F A S C I C U L I M A T H E M A T I C I

Nr 52 2014

Mostefa Nadir and Bachir Gagui

NUMERICAL SOLUTION OF HAMMERSTEIN INTEGRAL EQUATIONS IN L^p SPACES

Abstract. In this work, we give conditions guarantee the boundedness of the Hammerstein integral operator in L^p spaces. The existence and the uniqueness of the solution of Hammerstein integral equation are treated under some assumptions affected to the successive approximation, so that we obtain the convergence of the approximate solution to the exact one. Finally, we treat numerical examples to confirm our results.

Key words: Hammerstein integral equation, successive approximation.

AMS Mathematics Subject Classification: 45D05, 45E05, 45L05, 45L10, 65R20.

1. Introduction

In recent years there has been a growing interest in the integral equations. In particular, Hammerstein integral equations where we find this equation frequently in many applied areas, which include engineering, mechanics, potential theory and electostatics. Also this type of equations occur of scattering and radiation of surface water wave, where due to the Green's function we can transform any ordinary differential equation of the second order with boundary conditions into an Hammerstein integral equation of the general form

$$\varphi(t_0) = \int_0^1 k(t_0, t)l(t, \varphi(t))dt, \tag{1}$$

where $k(t_0, t)$ is a map from $[a, b] \times [a, b]$, into $\mathbb{R}$ and $l(t, \varphi(t)$ a nonlinear map from $[a, b] \times \mathbb{R}$, into $\mathbb{R}$.

The goal of this paper is to give sufficient conditions for the existence and uniqueness of a solution $\varphi \in L^p([a, b])$ of the equation (1) under weaker hypotheses; where we shall assume that

1) the function$l(t, \varphi(t))$ is strongly measurable in t and continuous in φ

2) $\|l(t,\varphi(t))\| \leq a_0(t)+b_0 \|\varphi\|$ for $t \in [a,b]$ and $\varphi \in \mathbb{R}$, where $a_0 \in L^q([a,b])$ sutch that $\frac{1}{p}+\frac{1}{q}=1$ and $b_0 \geq 0$.

Let us recall that, another existence theorems for L^p-solutions of (1) with a kernel $k \in L^p$ were proved in the papers [4] and [8]. Obviously, in this paper the kernel k is not necessary L^p-integrable.

2. Main results

Theorem 1. *Suppose that the functions $k(t_0,t)$ and $l(t,\varphi t))$ satisfy the following conditions*

$(A1)$ The kernel $k(t_0,t)$ is measurable on $[a,b]\times[a,b]$ and such that

$$\left(\int_a^b |k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\, dt_0\right)^{\frac{q}{(q-\sigma)p}} \leq M_1, \quad \text{for all } t \in [a,b],$$

where $\sigma \leq q$ and $\sigma, q > 1$.

$(A2)$ The kernel $k(t_0,t)$ is measurable on $[a,b]\times[a,b]$ and such that

$$\left(\int_a^b |k(t_0,t)|^{\frac{p\sigma}{q}}\, dt\right)^{\frac{q}{\sigma p}} \leq M_2, \quad \text{for all } t_0 \in [a,b].$$

$(A3)$ The function $l(t,\varphi(t))$ is a nonlinar map from $[a,b]\times\mathbb{R}$, into $\mathbb{R}$ satisfying the Carathéodory condition and such that

$$|l(t,\varphi(t))| \leq a_0(t) + b_0 |\varphi(t)|^{\frac{p}{q}}$$

where $a_0(t) \in L^q([a,b],\mathbb{R})$, $b_0 > 0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Under conditions $(A1)$, $(A2)$ and $(A3)$ and $q \leq p$ the operator

$$A\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt \tag{2}$$

is a map from L^p into L^p.

Proof. From the condition $(A3)$, we can write

$$|l(t,\varphi(t)|^q \leq \left(|a_0(t)| + b_0 |\varphi(t)|^{\frac{p}{q}}\right)^q$$

and therefore

$$\|l(t,\varphi(t)\|_q = \left(\int_a^b |l(t,\varphi(t)|^q\, dt\right)^{\frac{1}{q}} \leq \left(\int_a^b \left(|a_0(t)| + b_0 |\varphi(t)|^{\frac{p}{q}}\right)^q dt\right)^{\frac{1}{q}}.$$

Using Minkovski's inequality, it comes

$$\|l(t,\varphi(t)\|_q \le c\left(\left(\int_a^b |a_0(t)|^q\right)^{\frac{1}{q}} + \left(\int_a^b b_0^q\,|\varphi(t)|^p\right)^{\frac{1}{q}}\right)$$
$$\le c\left(\|a(t)\|_q + b_0\,\|\varphi(t)\|_p^{\frac{p}{q}}\right).$$

Hence the operator $l(t,\varphi(t))$ is a continuous element of $L^q([a,b],\mathbb{R})$ [9]. However, on the space $L^p([a,b],\mathbb{R})$ we consider,

$$A\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt,$$

where following [3], we have

$$\begin{aligned}
|A\varphi(t_0)| &= \left|\int_a^b k(t_0,t)l(t,\varphi(t)dt\right| \le \int_a^b |k(t_0,t)l(t,\varphi(t)|\,dt \\
&= \int_a^b \left(|k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\,|l(t,\varphi(t)|^q\right)^{\frac{1}{p}} |k(t_0,t)|^{\frac{\sigma}{q}}\,|l(t,\varphi(t)|^{1-\frac{q}{p}}\,dt \\
&\le \left(\int_a^b |k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\,|l(t,\varphi(t)|^q\,dt\right)^{\frac{1}{p}} \\
&\quad\times \left(\int_a^b |k(t_0,t)|^{\frac{p\sigma}{q}}\,dt\right)^{\frac{1}{p}} \left(\int_a^b |l(t,\varphi(t)|^q\,dt\right)^{\frac{p-q}{pq'}}
\end{aligned}$$

$$|A\varphi(t_0)| \le M_2^{\frac{\sigma}{q}}\,\|l(t,\varphi(t)\|^{\frac{(p-q)}{p}} \left(\int_a^b |k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\,|l(t,\varphi(t)|^q\,dt\right)^{\frac{1}{p}},$$

or again,

$$|A\varphi(t_0)|^p \le \left(M_2^{\frac{\sigma}{q}}\,\|l(t,\varphi(t)\|^{\frac{(p-q)}{p}} \left(\int_a^b |k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\,|l(t,\varphi(t)|^q\,dt\right)^{\frac{1}{p}}\right)^p$$

$$\begin{aligned}
\left(\int_a^b |A\varphi(t_0)|^p\,dt_0\right)^{\frac{1}{p}} &\le M_2^{\frac{\sigma}{q}}\,\|l(t,\varphi(t)\|^{\frac{(p-q)}{p}} \\
&\quad\times \left(\int_a^b \left(\int_a^b |k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\,|l(t,\varphi(t)|^q\,dt\right)dt_0\right)^{\frac{1}{p}} \\
&\le M_2^{\frac{\sigma}{q}}\,\|l(t,\varphi(t)\|^{\frac{p-q}{p}} \\
&\quad\times \left(\int_a^b |k(t_0,t)|^{\frac{(q-\sigma)p}{q}}\,dt_0\right)^{\frac{1}{p}} \left(\int_a^b |l(t,\varphi(t)|^q\,dt\right)^{\frac{1}{p}} \\
&\le M_2^{\frac{\sigma}{q}}\,\|l(t,\varphi(t)\|^{1-\frac{q}{p}}\,M_1^{1-\frac{\sigma}{q}}\,\|l(t,\varphi(t)\|^{\frac{q}{p}},
\end{aligned}$$

$$\|A\varphi(t_0)\|_p \leq M_2^{\frac{\sigma}{q}} M_1^{1-\frac{\sigma}{q}} \|l(t,\varphi(t)\|_q .$$

Hence, the operator $A\varphi(t_0)$ is well defined from L^p to L^p. ■

We present now the theorem of the existence and uniqueness of the L^p-solution of the equation (1).

Theorem 2. *Suppose that the functions $k(t_0,t)$ and $l(t,\varphi t))$ satisfy the following conditions*

$(B1)$ *The kernel $k(t_0,t)$ belongs to the space L^p for all $t_0 \in [a,b]$*

$$\left(\int_a^b |k(t_0,t)|^p \, dt\right)^{\frac{1}{p}} \leq N_1(t_0), \ \ \forall t_0 \in [a,b]$$

$(B2)$ *the function $l(t,\varphi(t))$ belongs to the space L^q for all $t \in [a,b]$*

$$\left(\int_a^b |l(t,\varphi(t))|^q \, dt\right)^{\frac{1}{q}} \leq C$$

and satisfying the Lipschitz condition

$$|l(t,\varphi_1(t)) - l(t,\varphi_2(t))| \leq L(t) \, |\varphi_1(t) - \varphi_2(t)| \, ,$$

with the function $L(t)$ belongs to the space $L^{\frac{pq}{p-q}}$ with $q \leq p$,

$$\left(\int_a^b |L(t)|^{\frac{pq}{p-q}} \, dt\right)^{\frac{p-q}{pq}} \leq N_2.$$

Under assumptions $(B1)$ and $(B2)$, the successive approximation

$$\varphi_{n+1}(t_0) = \int_a^b k(t_0,t) l(t,\varphi_n(t))dt,$$

converges almost everywhere to the solution of the equation (1) provided

$$N_2^p \int_a^b N_1^p(t)dt = N^p < 1.$$

Proof. For this method we put $\varphi_0(t)$ as an identically null function and successively

$$\varphi_{n+1}(t_0) = \int_a^b k(t_0,t) l(t,\varphi_n(t))dt, \ \ n = 0,1,2,\ldots,n,\ldots,$$

and therefore, we obtain

$$
\begin{aligned}
(3) \qquad |\varphi_{n+1} - \varphi_n| &\le \int_a^b |k(t_0,t)|\,|l(t,\varphi_n(t)) - l(t,\varphi_{n-1}(t)|\,dt,\\
|\varphi_{n+1} - \varphi_n| &\le \int_a^b |k(t_0,t)|\,L(t)\,|\varphi_n - \varphi_{n-1}|\,dt,\\
&\le \left(\int_a^b |k(t_0,t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |L(t)|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}\\
&\quad \times \left(\int_a^b |\varphi_n - \varphi_{n-1}|^p\,dt\right)^{\frac{1}{p}}\\
|\varphi_{n+1} - \varphi_n|^p &\le N_1^p(t_0) N_2^p \int_a^b |\varphi_{n+1} - \varphi_n|^p\,dt,
\end{aligned}
$$

using the condition $\varphi_0(t) = 0$, we get

$$
|\varphi_1(t_0)|^p \le N_1^p(t_0) \left(\int_a^b |l(t,0)|^q\,dt\right)^{\frac{p}{q}} = N_1^p(t_0) C^p
$$

and from (3) it comes

$$
\begin{aligned}
|\varphi_2(t_0) - \varphi_1(t_0)|^p &\le N_1^p t(_0) N_2^p \int_a^b N_1^p(t_0) C^p dt_0 = C^p N^p N_1^p(t_0),\\
|\varphi_3(t_0) - \varphi_2(t_0)|^p &\le N_1^p t(_0) N_2^p \int_a^b C^p N_1^p(t_0) N^p dt_0 = C^p N^{2p} N_1^p(t_0),
\end{aligned}
$$

more generally

$$
|\varphi_{n+1}(t_0) - \varphi_n(t_0)|^p \le C^p N^{2np} N_1^p(t_0).
$$

or again after simplification

$$
|\varphi_{n+1}(t_0) - \varphi_n(t_0)| \le C N^{2n} N_1(t_0).
$$

This expression gives that the sequence $\varphi_n(t_0)$ taken by the series

$$
\varphi_1(t_0) + (\varphi_2(t_0) - \varphi_1(t_0)) + \ldots + (\varphi_p(t_0) - \varphi_{p-1}(t_0)) + \ldots,
$$

has the majorant

$$
C N_1(t_0)(1 + N + N^2 + \ldots + N^{p-1} + \ldots
$$

Naturally, this series converges. Hence the sequence $\varphi_n(t_0)$ converges to the solution of (1). ■

3. Numerical experiments

In this section we describe some of the numerical experiments performed in solving the Hammerstein integral equations (1). In all cases, the interval is [0,1] and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with the method is correct [6], [7].

In each table, φ represents the given exact solution of the Hammerstein equation and $\widetilde{\varphi}$ corresponds to the approximate solution of the equation produced by the iterative method.

Example 1. Consider the Hammerstein integral equation

$$\varphi(t_0) - \int_0^1 \frac{4tt_0 + \pi \sin(\pi t)}{(\varphi(t))^2 + t^2 + 1} dt = \sin(\frac{\pi}{2} t_0) - 2t_0 \ln(3),$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \sin(\frac{\pi}{2} t).$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation method.

Table 1. we present exact and approximate solutions of Example 1 in some arbitrary points. As proved in Theorem 2.

Points of t	Exact solution	Approx solution	Error
0.000000	0.000000e+000	0.000000e+000	0.000000e+000
0.200000	3.090170e-001	3.090018e-001	1.522598e-005
0.400000	5.877853e-001	5.877548e-001	3.045196e-005
0.600000	8.090170e-001	8.089713e-001	4.567793e-005
0.800000	9.510565e-001	9.509956e-001	6.090391e-005

Example 2. Consider the Hammerstein integral equation

$$\varphi(t_0) - \int_0^1 tt_0(\varphi(t))^3 dt = \frac{1}{t_0^2 + 1} - \frac{3}{16} t_0,$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t^2 + 1}.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation.

Table 2. we present exact and approximate solutions of Example 2 in some arbitrary points. As proved in Theorem 2, the error is compared with the ones treated in [2].

Points of t	Exact solution	Approx solution	Error	Error [2]
0.000000	1.000000e+000	1.000000e+00	0.000000e+00	0.000000e+00
0.200000	9.615385e-001	9.615348e-001	3.642846e-006	1.194620e-004
0.400000	8.620690e-001	8.620617e-001	7.285693e-006	2.389660e-004
0.600000	7.352941e-001	7.352832e-001	1.092854e-005	3.581180e-004
0.800000	6.097561e-001	6.097415e-001	1.457139e-005	4.780980e-004

Example 3. Consider the Hammerstein integral equation

$$\varphi(t_0) - \frac{1}{5}\int_0^1 \cos(\pi t_0)\sin(\pi t)(\varphi(t))^3 dt = \sin(\pi t_0),$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \sin(\pi t) + \frac{20 - \sqrt{391}}{3}.\cos(\pi t).$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation.

Table 3. we present exact and approximate solutions of Example 3 in some arbitrary points. As proved in Theorem 2, the error is compared with the ones treated in [1].

Points of t	Exact solution	Approx solution	Error	Error [1]
0.000000	7.542669e-002	7.542669e-002	2.498002e-016	5.537237e - 15
0.200000	6.488067e-001	6.488067e-001	2.220446e-016	4.551914e - 15
0.400000	9.743646e-001	9.743646e-001	1.110223e-016	1.776356e - 15
0.600000	9.277484e-001	9.277484e-001	1.110223e-016	1.776356e - 15
0.800000	5.267638e-001	5.267638e-001	2.220446e-016	4.551914e - 15

Example 4. Consider the Hammerstein integral equation

$$\varphi(t_0) - \int_0^1 \sin(t + t_0)\ln(\varphi(t))dt = \exp(t_0) - 0.382\sin(t_0) - 0.301\cos(t_0),$$

$0 \leq t_0 \leq 1$, where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \exp(t).$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation.

Table 4. we present exact and approximate solutions of Example 4 in some arbitrary points. As proved in Theorem 2, the error is compared with the ones treated in [5].

Points of t	Exact solution	Approx solution	Error	Error [5]
0.000000	1.000000e+000	1.000195e+000	1.953229e-004	0.000000e+00
0.200000	1.221403e+000	1.221559e+000	1.567282e-004	1.940000e-004
0.400000	1.491825e+000	1.491937e+000	1.118852e-004	5.410000e-004
0.600000	1.822119e+000	1.822181e+000	6.258175e-005	3.360000e-004
0.800000	2.225541e+000	2.225552e+000	1.078332e-005	2.890000e-004

4. Conclusion

Under conditions of the Theorem 1, the boundedness of the Hammerstein integral operator in L^p spaces is assured. Also with assumptions of the Theorem 2 the existence and the uniqueness of the solution of the Hammerstein integral equation are assured. Our numerical results show that for the convergence of the solution of this equation to the exact one with a considerable accuracy improves with increasing of the number of iterations. Finally, we confirm that, the theorems cited above lead us to the good approximation of the exact solution.

References

[1] Awawdeh F., Adawi A., A numerical method for solving nonlinear integral equations, *International Mathematical Forum,* 4(1)(2009), 805-817.

[2] Ezzati R., Shakibi K., On approximation and numerical solution of Fredholm-Hammerstein integral equations using multiquadric quasi-interpolation, *Communication in Numerical Analysis,* 112(2012).

[3] Kantorovitch L.,Akilov G., *Functional analysis,* Pergamon Press, University of Michigan, 1982.

[4] Maleknejad K., Derili M., The collocation method for Hammerstein equations by Daubechies wavelets, *Applied Mathematics and Computation,* 172(2006), 846-864.

[5] Maleknejad K., Nouri K., Nosrati M., Convergence of approximate solution of nonlinear Fredholm-Hammerstein integral equations, *Commun Nonlinear Sci Numer Simulat,* 15(2010), 1432-1443.

[6] Nadir M., Gagui B., Two Points for the Adaptive Method for the Numerical Solution of Volterra Integral Equations, *International Journal Mathematical Manuscripts* (*IJMM*)*,* 1(2)(2007).

[7] Nadir M., Rahmoune A., Solving linear Fredholm integral equations of the second kind using Newton divided difference interpolation polynomial, *International Journal of Mathematics and Computation* (*IJMC*)*,* 7(10)(2010), 1-6.

[8] SZUFLA S., On the Hammerstein integral equation in Banach spaces, *Math. Nachr.*, 124(1985), 7-14.
[9] F.G. TRICOMI, *Integral Equations*, University Press, University of Cambridge, 1957.

MOSTEFA NADIR
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MSILA
28000 ALGERIA
e-mail: mostefanadir@yahoo.fr

BACHIR GAGUI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MSILA
28000 ALGERIA
e-mail: gagui_bachir@yahoo.fr

Received on 03.11.2012 and, in revised form, on 22.02.2013.