cap\ell_\infty^F(M,\Delta_n^r,p)$$

and

$$m_0^{IF}(M,\Delta_n^r,p)=c_0^{IF}(M,\Delta_n^r,p)\cap\ell_\infty^F(M,\Delta_n^r,p).$$

Throughout the article I denote a non-trivial admissible ideal of $\mathbb{N}$.

Theorem 1. *The spaces $c_0^{IF}(M,\Delta_n^r,p)$, $c^{IF}(M,\Delta_n^r,p)$, $\ell_\infty^F(M,\Delta_n^r,p)$, $m^{IF}(M,\Delta_n^r,p)$ and $m_0^{IF}(M,\Delta_n^r,p)$ are closed with respect to addition and scalar multiplication.*

Proof. We shall proof the result only for the space $c^{IF}(M,\Delta_n^r,p)$. The others can be treated similarly. Let $X=(X_k)$ and $Y=(Y_k)$ be two elements in $c^{IF}(M,\Delta_n^r,p)$ and α_1,α_2 be scalars. Let $\varepsilon>0$ be given. Then there exist some positive numbers ρ_1,ρ_2 such that

$$P=\left\{k\in\mathbb{N}:\left[M\left(\frac{\overline{d}(\Delta_n^rX_k,X_0)}{\rho_1}\right)\right]^{p_k}\geq\frac{\varepsilon}{2}\right\}\in I$$

and

$$Q=\left\{k\in\mathbb{N}:\left[M\left(\frac{\overline{d}(\Delta_n^rY_k,Y_0)}{\rho_2}\right)\right]^{p_k}\geq\frac{\varepsilon}{2}\right\}\in I.$$

Let $\rho_3=\max(2|\alpha_1|\rho_1,2|\alpha_2|\rho_2)$. Since M is non-decreasing and convex function, we have

$$\begin{aligned}&\left[M\left(\frac{\overline{d}(\Delta_n^r(\alpha_1X_k+\alpha_2Y_k),\alpha_1X_0+\alpha_2Y_0)}{\rho_3}\right)\right]^{p_k}\\&\qquad\leq\left[M\left(\frac{\alpha_1\overline{d}(\Delta_n^rX_k,X_0)}{\rho_3}\right)\right]^{p_k}+\left[M\left(\frac{\alpha_2\overline{d}(\Delta_n^rY_k,Y_0)}{\rho_3}\right)\right]^{p_k}\\&\qquad\leq\left[M\left(\frac{\overline{d}(\Delta_n^rX_k,X_0)}{\rho_1}\right)\right]^{p_k}+\left[M\left(\frac{\overline{d}(\Delta_n^rY_k,Y_0)}{\rho_2}\right)\right]^{p_k}.\end{aligned}$$

Now,

$$\left\{k\in\mathbb{N}:\left[M\left(\frac{\overline{d}(\Delta_n^r(\alpha_1X_k+\alpha_2Y_k),\alpha_1X_0+\alpha_2Y_0)}{\rho_3}\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq P\cup Q\in I.$$

Therefore $(\alpha_1X+\alpha_2Y)\in c^{IF}(M,\Delta_n^r,p)$. This completes the proof. ■

Theorem 2. *For an Orlicz function M, $c_0^{IF}(M,\Delta_n^r,p)$, $c^{IF}(M,\Delta_n^r,p)$, $m^{IF}(M,\Delta_n^r,p)$, $m_0^{IF}(M,\Delta_n^r,p)$ and $\ell_\infty^F(M,\Delta_n^r,p)$ are complete metric spaces with the metric*

$$g_\Delta(X,Y)=\sum_{k=1}^{nr}\overline{d}(X_k,Y_k)+\inf\left\{\rho^{\frac{p_k}{H}}>0:\sup_k M\left(\frac{\overline{d}(\Delta_n^rX_k,\Delta_n^rY_k)}{\rho}\right)\leq1\right\},$$

where $H=\sup_k p_k$.

Proof. We shall prove only for the space $c^{IF}(M,\Delta_n^r,p)$. The other can be treated, similarly. It can shown that g_Δ is a metric on $c^{IF}(M,\Delta_n^r,p)$. Let (X_k) be a Cauchy sequence in $c^{IF}(M,\Delta_n^r,p)$. Let $\varepsilon>0$ be given. For a fixed $X_0>0$ and choose $t>0$ such that $M\left(\frac{tX_0}{2}\right)\geq1$. Then there exists $n_0\in\mathbb{N}$ such that

$$\begin{aligned}&g_\Delta(X^i,X^j)<\frac{\varepsilon}{tX_0}\quad\text{for all }\ i,j\geq n_0.\\&\Rightarrow\sum_{k=1}^{nr}\overline{d}(X_k^i,X_k^j)+\inf\left\{\rho^{\frac{p_k}{H}}>0:\sup_k M\left(\frac{\overline{d}(\Delta_n^rX_k^i,\Delta_n^rX_k^j)}{\rho}\right)\leq1\right\}<\varepsilon\end{aligned}$$

for all $i, j \geq n_0$, which implies

$$\sum_{k=1}^{nr} \overline{d}(X_k^i, X_k^j) < \varepsilon \text{ for all } i, j \geq n_0$$
$$\Rightarrow \overline{d}(X_k^i, X_k^j) < \varepsilon \text{ for all } i, j \geq n_0,\ k = 1, 2, 3..., nr.$$

Hence (X_k^i) for $k = 1, 2, 3, \ldots, nr$ are Cauchy sequences in $\mathbb{R}(J)$ and hence are convergent in $\mathbb{R}(J)$, since $\mathbb{R}(J)$ is a complete metric space.

Let

$$\lim_{i\to\infty} X_k^i = X_k \text{ for } k = 1, 2, 3, ..., nr. \tag{2}$$

Also

$$\sup_k M\left(\frac{\overline{d}(\Delta_n^r X_k^i, \Delta_n^r X_k^j)}{\rho}\right) \leq 1 \text{ for all } i, j \geq n_0 \text{ and } k \in \mathbb{N}. \tag{3}$$

$$M\left(\frac{\overline{d}(\Delta_n^r X_k^i, \Delta_n^r X_k^j)}{g_\Delta(X^i, X^j)}\right) \leq 1 \leq M\left(\frac{tX_0}{2}\right) \text{ for all } i, j \geq n_0 \text{ and } k \in \mathbb{N}.$$

$$\overline{d}(\Delta_n^r X_k^i, \Delta_n^r X_k^j) < \frac{\varepsilon}{2} \text{ for all } i, j \geq n_0 \text{ and } k \in \mathbb{N}.$$

Thus $(\Delta_n^r X_k^i)$ is Cauchy sequence of fuzzy numbers. Let $\lim_{i\to\infty} \Delta_n^r X_k^i = X_k$ for each $k \in \mathbb{N}$. For $k = 1$ we have, from (1) and (3),

$$\lim_{i\to\infty} X_{1+rn}^i = X_{1+rn} \text{ for } r \geq 1,\ n \geq 1.$$

Proceeding in this way inductively we conclude that

$$\lim_{i\to\infty} X_k^i = X_k \text{ for each } k \in \mathbb{N}.$$

Also

$$\lim_{i\to\infty} \Delta_n^r X_k^i = X_k \text{ for each } k \in \mathbb{N}.$$

By the continuity of M, we have the following from (4)

$$\sup_k M\left(\frac{\overline{d}(\Delta_n^r X_k^i, X_k)}{\rho}\right) \leq 1 \text{ for all } i \geq n_0,\ j \to \infty.$$

$$\Rightarrow \inf\left\{\rho^{\frac{p_k}{H}} > 0 : \sup_k M\left(\frac{\overline{d}(\Delta_n^r X_k^i, X_k)}{\rho}\right) \leq 1\right\} < \varepsilon \text{ for all } i \geq n_0.$$

Hence from (2) on taking limit as $j \to \infty$, we get

$$\sum_{k=1}^{nr} \overline{d}(X_k^i, X_k) + \inf\left\{\rho^{\frac{p_k}{H}} > 0 : \sup_k M\left(\frac{\overline{d}(\Delta_n^r X_k^i, \Delta_n^r X_k)}{\rho}\right) \le 1\right\}$$
$$< \varepsilon + \varepsilon = 2\varepsilon \quad \text{for all} \quad i \ge n_0.$$

i.e. $g_\Delta(X^i, X) < \varepsilon$ for all $i \ge n_0$. Then the inequality

$$g_\Delta(X, \overline{0}) \le g_\Delta(X, \Delta_n^r X^i) + g_\Delta(\Delta_n^r X^i, \overline{0}) \quad \text{for all} \quad i \ge n_0$$

implies that $(X_k) \in c^{IF}(M, \Delta_n^r, p)$. This completes the proof. ■

Following standard techniques, one can easily prove the results.

Theorem 3. *The spaces $c_0^{IF}(\Delta)$, $c^{IF}(\Delta)$, $m_0^{IF}(\Delta)$ and $m^{IF}(\Delta)$ are closed linear subspaces of the complete metric space $\ell_\infty^F(\Delta)$ with the metric*

$$f_\Delta(X, Y) = \overline{d}(X_1, Y_1) + \sup_k \overline{d}(\Delta X_k, \Delta Y_k).$$

Theorem 4. *The spaces $c_0^{IF}(\Delta)$, $c^{IF}(\Delta)$, $m_0^{IF}(\Delta)$, $m^{IF}(\Delta)$ and $\ell_\infty^F(\Delta)$ are complete metric spaces with the metric f_Δ.*

Theorem 5. *Let M_1 and M_2 be two Orlicz functions. Then*
(i) $Z(M_2, \Delta_n^r, p) \subseteq Z(M_1 M_2, \Delta_n^r, p)$.
(ii) $Z(M_1, \Delta_n^r, p) \cap Z(M_2, \Delta_n^r, p) \subseteq Z(M_1 + M_2, \Delta_n^r, p)$,
for $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}, \ell_\infty^F$.

Proof. (i) Let $X = (X_k) \in c^{IF}(M_2, \Delta_n^r, p)$. For some $\rho > 0$ we have

$$\left\{k \in \mathbb{N} : \left[M_2\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right]^{p_k} \ge \varepsilon\right\} \in I \text{ for every } \varepsilon > 0. \tag{4}$$

Let $\varepsilon > 0$ and choose λ with $0 < \lambda < 1$ such that $M_1(t) < \varepsilon$ for $0 \le t \le \lambda$. We define

$$Y_k = \frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}$$

and consider

$$\lim_{k \in \mathbb{N}; 0 \le Y_k \le \lambda} [M_1(Y_k)]^{p_k} = \lim_{k \in \mathbb{N}; Y_k \le \lambda} [M_1(Y_k)]^{p_k} + \lim_{k \in \mathbb{N}; Y_k > \lambda} [M_1(Y_k)]^{p_k}.$$

We have

$$\lim_{k \in \mathbb{N}; Y_k \le \lambda} [M_1(Y_k)]^{p_k} \le [M_1(2)]^H \lim_{k \in \mathbb{N}; Y_k \le \lambda} [Y_k]^{p_k}, \quad H = \sup_k p_k. \tag{5}$$

For the second summation (i.e. $Y_k > \lambda$), we go through the following procedure. We have

$$Y_k < \frac{Y_k}{\lambda} < 1 + \frac{Y_k}{\lambda}.$$

Since M_1 is non-decreasing and convex, it follows that

$$M_1(Y_k) < M_1\left(1 + \frac{Y_k}{\lambda}\right) \leq \frac{1}{2}M_1(2) + \frac{1}{2}M_1\left(\frac{2Y_k}{\lambda}\right).$$

Since M_1 satisfies Δ_2-condition, we can write

$$M_1(Y_k) < \frac{1}{2}K\frac{Y_k}{\lambda}M_1(2) + \frac{1}{2}K\frac{Y_k}{\lambda}M_1(2) = K\frac{Y_k}{\lambda}M_1(2).$$

We get the following estimates:

$$\lim_{k\in\mathbb{N};Y_k>\lambda}[M_1(Y_k)]^{p_k} \leq \max\left\{1, (K\lambda^{-1}M_1(2))^H\right\} \lim_{k\in\mathbb{N};Y_k>\lambda}[Y_k]^{p_k}. \tag{6}$$

From (5), (6) and (7), it follows that $(X_k) \in c^{IF}(M_1M_2, \Delta_n^r, p)$.

Hence $c^{IF}(M_2, \Delta_n^r, p) \subseteq c^{IF}(M_1M_2, \Delta_n^r, p)$.

(ii) Let $(X_k) \in c^{IF}(M_1, \Delta_n^r, p) \cap c^{IF}(M_2, \Delta_n^r, p)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M_1\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I$$

and

$$\left\{k \in \mathbb{N} : \left[M_2\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

The rest of the proof follows from the following relation:

$$\begin{aligned}
&\left\{k \in \mathbb{N} : \left[(M_1 + M_2)\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \\
&\subseteq \left\{k \in \mathbb{N} : \left[M_1\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \\
&\quad \cup \left\{k \in \mathbb{N} : \left[M_2\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}.
\end{aligned}$$

■

Taking $M_2(x) = x$ and $M_1(x) = M(x)$ for all $x \in [0, \infty)$, we have the following result.

Corollary 1. *If I is an admissible ideal, then $Z(\Delta_n^r, p) \subseteq Z(M, \Delta_n^r, p)$ for $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}, \ell_\infty^F$.*

Following standard techniques, one can easily prove the results.

Theorem 6. *If $M_1(x) \leq M_2(x)$ for all $x \in [0, \infty)$, then $Z(M_2, \Delta_n^r, p) \subseteq Z(M_1, \Delta_n^r, p)$ for $Z = c_0^{IF}, c^{IF}$ and ℓ_∞^F.*

Theorem 7. *Let M be an Orlicz function. Then*

$$c_0^{IF}(M, \Delta_n^r) \subset c^{IF}(M, \Delta_n^r) \subset \ell_\infty^F(M, \Delta_n^r)$$

and the inclusions are proper.

Proof. Let $(X_k) \in c^{IF}(M, \Delta_n^r)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right)\right] \geq \varepsilon\right\} \in I.$$

Since

$$M\left(\frac{\overline{d}(\Delta_n^r X_k, \overline{0})}{\rho}\right) \leq \frac{1}{2} M\left(\frac{\overline{d}(\Delta_n^r X_k, X_0)}{\rho}\right) + \frac{1}{2} M\left(\frac{\overline{d}(X_0, \overline{0})}{\rho}\right).$$

Taking supremum over k on both sides implies that $(X_k) \in \ell_\infty^F(M, \Delta_n^r)$. ■

The inclusion $c_0^{IF}(M, \Delta_n^r) \subset c^{IF}(M, \Delta_n^r)$ is obvious. The inclusion is strict, for this consider the following example.

Example 3. Let $M(x) = x^2$ for all $x \in [0, \infty)$ and $r = 1$, $n = 1$. Consider the sequence (X_k) of fuzzy numbers be defined as follows:

For $k = 2^i$, $i = 1, 2, 3, \ldots$

$$X_k(t) = \begin{cases} \frac{4}{k}t + 1, & \text{if } -\frac{k}{4} \leq t \leq 0; \\ -\frac{4}{k}t + 1, & \text{if } 0 < t \leq \frac{k}{4}; \\ 0, & \text{otherwise} \end{cases}$$

otherwise, $X_k(t) = \overline{0}$.

For $\alpha \in (0, 1]$, the α-level sets of X_k and ΔX_k are

$$[X_k]^\alpha = \begin{cases} [\frac{k}{4}(\alpha - 1), \frac{k}{4}(1 - \alpha)], & \text{if } k = 2^i,\ i = 1, 2, 3, \ldots \\ [0, 0], & \text{otherwise} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [\frac{k}{4}(\alpha - 1), \frac{k}{4}(1 - \alpha)], & \text{for } k = 2^i; \\ [\frac{k}{4}(\alpha - 1), \frac{k}{4}(1 - \alpha)], & \text{for } k + 1 = 2^i (i > 1); \\ [0, 0], & \text{otherwise }. \end{cases}$$

It is easy to prove that the sequences (X_k) and (ΔX_k) are bounded but these are not I-convergent.

Theorem 8. *The inclusions $Z(M,\Delta_n^{r-1},p) \subseteq Z(M,\Delta_n^r,p)$ are strict for $r \geq 1$. In general $Z(M,\Delta_n^i,p) \subseteq Z(M,\Delta_n^r,p)$ for $i = 1,2,\ldots,r-1$ and the inclusion is strict, for $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}, \ell_\infty^F$.*

Proof. Let $X = (X_k) \in c_0^{IF}(M,\Delta_n^{r-1},p)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^{r-1}X_k,\overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Since M is non-decreasing and convex it follows that

$$\begin{aligned}\left[M\left(\frac{\overline{d}(\Delta_n^{r}X_k,\overline{0})}{2\rho}\right)\right]^{p_k} &\leq \left[M\left(\frac{\overline{d}(\Delta_n^{r-1}X_k - \Delta_n^{r-1}X_{k+1},\overline{0})}{2\rho}\right)\right]^{p_k}\\ &\leq D\left[\frac{1}{2}M\left(\frac{\overline{d}(\Delta_n^{r-1}X_k,\overline{0})}{\rho}\right)\right]^{p_k} + D\left[\frac{1}{2}M\left(\frac{\overline{d}(\Delta_n^{r-1}X_{k+1},\overline{0})}{\rho}\right)\right]^{p_k}\\ &\leq DK\left[M\left(\frac{\overline{d}(\Delta_n^{r-1}X_k,\overline{0})}{\rho}\right)\right]^{p_k} + DK\left[M\left(\frac{\overline{d}(\Delta_n^{r-1}X_{k+1},\overline{0})}{\rho}\right)\right]^{p_k},\end{aligned}$$

where $K = \max\{1, \left(\frac{1}{2}\right)^H\}$.

Therefore we have

$$\begin{aligned}&\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^{r}X_k,\overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\}\\ &\subseteq \left\{k \in \mathbb{N} : DK\left[M\left(\frac{\overline{d}(\Delta_n^{r-1}X_k,\overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}\\ &\quad\cup \left\{k \in \mathbb{N} : DK\left[M\left(\frac{\overline{d}(\Delta_n^{r-1}X_{k+1},\overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}.\\ &\text{i.e. } \left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^{r}X_k,\overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.\end{aligned}$$

Hence $(X_k) \in c_0^{IF}(M,\Delta_n^r,p)$. ∎

The inclusion is strict follows from the following example.

Example 4. Let $M(x) = x$ for all $x \in [0,\infty), r = 3, n = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (X_k) of fuzzy numbers as follows:

$$X_k(t) = \begin{cases} -\frac{t}{k^3-1} + 1, & \text{if } k^3 - 1 \leq t \leq 0;\\ -\frac{t}{k^3+1} + 1, & \text{if } 0 < t \leq k^3 + 1;\\ 0, & \text{otherwise.}\end{cases}$$

For $\alpha \in (0,1]$, the α-level sets of X_k, ΔX_k, $\Delta^2 X_k$ and $\Delta^3 X_k$ are

$$\begin{aligned}
[X_k]^\alpha &= [(1-\alpha)(k^3-1),(1-\alpha)(k^3+1)]\\
[\Delta X_k]^\alpha &= [(1-\alpha)(-3k^2-3k-3),(1-\alpha)(-3k^2-3k+1)]\\
[\Delta^2 X_k]^\alpha &= [(1-\alpha)(6k+2),(1-\alpha)(6k+10)]\\
[\Delta^3 X_k]^\alpha &= [-14(1-\alpha),2(1-\alpha)],
\end{aligned}$$

respectively. It is easy to see that the sequence $[\Delta^2 X_k]^\alpha$ is not I-bounded but $[\Delta^3 X_k]^\alpha$ is I-bounded.

Theorem 9. *Let $0 < p_k \le q_k < \infty$ for each k. Then $Z(M,\Delta_n^r,p) \subseteq Z(M,\Delta_n^r,q)$ for $Z = c_0^{IF}$ and c^{IF}.*

Proof. Let $(X_k) \in c_0^{IF}(M,\Delta_n^r,p)$. Then there exists a number $\rho > 0$ such that

$$\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^r X_k,\overline{0})}{\rho}\right)\right]^{p_k} \ge \varepsilon\right\} \in I.$$

For sufficiently large k. Since $p_k \le q_k$ for each k, therefore we get

$$\begin{aligned}
&\left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^r X_k,\overline{0})}{\rho}\right)\right]^{q_k} \ge \varepsilon\right\}\\
&\subseteq \left\{k \in \mathbb{N} : \left[M\left(\frac{\overline{d}(\Delta_n^r X_k,\overline{0})}{\rho}\right)\right]^{p_k} \ge \varepsilon\right\} \in I.
\end{aligned}$$

i.e. $(X_k) \in c_0^{IF}(M,\Delta_n^r,q)$. This completes the proof. ∎

Similarly, it can be shown that $c^{IF}(M,\Delta_n^r,p) \subseteq c^{IF}(M,\Delta_n^r,q)$.

Corollary 2. (*a*) *Let $0 < \inf_k p_k \le p_k \le 1$. Then $Z(M,\Delta_n^r,p) \subseteq Z(M,\Delta_n^r)$ for $Z = c_0^{IF}$ and c^{IF}.*

(*b*) *Let $1 \le p_k \le \sup_k p_k < \infty$. Then $Z(M,\Delta_n^r) \subseteq Z(M,\Delta_n^r,p)$ for $Z = c_0^{IF}$ and c^{IF}.*

Theorem 10. *If I is an admissible ideal and $I \ne I_f$, then the sequence spaces $c_0^{IF}(M,\Delta_n^r,p)$, $c^{IF}(M,\Delta_n^r,p)$, $m^{IF}(M,\Delta_n^r,p)$ and $m_0^{IF}(M,\Delta_n^r,p)$ are neither normal nor monotone.*

Proof. We prove this result with the help of following example. ∎

Example 5. Let $M(x) = x^2$ for all $x \in [0,\infty)$ and $r = 1$, $n = 1$. For $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (X_k) of fuzzy numbers as follows:

$$X_k(t) = \begin{cases} t-3k+1, & \text{if } t \in [3k-1,3k];\\ -t+3k+1, & \text{if } t \in [3k,3k+1];\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$\alpha_k = \begin{cases} 1, & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

Thus $(\alpha_k X_k) \notin Z(M, \Delta_n^r)$ for $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}$. Therefore $c_0^{IF}(M, \Delta_n^r, p)$, $c^{IF}(M, \Delta_n^r, p)$, $m^{IF}(M, \Delta_n^r, p)$ and $m_0^{IF}(M, \Delta_n^r, p)$ are not normal. By Lemma 1, these spaces are not monotone.

Theorem 11. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $Z(M, \Delta_n^r, p)$ are not symmetric, where $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}$.*

Proof. We prove of the result only for $c^{IF}(M, \Delta_n^r, p)$ with the help of the following example. The rest of the results follow similar way. ■

Example 6. Let $M(x) = x^2$ for all $x \in [0, \infty)$ and $r = 1$, $n = 1$. For $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (X_k) of fuzzy numbers as follows:

$$X_k(t) = \begin{cases} t - 2k + 1, & \text{if } t \in [2k-1, 2k]; \\ -t + 2k + 1, & \text{if } t \in [2k, 2k+1]; \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have $(X_k) \in c^{IF}(M, \Delta_n^r, p)$. But the rearrangement (Y_k) of (X_k) defined as

$$Y_k = \{X_1, X_4, X_2, X_9, X_3, X_{16}, X_5, X_{25}, X_6, \ldots\}.$$

This implies that $(Y_k) \notin c^{IF}(M, \Delta_n^r, p)$. Hence $c^{IF}(M, \Delta_n^r, p)$ is not symmetric.

We give the following proposition without proof.

Proposition 1. *The spaces $Z(M)$ are normal as well as monotone and symmetric, where $Z = c_0^{IF}, c^{IF}, m_0^{IF}, m^{IF}$ and ℓ_∞^F.*

Theorem 12. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $c_0^{IF}(M, \Delta_n^r, p)$, $c^{IF}(M, \Delta_n^r, p)$, $m^{IF}(M, \Delta_n^r, p)$ and $m_0^{IF}(M, \Delta_n^r, p)$ are not convergence free.*

Proof. It follows from the following example that these spaces are not convergence free. ■

Example 7. Let $M(x) = x^2$ for all $x \in [0, \infty)$ and $r = 1$, $n = 1$. For $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence (X_k) of fuzzy numbers as follows:

$$X_k(t) = \overline{0} \text{ for } k = 2^i, \ i = 1, 2, 3, \ldots$$

Otherwise

$$X_k(t) = \begin{cases} \frac{k}{3}(t-2)+1, & \text{if } t \in [\frac{2k-3}{k}, 2]; \\ -\frac{k}{3}(t-2)+1, & \text{if } t \in [2, \frac{2k+3}{k}]; \\ 0, & \text{otherwise.} \end{cases}$$

Then the α-level sets of (X_k) and (ΔX_k) are

$$[X_k]^\alpha = \begin{cases} [0,0], & \text{if } k = 2^i,\ i = 1,2,3,\dots \\ [2 - \frac{3}{k}(1-\alpha), 2 + \frac{3}{k}(1-\alpha)], & \text{otherwise .} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-2 - \frac{3}{k}(1-\alpha), -2 + \frac{3}{k}(1-\alpha)], & \text{for } k = 2^i \\ [2 - \frac{3}{k}(1-\alpha), 2 + \frac{3}{k}(1-\alpha)], & \text{for } k+1 = 2^i (i > 1) \\ [(\alpha-1)(\frac{3}{k} + \frac{3}{k+1}), (1-\alpha)(\frac{3}{k} + \frac{3}{k+1})], & \text{otherwise.} \end{cases}$$

Thus we have $(X_k) \in Z(M, \Delta_n^r, p)$ for $Z = c_0^{IF}, c^{IF}, m^{IF}, m_0^{IF}$.

Define a sequence (Y_k) of fuzzy numbers as follows:

$$Y_k = \overline{0} \text{ for } k = 2^i,\ i = 1,2,3,\dots$$

$$Y_k(t) = \begin{cases} t-(k+1), & \text{if } t \in [k+1, k+2]; \\ \frac{1}{k-k^2-1}t - \frac{k^2+3}{k-k^2-1}, & \text{if } t \in [k+2, k^2+3]; \\ 0, & \text{otherwise.} \end{cases}$$

Then the α-level sets of (Y_k) and (ΔY_k) are

$$[Y_k]^\alpha = \begin{cases} [0,0], & \text{if } k = 2^i, i = 1,2,3,\dots \\ [k+1+\alpha, k^2+3+\alpha(k-k^2-1)], & \text{otherwise .} \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [-(k+2)^2 + \alpha(k+k^2+1) - 3, -(k+2+\alpha)], & \text{for } k = 2^i \\ [k+1+\alpha, k^2+3+\alpha(k-k^2-1)], & \text{for } k+1 = 2^i \\ & (i>1) \\ [k+1-(k+2)^2 - 3 + \alpha(k+k^2+3), & \\ k^2-k+1+\alpha(k-k^2-3)], & \text{otherwise .} \end{cases}$$

This implies that $(Y_k) \notin Z(M, \Delta_n^r, p)$ for $Z = c_0^{IF}, c^{IF}, m^{IF}, m_0^{IF}$. Hence $c_0^{IF}(M, \Delta_n^r, p)$, $c^{IF}(M, \Delta_n^r, p)$, $m^{IF}(M, \Delta_n^r, p)$ and $m_0^{IF}(M, \Delta_n^r, p)$ are not convergence free.

Theorem 13. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $c_0^{IF}(M, \Delta_n^r, p)$, $c^{IF}(M, \Delta_n^r, p)$, $m^{IF}(M, \Delta_n^r, p)$ and $m_0^{IF}(M, \Delta_n^r, p)$ are not sequence algebra.*

Proof. These spaces are sequence algebra which follows from the following example. ■

Example 8. Let $M(x) = x^2$ for all $x \in [0, \infty)$ and $r = 1$, $n = 1$. For $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequences (X_k) and (Y_k) of fuzzy numbers as follows:

For $k = 2^i$, $i = 1, 2, 3, \dots$

$$X_k(t) = \begin{cases} \frac{k}{2k-2}t - \frac{k}{2k-2}, & \text{if } t \in [1, \frac{3k-2}{k}]; \\ 1, & \text{if } t \in [\frac{3k-2}{k}, \frac{3k+2}{k}]; \\ -\frac{k}{2k-2}t + \frac{5k}{2k-2}, & \text{if } t \in [\frac{3k+2}{k}, 5]; \\ 0, & \text{otherwise.} \end{cases}$$

otherwise

$$X_k(t) = \begin{cases} kt - 7k + 1, & \text{if } t \in [7 - \frac{1}{k}, 7]; \\ -t + 8, & \text{if } t \in [7, 8]; \\ 0, & \text{otherwise.} \end{cases}$$

and for $k = 2^i$, $i = 1, 2, 3, \dots$

$$Y_k(t) = \begin{cases} t - k - 1, & \text{if } t \in [k+1, k+2]; \\ -t + k + 3, & \text{if } t \in [k+2, k+3]; \\ 0, & \text{otherwise.} \end{cases}$$

otherwise

$$Y_k(t) = \begin{cases} t - k, & \text{if } t \in [k, k+1]; \\ -2t + 2k + 5, & \text{if } t \in [k+2, k + \frac{5}{2}]; \\ 0, & \text{otherwise.} \end{cases}$$

Then the α-level sets of (X_k) and (Y_k) are

$$[X_k]^\alpha = \begin{cases} [1 + \alpha, 5 - \alpha], & \text{if } k = 2^i, i = 1, 2, 3, ... \\ [7 - \frac{1}{k}(1 - \alpha), 8 - \alpha], & \text{otherwise .} \end{cases}$$

and

$$[Y_k]^\alpha = \begin{cases} [k + 1 + \alpha, k + 3 - \alpha], & \text{if } k = 2^i, i = 1, 2, 3, ... \\ [k + \alpha, k + \frac{1}{2}(5 - \alpha)], & \text{otherwise .} \end{cases}$$

Therefore the α-level sets of (ΔX_k) and (ΔY_k) are

$$[\Delta X_k]^\alpha = \begin{cases} [-7+2\alpha, -2-\alpha+\frac{1-\alpha}{k+1}], & \text{for } k=2^i \\ [2-\frac{1-\alpha}{k}+\alpha, 7-2\alpha], & \text{for } k+1=2^i (i>1) \\ [-1-\frac{1-\alpha}{k}+\alpha, 1-\alpha+\frac{1-\alpha}{k+1}], & \text{otherwise .} \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [-\frac{5}{2}+\frac{3\alpha}{2}, 2-2\alpha], & \text{for } k=2^i \\ [-4+2\alpha, \frac{1}{2}-\frac{3\alpha}{2}], & \text{for } k+1=2^i (i>1) \\ [-\frac{7}{2}+\frac{3\alpha}{2}, \frac{3}{2}-\frac{3\alpha}{2}], & \text{otherwise .} \end{cases}$$

Thus for the sequences (X_k) and (Y_k) we have

$$(X_k), (Y_k) \in m^{IF}(M, \Delta_n^r, p) (\subset c^{IF}(M, \Delta_n^r, p)).$$

But

$$(X_k \otimes Y_k) \notin c^{IF}(M, \Delta_n^r, p) (\supset m^{IF}(M, \Delta_n^r, p)).$$

This completes the proof of the result.

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