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COMMON FIXED POINTS OF A THREE-STEP ITERATION WITH ERRORS OF ASYMPTOTICALLY QUASI-NONEXPANSIVE NONSELF-MAPPINGS IN THE INTERMEDIATE SENSE IN BANACH SPACES

Abstract. In this paper, we extend the results of Inprasit and Wattanataweekul [7] to the class of asymptotically quasi-nonexpansive nonself-mappings in the intermediate sense. We prove some strong convergence theorems for asymptotically quasi-nonexpansive nonself-mappings in the intermediate sense using a three-step iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Our results extends, improves, unifies and generalizes the results of [13], [25] and [27].

Key words: nonexpansive mappings, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings in the intermediate sense, nonself-mappings, Banach spaces.

AMS Mathematics Subject Classification: 47H10.

1. Introduction

In the sequel, we give the following definitions of some of the concepts that will feature prominently in this study.

Definition 1. *Suppose that X is a normed space and C is a nonempty subset of X. Let $T : C \to C$ be a mapping. T is said to be*

(a) asymptotically nonexpansive *[6] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1, \ x, y \in C. \tag{1}$$

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In 1972, Goebel and Kirk [6] introduced the class of asymptotically nonexpansive mappings as a generalization of the class of nonexpansive mappings.

(b) *asymptotically quasi-nonexpansive* [7] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - q\| \leq k_n \|x - q\|$ for all $x \in C$, $q \in F(T)$, $n \geq 1$, where $F(T)$ is the set of fixed points of T.

(c) *uniformly L-Lipschitzian* [7] if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in C$ and each $n \geq 1$.

(d) A subset C of X is said to be a retract of X [7] if there exists a continuous map $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : X \to C$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

(e) A mapping $T : C \to X$ is said to be *asymptotically quasi-nonexpansive* with respect to a nonexpansive retraction P [7] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}x - q\| \leq k_n \|x - q\| \tag{2}$$

for all $x, y \in C$, $q \in F(T)$, $n \geq 1$, where $F(T)$ is the set of fixed points of T and $(PT)^0 = I$, the identity operator on C.

(f) The mapping $T : C \to X$ is called *uniformly L-Lipschitzian* with respect to a nonexpansive retraction P [7] if there exists a positive constant L such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \tag{3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(g) *asymptotically quasi-nonexpansive in the intermediate sense* with respect to a nonexpansive retraction P provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x\in C} \left(\|T(PT)^{n-1}x - q\| - \|x - q\|\right) \leq 0 \quad \forall q \in F(T). \tag{4}$$

Putting

$$\tau_n = \max\left\{0, \sup_{x\in C, q\in F(T)} (\|T(PT)^{n-1}x - q\| - \|x - q\|)\right\}, \tag{5}$$

we see that $\tau_n \to 0$ as $n \to \infty$. Then (4) is reduced to the following:

$$\|T(PT)^{n-1}x - q\| \leq \|x - q\| + \tau_n, \quad \forall x \in C, \ q \in F(T), \ n \geq 1. \tag{6}$$

Definition 2 ([20]). *A Banach space X is said to satisfy* Opial's condition *if for each sequence $\{x_n\}$ and $x, y \in X$ with $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|. \tag{7}$$

The following lemmas will be useful in this study.

Lemma 1 ([24]). *Let $\{a_n\}$, $\{b_n\}$, $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad \text{for all } n = 1, 2, \cdots, \tag{8}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(i) $\lim_{n\to\infty} a_n$ *exists, and*
(ii) $\lim_{n\to\infty} a_n = 0$ *whenever,* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2 ([11]). *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$ be a closed ball of X. Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 - \lambda\mu g(\|x - y\|) \tag{9}$$

for all $x, y, z, w \in B_r$ and all $\lambda, \mu, \xi, \vartheta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

Lemma 3 ([7]). *Let X be a uniformly convex Banach space and let B_r be a closed ball of X. Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\begin{aligned} \|\lambda x + \mu y + \xi z + \vartheta w + \zeta s\|^2 \leq\ & \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 \\ & + \zeta\|s\|^2 - \lambda\mu g(\|x - y\|) \end{aligned} \tag{10}$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \vartheta, \zeta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta + \zeta = 1$.

Lemma 4 ([23]). *Let X be a Banach space which satisfies* Opial's condition *and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be so that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converges weakly to u and v, respectively, then $u = v$.*

In [7], U. Inprasit and H. Wattanataweekul introduced the following new three-step iterative scheme.

Let C be a nonempty closed convex subset of X and $P : X \to C$ a nonexpansive retraction of X onto C, and $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings and F is the set of all common fixed points of T_i i.e., $F = \cap_{i=1}^{3} F(T_i)$, where $F(T_i) = \{x \in C : T_i x = x\}$ for all $i = 1, 2, 3$.

Then, for arbitrary $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ by the iterative scheme

$$\begin{aligned}
(11) \quad z_n &= P[a_n T_1 (PT_1)^{n-1} x_n + (1 - a_n - \delta_n) x_n + \delta_n u_n], \\
y_n &= P[b_n T_2 (PT_2)^{n-1} z_n + c_n T_1 (PT_1)^{n-1} x_n \\
&\quad + (1 - b_n - c_n - \sigma_n) x_n + \sigma_n v_n], \\
x_{n+1} &= P[\alpha_n T_3 (PT_3)^{n-1} y_n + \beta_n T_2 (PT_2)^{n-1} z_n + \gamma_n T_1 (PT_1)^{n-1} x_n \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n) x_n + \rho_n w_n]
\end{aligned}$$

for all $n \geq 1$, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C.

It was established by Bnouhachem *et al.* [1] that three-step method performs better than two-step and one-step methods for solving variational inequalities. Glowinski and P. Le Tallec [5] applied three-step iterative sequences for finding the approximate solutions of the elastoviscoplasticity problem, eigenvalue problems and in the liquid crystal theory. Moreover, three-step schemes are natural generalization of the splitting methods to solve partial differential equations, [23, 24, 25]. What this means is that Noor three-step methods are robust and more efficient than the Mann (one-step) and Ishikawa (two-step) type schemes for solving problems in pure and applied sciences.

The following strong convergence results was also established by U. Inprasit and H. Wattanataweekul [7].

Theorem IW ([7]). *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}$, $\{l_n\}$, $\{m_n\}$, respectively, such that $F \neq \emptyset$, $k_n \geq 1$, $l_n \geq 1$, $m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded sequences in C. Assume that T_1, T_2, T_3 are uniformly L-Lipschitzian. If one of T_i, $(i = 1, 2, 3)$ is a completely continuous and one of the following conditions $(C1)$-$(C5)$ is satisfied:*

$(C1)$ $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \delta_n) < 1,$
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C2)$ $0 < \liminf_{n\to\infty} b_n, \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$ *and*

$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C3)$ $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C4)$ $\liminf_{n\to\infty} b_n > 0,$ *and* $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \delta_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C5)$ $0 < \liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n, \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n+\rho_n) < 1.$

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *defined as in* (11) *converge strongly to a common fixed point of* T_1, T_2 *and* T_3.

2. Main Results

Lemma 5. *Let* X *be a uniformly convex Banach space and* C *a nonempty closed convex nonexpansive retract of* X *with* P *as a nonexpansive retraction. Let* $T_1, T_2, T_3 : C \to X$ *be asymptotically quasi-nonexpansive in the intermediate sense with respect to sequences* $\{\tau_n\}$, $\{\ell_n\}$, $\{\eta_n\}$, *respectively such that* $F \neq \emptyset$, $\tau_n \to 0$ *as* $n \to \infty$, $\ell_n \to 0$ *as* $n \to \infty$, $\eta_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty} \ell_n < \infty$, $\sum_{n=1}^{\infty} \eta_n < \infty$. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ *be real sequences in* $[0,1]$ *such that* $a_n + \delta_n$, $b_n + c_n + \sigma_n$ *and* $\alpha_n + \beta_n + \gamma_n + \rho_n$ *are in* $[0,1]$ *for all* $n \ge 1$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ *and let* $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ *be bounded sequences in* C. *For a given* $x_1 \in C$, *let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *be the sequences as in (11). Then*

(i) $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in F$.

(ii) *If one of the following conditions* (a), (b), (c) *and* (d) *holds, then* $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0.$

(a) $\liminf_{n\to\infty} \beta_n > 0$ *and* $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \delta_n) < 1.$

(b) $\liminf_{n\to\infty} \alpha_n$, $\liminf_{n\to\infty} b_n > 0$ *and*
$0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \delta_n) < 1.$

(c) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

(d) $0 < \liminf_{n\to\infty} \alpha_n$ *and* $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1.$

(iii) *If either* (a) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$
or (b) $\liminf_{n\to\infty} \alpha_n > 0$ *and* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$
then $\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0.$

(iv) *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1,$
then $\lim_{n\to\infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0.$

Proof. Let $q \in F$. Using (6) and (11), we have:

$$
\begin{aligned}
(12) \quad \|z_n - q\| &= \|P[a_n T_1(PT_1)^{n-1}x_n + (1-a_n-\delta_n)x_n \\
&\quad + \delta_n u_n] - P(q)\| \\
&\leq \|a_n T_1(PT_1)^{n-1}x_n - q + (1-a_n-\delta_n)x_n \\
&\quad - q + \delta_n u_n - q\| \\
&\leq a_n\|T_1(PT_1)^{n-1}x_n - q\| + (1-a_n-\delta_n)\|x_n - q\| \\
&\quad + \delta_n\|u_n - q\| \\
&\leq a_n(\|x_n - q\| + \tau_n) + (1-a_n-\delta_n)\|x_n - q\| \\
&\quad + \delta_n\|u_n - q\| \\
&= (1-\delta_n)\|x_n - q\| + \delta_n\|u_n - q\| + a_n\tau_n.
\end{aligned}
$$

$$
\begin{aligned}
(13) \quad \|y_n - q\| &= \|P[b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n \\
&\quad + (1-b_n-c_n-\sigma_n)x_n + \sigma_n v_n] - P(q)\| \\
&\leq \|b_n T_2(PT_2)^{n-1}z_n - q + c_n T_1(PT_1)^{n-1}x_n \\
&\quad - q + (1-b_n-c_n-\sigma_n)x_n - q \ + \sigma_n v_n - q\| \\
&\leq b_n\|T_2(PT_2)^{n-1}z_n - q\| + c_n\|T_1(PT_1)^{n-1}x_n \\
&\quad - q\| + (1-b_n-c_n-\sigma_n)\|x_n - q\| + \sigma_n\|v_n - q\| \\
&\leq b_n(\|z_n - q\| + \ell_n) + c_n(\|x_n - q\| + \tau_n) \\
&\quad + (1-b_n-c_n-\sigma_n)\|x_n - q\| + \sigma_n\|v_n - q\| \\
&= b_n\|z_n - q\| + c_n\|x_n - q\| + (1-b_n-c_n-\sigma_n) \\
&\quad \times \|x_n - q\| + \sigma_n\|v_n - q\| + b_n\ell_n + c_n\tau_n \\
&= b_n\|z_n - q\| + (1-b_n-\sigma_n)\|x_n - q\| \\
&\quad + \sigma_n\|v_n - q\| + b_n\ell_n + c_n\tau_n.
\end{aligned}
$$

Using (6), (11), (5) and (6), we obtain:

$$
\begin{aligned}
(14) \quad \|x_{n+1} - q\| &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n \\
&\quad + \gamma_n T_1(PT_1)^{n-1}x_n \\
&\quad + (1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n + \rho_n w_n] - P(q)\| \\
&\leq \|\alpha_n T_3(PT_3)^{n-1}y_n - q + \beta_n T_2(PT_2)^{n-1}z_n - q \\
&\quad + \gamma_n T_1(PT_1)^{n-1}x_n - q \\
&\quad + (1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n - q + \rho_n w_n - q\| \\
&\leq \alpha_n\|T_3(PT_3)^{n-1}y_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - q\| \\
&\quad + \gamma_n\|T_1(PT_1)^{n-1}x_n - q\| \\
&\quad + (1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n - q\| + \rho_n\|w_n - q\| \\
&\leq \alpha_n(\|y_n - q\| + \eta_n) + \beta_n(\|z_n - q\| + \ell_n)
\end{aligned}
$$

$$
\begin{aligned}
&+ \gamma_n(\|x_n - q\| + \tau_n) \\
&+ (1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n - q\| + \rho_n\|w_n - q\| \\
= \; & \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| + \gamma_n\|x_n - q\| \\
&+ (1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n \\
= \; & \alpha_n\|y_n - q\| + \beta_n\|z_n - q\| \\
&+ (1-\alpha_n-\beta_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n \\
\leq \; & \alpha_n\{b_n\|z_n - q\| + (1-b_n-\sigma_n)\|x_n - q\| \\
&+ \sigma_n\|v_n - q\| + b_n\ell_n + c_n\tau_n\} \\
&+ \beta_n\{(1-\delta_n)\|x_n - q\| + \delta_n\|u_n - q\| + a_n\tau_n\} \\
&+ (1-\alpha_n-\beta_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n \\
= \; & \alpha_n b_n\|z_n - q\| + \alpha_n(1-b_n-\sigma_n)\|x_n - q\| \\
&+ \alpha_n\sigma_n\|v_n - q\| + \alpha_n b_n\ell_n \\
&+ \alpha_n c_n\tau_n + \beta_n(1-\delta_n)\|x_n - q\| \\
&+ \beta_n\delta_n\|u_n - q\| + \beta_n a_n\tau_n \\
&+ (1-\alpha_n-\beta_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n \\
\leq \; & \alpha_n b_n\{(1-\delta_n)\|x_n - q\| + \delta_n\|u_n - q\| + a_n\tau_n\} \\
&+ \alpha_n(1-b_n-\sigma_n)\|x_n - q\| + \alpha_n\sigma_n\|v_n - q\| \\
&+ \alpha_n b_n\ell_n + \alpha_n c_n\tau_n + \beta_n(1-\delta_n)\|x_n - q\| \\
&+\beta_n\delta_n\|u_n - q\| + \beta_n a_n\tau_n + (1-\alpha_n-\beta_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n \\
= \; & \alpha_n b_n(1-\delta_n)\|x_n - q\| + \alpha_n b_n\delta_n\|u_n - q\| + \alpha_n b_n a_n\tau_n \\
&+ \alpha_n(1-b_n-\sigma_n)\|x_n - q\| + \alpha_n\sigma_n\|v_n - q\| \\
&+ \alpha_n b_n\ell_n + \alpha_n c_n\tau_n + \beta_n(1-\delta_n)\|x_n - q\| \\
&+ \beta_n\delta_n\|u_n - q\| + \beta_n a_n\tau_n + (1-\alpha_n-\beta_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n \\
\leq \; & \|x_n - q\| + \alpha_n b_n\delta_n\|u_n - q\| + \alpha_n b_n a_n\tau_n \\
&+ \alpha_n(1-b_n-\sigma_n)\|x_n - q\| + \alpha_n\sigma_n\|v_n - q\| \\
&+ \alpha_n b_n\ell_n + \alpha_n c_n\tau_n + \beta_n(1-\delta_n)\|x_n - q\| \\
&+\beta_n\delta_n\|u_n - q\| + \beta_n a_n\tau_n + (1-\alpha_n-\beta_n-\rho_n)\|x_n - q\| \\
&+ \rho_n\|w_n - q\| + \alpha_n\eta_n + \beta_n\ell_n + \gamma_n\tau_n
\end{aligned}
$$

$$
\begin{aligned}
&= \|x_n - q\| + \{\alpha_n(1-b_n-\gamma_n) + \beta_n(1-\delta_n) \\
&\quad + (1-\alpha_n-\beta_n-\rho_n)\}\|x_n - q\| \\
&\quad + \delta_n(\alpha_n b_n + \beta_n)\|u_n - q\| + \alpha_n\sigma_n\|v_n - q\| + \rho_n\|w_n - q\| \\
&\quad + \tau_n(\alpha_n b_n a_n + \alpha_n c_n + \beta_n a_n + \gamma_n) + \ell_n(\alpha_n b_n + \beta_n) + \alpha_n\eta_n.
\end{aligned}
$$

But $\{\tau_n\}$, $\{\ell_n\}$, $\{\eta_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded, there exists a constant $K > 0$ such that $\alpha_n(1-b_n-\gamma_n) \leq K$, $\beta_n(1-\delta_n) \leq K$, $(1-\alpha_n-\beta_n-\rho_n) \leq K$, $(\alpha_n b_n + \beta_n)\|u_n - q\| \leq K$, $\alpha_n\|v_n - q\| \leq K$, $\|w_n - q\| \leq K$, $(\alpha_n b_n a_n + \alpha_n c_n + \beta_n a_n + \gamma_n) \leq K$, $(\alpha_n b_n + \beta_n) \leq K$ and $\alpha_n \leq K$. Hence, we have

$$
(15) \quad \|x_{n+1} - q\| \leq (1+K)\|x_n - q\| + K(\delta_n + \sigma_n + \rho_n + \tau_n + \ell_n + \eta_n).
$$

Hence, by Lemma 1, we obtain $\lim_{n\to\infty}\|x_n - q\|$ exists. ■

We now prove (ii), (iii) and (iv). But from (i), we observe that $\{x_n - q\}$, $\{T_1(PT_1)^{n-1}x_n - q\}$, $\{y_n - q\}$, $\{T_3(PT_3)^{n-1}y_n - q\}$, $\{z_n - q\}$ and $\{T_2(PT_2)^{n-1} z_n - q\}$ are all bounded. Now, let

$$
\begin{aligned}
(16) \quad M &= \max\{\sup_{n\geq 1}\|x_n - 1\|, \sup_{n\geq 1}\|T_1(PT_1)^{n-1}x_n - q\|, \sup_{n\geq 1}\|y_n - q\|, \\
&\quad \sup_{n\geq 1}\|T_3(PT_3)^{n-1}y_n - q\|, \sup_{n\geq 1}\|z_n - q\|, \sup_{n\geq 1}\|u_n - q\|, \\
&\quad \sup_{n\geq 1}\|T_2(PT_2)^{n-1}z_n - q\|, \sup_{n\geq 1}\|v_n - q\|, \sup_{n\geq 1}\|w_n - q\|\}.
\end{aligned}
$$

Using Lemma 3, there exists a continuous, strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$
\begin{aligned}
(17) \quad \|\lambda x + \mu y + \xi z + \vartheta w + \zeta s\|^2 &\leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 \\
&\quad + \vartheta\|w\|^2 + \zeta\|s\|^2 - \lambda\mu g(\|x - y\|)
\end{aligned}
$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \vartheta, \zeta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta + \zeta = 1$. Using (17), we obtain the followings:

$$
\begin{aligned}
(18) \quad \|z_n - q\|^2 &= \|P[a_n T_1(PT_1)^{n-1}x_n \\
&\quad + (1-a_n-\delta_n)x_n + \delta_n u_n] - P(q)\|^2 \\
&\leq \|a_n(T_1(PT_1)^{n-1}x_n - q) \\
&\quad + (1-a_n-\delta_n)(x_n - q) + \delta_n(u_n - q)\|^2 \\
&\leq a_n\|T_1(PT_1)^{n-1}x_n - q\|^2 \\
&\quad + (1-a_n-\delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 \\
&\quad - a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\leq a_n(\|x_n - q\|^2 + \tau_n^2) + (1-a_n-\delta_n)\|x_n - q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \delta_n\|u_n - q\|^2 - a_n(1 - a_n - \delta_n) \\
&\times g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
= \;& (1 - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 + a_n\tau_n^2 \\
&- a_n(1 - a_n - \delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|),
\end{aligned}
$$

$$
\begin{aligned}
(19) \quad \|y_n - q\|^2 = \;& \|P[b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n \\
&+ (1 - b_n - c_n - \sigma_n)x_n + \sigma_n v_n] - P(q)\|^2 \\
\le \;& \|b_n(T_2(PT_2)^{n-1}z_n - q) + c_n(T_1(PT_1)^{n-1}x_n - q) \\
&+ (1 - b_n - c_n - \sigma_n)(x_n - q) + \sigma_n(v_n - q)\|^2 \\
\le \;& b_n\|T_2(PT_2)^{n-1}z_n - q\|^2 \\
&+ (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&+ c_n\|T_1(PT_1)^{n-1}x_n - q\|^2 + \sigma_n\|v_n - q\|^2 \\
&- b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
\le \;& b_n(\|z_n - q\|^2 + \ell_n^2) + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&+ c_n(\|x_n - q\|^2 + \tau_n^2) + \sigma_n\|v_n - q\|^2 \\
&- b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
= \;& b_n\|z_n - q\|^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&+ c_n\|x_n - q\|^2 + \sigma_n\|v_n - q\|^2 + b_n\ell_n^2 + c_n\tau_n^2 \\
&- b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|).
\end{aligned}
$$

Using (17), (18) and (19), we obtain:

$$
\begin{aligned}
(20) \quad \|x_{n+1} - q\|^2 = \;& \|P[\alpha_nT_3(PT_3)^{n-1}y_n \\
&+ \beta_nT_2(PT_2)^{n-1}z_n + \gamma_nT_1(PT_1)^{n-1}x_n \\
&+ (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\|^2 \\
\le \;& \alpha_n\|T_3(PT_3)^{n-1}y_n - q\|^2 \\
&+ \beta_n\|T_2(PT_2)^{n-1}z_n - q\|^2 \\
&+ \gamma_n\|T_1(PT_1)^{n-1}x_n - q\|^2 \\
&+ (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&+ \rho_n\|w_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\times g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
\le \;& \alpha_n(\|y_n - q\|^2 + \eta_n^2) + \beta_n(\|z_n - q\|^2 + \ell_n^2) \\
&+ \gamma_n(\|x_n - q\|^2 + \tau_n^2) \\
&+ (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&+ \rho_n\|w_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\times g(\|T_3(PT_3)^{n-1}y_n - x_n\|)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_n\|y_n - q\|^2 + \alpha_n\eta_n^2 + \beta_n\|z_n - q\|^2 + \beta_n\ell_n^2 \\
&\quad + \gamma_n\|x_n - q\|^2 + \gamma_n\tau_n^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&\quad + \rho_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\quad \times g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\leq \alpha_n\{b_n\|z_n - q\|^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&\quad + c_n\|x_n - q\|^2 + \sigma_n\|v_n - q\|^2 + b_n\ell_n^2 + c_n\tau_n^2 \\
&\quad - b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|)\} \\
&\quad + \alpha_n\eta_n^2 + \beta_n\|z_n - q\|^2 + \beta_n\ell_n^2 + \gamma_n\|x_n - q\|^2 \\
&\quad + \gamma_n\tau_n^2 + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&\quad + \rho_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\quad \times g(\|T_3(PT_3)^{n-1}y_n - x_n) \\
&= \alpha_n b_n\|z_n - q\|^2 + \alpha_n(1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&\quad + \alpha_n c_n\|x_n - q\|^2 + \alpha_n\sigma_n\|v_n - q\|^2 + \alpha_n b_n\ell_n^2 + \alpha_n c_n\tau_n^2 \\
&\quad - \alpha_n b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n \\
&\quad - x_n\|) + \alpha_n\eta_n^2 + \beta_n\|z_n - q\|^2 \\
&\quad + \beta_n\ell_n^2 + \gamma_n\|x_n - q\|^2 + \gamma_n\tau_n^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 + \rho_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\leq (\alpha_n b_n + \beta_n)\{(1 - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 + a_n\tau_n^2 \\
&\quad -a_n(1 - a_n - \delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|)\} \\
&\quad + \alpha_n(1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 + \alpha_n c_n\|x_n - q\|^2 \\
&\quad + \alpha_n\sigma_n\|v_n - q\|^2 + \alpha_n b_n\ell_n^2 + \alpha_n c_n\tau_n^2 \\
&\quad - \alpha_n b_n(1 - b_n - c_n - \sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad + \alpha_n\eta_n^2 + \beta_n\ell_n^2 + \gamma_n\|x_n - q\|^2 + \gamma_n\tau_n^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 + \rho_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&= (\alpha_n b_n + \beta_n)(1 - \delta_n)\|x_n - q\|^2 + \delta_n(\alpha_n b_n + \beta_n)\|u_n - q\|^2 \\
&\quad + (\alpha_n b_n + \beta_n)a_n\tau_n^2 + \alpha_n(1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&\quad + \alpha_n c_n\|x_n - q\|^2 + \alpha_n\sigma_n\|v_n - q\|^2 + \alpha_n b_n\ell_n^2 + \alpha_n c_n\tau_n^2 \\
&\quad + \alpha_n\eta_n^2 + \beta_n\ell_n^2 + \gamma_n\|x_n - q\|^2 + \gamma_n\tau_n^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 + \rho_n\|w_n - q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha_n b_n a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \beta_n a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \alpha_n b_n(1-b_n-c_n-\sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&- \alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
\leq\ & \|x_n-q\|^2 + \{(\alpha_n b_n+\beta_n)(1-\delta_n)+\alpha_n c_n+\gamma_n \\
&+ (1-\alpha_n-\beta_n-\gamma_n-\rho_n)\}\|x_n-q\|^2 \\
&+ \delta_n(\alpha_n b_n+\beta_n)\|u_n-q\|^2 + \alpha_n\sigma_n\|v_n-q\|^2 \\
&+ \rho_n\|w_n-q\|^2 + \tau_n^2(a_n(\alpha_n b_n+\beta_n)+\alpha_n c_n+\gamma_n) \\
&+ \ell_n^2(\alpha_n b_n+\beta_n) + \alpha_n\eta_n^2 \\
&- \alpha_n b_n a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \beta_n a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \alpha_n b_n(1-b_n-c_n-\sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&- \alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|).
\end{aligned}
$$

Since $\{\tau_n\}$, $\{\ell_n\}$, $\{\eta_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded and $\{x_n\}$ is bounded, there exists $K_0 > 0$ such that $(\alpha_n b_n+\beta_n)(1-\delta_n)\|x_n-q\|^2 \leq K_0$, $\alpha_n c_n\|x_n-q\|^2 \leq K_0$, $\gamma_n\|x_n-q\|^2 \leq K_0$, $(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \leq K_0$, $(\alpha_n b_n+\beta_n)\|u_n-q\|^2 \leq K_0$, $\alpha_n\|v_n-q\|^2 \leq K_0$, $\|w_n-q\|^2 \leq K_0$, $(a_n(\alpha_n b_n+\beta_n)+\alpha_n c_n+\gamma_n) \leq K_0$, $(\alpha_n b_n+\beta_n) \leq K_0$, $\alpha_n \leq K_0$ for all $n \geq 1$. Hence,

$$
\begin{aligned}
(21)\quad & \alpha_n b_n a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\leq \|x_n-q\|^2 - \|x_{n+1}-q\|^2 + K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2)
\end{aligned}
$$

$$
\begin{aligned}
(22)\quad & \beta_n a_n(1-a_n-\delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\leq \|x_n-q\|^2 - \|x_{n+1}-q\|^2 + K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2)
\end{aligned}
$$

$$
\begin{aligned}
(23)\quad & \alpha_n b_n(1-b_n-c_n-\sigma_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\leq \|x_n-q\|^2 - \|x_{n+1}-q\|^2 + K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2)
\end{aligned}
$$

$$
\begin{aligned}
(24)\quad & \alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\leq \|x_n-q\|^2 - \|x_{n+1}-q\|^2 + K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2).
\end{aligned}
$$

Using (17) again, we obtain:

$$
\begin{aligned}
(25)\qquad \|y_n-q\|^2 &= \|P[b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n \\
&\quad + (1-b_n-c_n-\sigma_n)x_n + \sigma_n v_n] - P(q)\|^2 \\
&\leq c_n\|T_1(PT_1)^{n-1}x_n - q\|^2 \\
&\quad + (1-b_n-c_n-\sigma_n)\|x_n-q\|^2 \\
&\quad + b_n\|T_2(PT_2)^{n-1}z_n - q\|^2 + \sigma_n\|v_n-q\|^2 \\
&\quad - c_n(1-b_n-c_n-\sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|)
\end{aligned}
$$

$$
\begin{aligned}
&\leq c_n(\|x_n - q\|^2 + \tau_n^2) + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&\quad + b_n(\|z_n - q\|^2 + \ell_n^2) + \sigma_n\|v_n - q\|^2 \\
&\quad - c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&= c_n\|x_n - q\|^2 + c_n\tau_n^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&\quad + b_n\|z_n - q\|^2 + b_n\ell_n^2 + \sigma_n\|v_n - q\|^2 \\
&\quad - c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|).
\end{aligned}
$$

By (17), (8) and (25), we have

$$
\begin{aligned}
(26)\quad \|x_{n+1} - q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n \\
&\quad + \gamma_n T_1(PT_1)^{n-1}x_n \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\|^2 \\
&\leq \alpha_n\|T_3(PT_3)^{n-1}y_n - q\|^2 \\
&\quad + \beta_n\|T_2(PT_2)^{n-1}z_n - q\|^2 \\
&\quad + \rho_n\|w_n - q\|^2 + \gamma_n\|T_1(PT_1)^{n-1}x_n - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&\quad - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\quad \times g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\leq \alpha_n(\|y_n - q\|^2 + \eta_n^2) + \beta_n(\|z_n - q\|^2 + \ell_n^2) \\
&\quad + \rho_n\|w_n - q\|^2 + \gamma_n(\|x_n - q\|^2 + \tau_n^2) \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&\quad - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\quad \times g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&= \alpha_n\|y_n - q\|^2 + \alpha_n\eta_n^2 + \beta_n\|z_n - q\|^2 \\
&\quad + \beta_n\ell_n^2 + \rho_n\|w_n - q\|^2 + \gamma_n\|x_n - q\|^2 + \gamma_n\tau_n^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&\quad - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n) \\
&\quad \times g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\leq \alpha_n\{c_n\|x_n - q\|^2 + c_n\tau_n^2 \\
&\quad + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
&\quad + b_n\|z_n - q\|^2 + b_n\ell_n^2 + \sigma_n\|v_n - q\|^2 \\
&\quad - c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|)\} \\
&\quad + \alpha_n\eta_n^2 + \beta_n\|z_n - q\|^2 + \beta_n\ell_n^2 \\
&\quad + \rho_n\|w_n - q\|^2 + \gamma_n\|x_n - q\|^2 + \gamma_n\tau_n^2 \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
&\quad - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)
\end{aligned}
$$

$$
\begin{aligned}
& \times g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
= \; & \alpha_n c_n\|x_n - q\|^2 + \alpha_n c_n \tau_n^2 \\
& + \alpha_n(1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
& + \alpha_n b_n\|z_n - q\|^2 + \alpha_n b_n \ell_n^2 + \alpha_n \sigma_n\|v_n - q\|^2 \\
& - \alpha_n c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
& +\alpha_n \eta_n^2 + \beta_n\|z_n - q\|^2 + \beta_n \ell_n^2 \\
& + \rho_n\|w_n - q\|^2 + \gamma_n\|x_n - q\|^2 + \gamma_n \tau_n^2 \\
& + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
& - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
\leq \; & (\alpha_n b_n + \beta_n)\{(1 - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 + a_n \tau_n^2 \\
& - a_n(1 - a_n - \delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|)\} \\
& + \alpha_n c_n\|x_n - q\|^2 + \alpha_n c_n \tau_n^2 \\
& + \alpha_n(1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\
& + \alpha_n b_n \ell_n^2 + \alpha_n \sigma_n\|v_n - q\|^2 \\
& - \alpha_n c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
& + \alpha_n \eta_n^2 + \beta_n \ell_n^2 + \rho_n\|w_n - q\|^2 \\
& + \gamma_n\|x_n - q\|^2 + \gamma_n \tau_n^2 + \\
& (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\|^2 \\
& -\beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
\leq \; & \|x_n - q\|^2 + \{(\alpha_n b_n + \beta_n)(1 - \delta_n) + \alpha_n c_n + \gamma_n \\
& + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\}\|x_n - q\|^2 \\
& + \delta_n(\alpha_n b_n + \beta_n)\|u_n - q\|^2 \\
& + \alpha_n \sigma_n\|v_n - q\|^2 + \rho_n\|w_n - q\|^2 \\
& + \tau_n^2(a_n(\alpha_n b_n + \beta_n) + \alpha_n c_n + \gamma_n) \\
& + \ell_n^2(\alpha_n b_n + \beta_n) + \alpha_n \eta_n^2 \\
& - a_n(\alpha_n b_n + \beta_n)(1 - a_n - \delta_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
& - \alpha_n c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
& - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
\leq \; & \|x_n - q\|^2 + \{(\alpha_n b_n + \beta_n)(1 - \delta_n) + \alpha_n c_n + \gamma_n \\
& +(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\}\|x_n - q\|^2 \\
& + \delta_n(\alpha_n b_n + \beta_n)\|u_n - q\|^2 + \alpha_n \sigma_n\|v_n - q\|^2 \\
& + \rho_n\|w_n - q\|^2 + \tau_n^2(a_n(\alpha_n b_n + \beta_n) + \alpha_n c_n + \gamma_n) \\
& + \ell_n^2(\alpha_n b_n + \beta_n) + \alpha_n \eta_n^2 \\
& - \alpha_n c_n(1 - b_n - c_n - \sigma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
& - \beta_n(1 - \alpha_n - \beta_n - \gamma_n - \rho_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|).
\end{aligned}
$$

Hence,

$$\alpha_n c_n(1-b_n-c_n-\sigma_n)g(\|T_1(PT_1)^{n-1}x_n-x_n\|) \leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2). \tag{27}$$

$$\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_2(PT_2)^{n-1}z_n-x_n\|) \leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2). \tag{28}$$

Using (17), (18) and (19), we have:

$$\begin{aligned} \|x_{n+1}-q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n+\beta_n T_2(PT_2)^{n-1}z_n \\ &\quad +\gamma_n T_1(PT_1)^{n-1}x_n \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_n w_n]-P(q)\|^2 \\ &\leq \gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2 \\ &\quad +\alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\rho_n\|w_n-q\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\ &\quad +\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2 \\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n) \\ &\quad \times g(\|T_1(PT_1)^{n-1}x_n-x_n\|) \\ &\leq \gamma_n(\|x_n-q\|^2+\tau_n^2)+\alpha_n(\|y_n-q\|^2+\eta_n^2) \\ &\quad +\rho_n\|w_n-q\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\ &\quad +\beta_n(\|z_n-q\|^2+\ell_n^2) \\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n) \\ &\quad \times g(\|T_1(PT_1)^{n-1}x_n-x_n\|) \\ &= \gamma_n\|x_n-q\|^2+\gamma_n\tau_n^2+\alpha_n\|y_n-q\|^2 \\ &\quad +\alpha_n\eta_n^2+\rho_n\|w_n-q\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2+ \\ &\quad +\beta_n\|z_n-q\|^2+\beta_n\ell_n^2 \\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n) \\ &\quad \times g(\|T_1(PT_1)^{n-1}x_n-x_n\|) \\ &\leq \gamma_n\|x_n-q\|^2+\gamma_n\tau_n^2+\alpha_n\{b_n\|z_n-q\|^2 \\ &\quad +(1-b_n-c_n-\sigma_n)\|x_n-q\|^2 \\ &\quad +c_n\|x_n-q\|^2+\sigma_n\|v_n-q\|^2+b_n\ell_n^2+c_n\tau_n^2\} \\ &\quad +\alpha_n\eta_n^2+\rho_n\|w_n-q\|^2 \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\ &\quad +\beta_n\|z_n-q\|^2+\beta_n\ell_n^2 \end{aligned} \tag{29}$$

$$
\begin{aligned}
&- \gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_1(PT_1)^{n-1}x_n-x_n\|)\\
\le\ & (\alpha_n b_n+\beta_n)\{(1-\delta_n)\|x_n-q\|^2\\
&+\delta_n\|u_n-q\|^2+a_n\tau_n^2\}+\gamma_n\|x_n-q\|^2\\
&+\gamma_n\tau_n^2+\alpha_n(1-b_n-c_n-\sigma_n)\|x_n-q\|^2\\
&+\alpha_n c_n\|x_n-q\|^2+\alpha_n\sigma_n\|v_n-q\|^2\\
&+\alpha_n b_n\ell_n^2+\alpha_n c_n\tau_n^2+\alpha_n\eta_n^2+\rho_n\|w_n-q\|^2\\
&+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2+\beta_n\ell_n^2\\
&-\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_1(PT_1)^{n-1}x_n-x_n\|)\\
\le\ & \|x_n-q\|^2+\{(\alpha_n b_n+\beta_n)(1-\delta_n)+\gamma_n+\alpha_n c_n\\
&+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\}\|x_n-q\|^2\\
&+(\alpha_n b_n+\beta_n)\delta_n\|u_n-q\|^2+(\alpha_n b_n+\beta_n)a_n\tau_n^2\\
&+\gamma_n\tau_n^2+\alpha_n\sigma_n\|v_n-q\|^2\\
&+\alpha_n b_n\ell_n^2+\alpha_n c_n\tau_n^2\\
&+\alpha_n\eta_n^2+\rho_n\|w_n-q\|^2+\beta_n\ell_n^2\\
&-\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_1(PT_1)^{n-1}x_n-x_n\|)\\
=\ & \|x_n-q\|^2+\{(\alpha_n b_n+\beta_n)(1-\delta_n)+\gamma_n+\alpha_n c_n\\
&+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\}\|x_n-q\|^2\\
&+\delta_n(\alpha_n b_n+\beta_n)\|u_n-q\|^2\\
&+\alpha_n\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\
&+\tau_n^2(a_n(\alpha_n b_n+\beta_n)+\gamma_n+\alpha_n c_n)\\
&+\ell_n^2(\alpha_n b_n+\beta_n)+\alpha_n\eta_n^2\\
&-\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_1(PT_1)^{n-1}x_n-x_n\|).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(30)\qquad &\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g(\|T_1(PT_1)^{n-1}x_n-x_n\|)\\
&\qquad\le \|x_n-q\|^2-\|x_{n+1}-q\|^2+K_0(1+\delta_n+\sigma_n+\rho_n).
\end{aligned}
$$

(ii) Let $\liminf_{n\to\infty}\beta_n>0$ and $0<\liminf_{n\to\infty}a_n\le\limsup(a_n+\delta_n)<1$. Hence, there exists a positive integer n_0 and $\lambda,\lambda'\in(0,1)$ such that $0<\lambda<\beta_n$, $0<\lambda<\beta_n$, $0<\lambda<a_n$ and $a_n+\delta_n<\lambda'<1$ for all $n\ge n_0$. This implies by (22) that

$$
\begin{aligned}
(31)\qquad &\lambda^2(1-\lambda')g(\|T_1(PT_1)^{n-1}x_n-x_n\|)\le\|x_n-q\|^2\\
&\qquad-\|x_{n+1}-q\|^2+K_0(1+\delta_n+\sigma_n+\rho_n+\tau_n^2+\ell_n^2+\eta_n^2)
\end{aligned}
$$

for each $n\ge n_0$. From (31), we obtain that for each $r\ge n_0$,

$$\sum_{n=n_0}^{r} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \tag{32}$$
$$\leq \frac{1}{\lambda^2(1-\lambda')}\{\sum_{n=n_0}^{r}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2)$$
$$+ K_0 \sum_{n=n_0}^{r}(1 + \delta_n + \sigma_n + \rho_n + \tau_n^2 + \ell_n^2 + \eta_n^2)\}$$
$$\leq \frac{1}{\lambda^2(1-\lambda')}\{\|x_{n_0} - q\|^2$$
$$+ K_0 \sum_{n=n_0}^{r}(1 + \delta_n + \sigma_n + \rho_n + \tau_n^2 + \ell_n^2 + \eta_n^2)\}.$$

Since δ_n, σ_n and ρ_n are all bounded and $\tau_n \to 0$ as $n \to \infty$, $\ell_n \to 0$ as $n \to \infty$, $\eta_n \to 0$ as $n \to \infty$, it follows that $\tau_n^2 \to 0$ as $n \to \infty$, $\ell_n^2 \to 0$ as $n \to \infty$ and $\eta_n^2 \to 0$ as $n \to \infty$. Hence, by (32), we obtain that as $r \to \infty$ $\sum_{n=n_0}^{\infty} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) < \infty$. Hence, $\lim_{n\to\infty} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, hence $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

Following same procedure as in (ii) part (a) and the results displayed in (21), (30), (27), (28), (23) and (24), the results in (ii) (b, c, d), (iii) (a, b) and (iv), can be proved.

Lemma 6. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive in the intermediate sense with respect to sequences $\{\tau_n\}$, $\{\ell_n\}$, $\{\eta_n\}$, respectively such that $F \neq \emptyset$, $\tau_n \to 0$ as $n \to \infty$, $\ell_n \to 0$ as $n \to \infty$, $\eta_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty} \ell_n < \infty$, $\sum_{n=1}^{\infty} \eta_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ be real sequences in $[0,1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0,1]$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences as in (11). Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian. If $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0$, then*

(i) $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$,

(ii) $\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0$, *and*

(iii) $\lim_{n\to\infty} \|T_3 x_n - x_n\| = 0$.

Proof. But

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T_3(PT_3)^{n-1}y_n - x_n\| \tag{33}$$

$$+ \beta_n \|T_2(PT_2)^{n-1} z_n - x_n\| + \gamma_n \|T_1(PT_1)^{n-1} x_n - x_n\|$$
$$+ (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - x_n\| + \rho_n \|w_n - x_n\|$$
$$\longrightarrow 0 \ \ as \ \ n \to \infty,$$

by triangle inequality, we obtain

$$\begin{aligned} \|T_1(PT_1)^{n-1}x_{n+1} - x_{n+1}\| &\leq \|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\| \\ &\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| + \|x_{n+1} - x_n\| \\ &\leq L\|x_{n+1} - x_n\| + \|x_n - x_n\| + \tau_n + \|x_{n+1} - x_n\| \\ &\quad \longrightarrow 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{34}$$

By (34), we obtain

$$\begin{aligned} |T_1 x_n - x_n\| &\leq \|T_1(PT_1)^{n-1}x_n - x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1 x_n\| \\ &\leq \|x_n - x_n\| + \tau_n + L\|T_1(PT_1)^{n-2}x_n - x_n\| \\ &\quad \longrightarrow 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{35}$$

Hence, $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$, this proves (i). Next, we prove (ii). Since

$$\begin{aligned} \|z_n - x_n\| &\leq a_n \|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n \|u_n - x_n\| \\ &\quad \longrightarrow 0 \ \ as \ \ n \to \infty, \end{aligned} \tag{36}$$

hence, we have

$$\begin{aligned} \|T_2(PT_2)^{n-1}x_{n+1} - x_{n+1}\| &\leq \|T_2(PT_2)^{n-1}x_{n+1} - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}z_n - x_n\| + \|x_{n+1} - x_n\| \\ &\leq L\|x_{n+1} - x_n\| + L\|z_n - x_n\| + \|z_n - x_n\| \\ &\quad + \ell_n + \|x_{n+1} - x_n\| \longrightarrow 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{37}$$

Thus, we obtain

$$\begin{aligned} \|T_2 x_n - x_n\| &\leq \|T_2(PT_2)^{n-1}x_n - x_n\| + \|T_2(PT_2)^{n-1}x_n - T_2 x_n\| \\ &\leq \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| + \|z_n - x_n\| \\ &\quad + \ell_n + L\|T_2(PT_2)^{n-2}x_n - x_n\| \longrightarrow 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{38}$$

Hence, $\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0$, and (ii) is obtained. ■

Now, we prove (iii)

$$\begin{aligned} \|y_n - x_n\| &\leq b_n \|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad +; c_n \|T_1(PT_1)^{n-1}x_n - x_n\| + \sigma_n \|v_n - x_n\| \longrightarrow 0 \end{aligned} \tag{39}$$

and $\|T_3(PT_3)^{n-1}y_n - x_n\| \longrightarrow 0$ as $n \to \infty$, we have

$$(40) \quad \begin{aligned} \|T_3(PT_3)^{n-1}x_n - x_n\| &\leq \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\leq L\|y_n - x_n\| + \|y_n - x_n\| + \eta_n \\ &\quad \longrightarrow 0 \;\; as \;\; n \to \infty. \end{aligned}$$

Hence,

$$(41) \quad \begin{aligned} \|T_3(PT_3)^{n-1}x_{n+1} - x_{n+1}\| &\leq \|T_3(PT_3)^{n-1}x_{n+1} - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| + \|x_{n+1} - x_n\| \\ &\leq L\|x_{n+1} - x_n\| + L\|y_n - x_n\| + \|y_n - x_n\| + \eta_n \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \; as \; n \to \infty. \end{aligned}$$

Thus, we have:

$$(42) \quad \begin{aligned} \|T_3x_n - x_n\| &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + \|T_3(PT_3)^{n-1}x_n - T_3x_n\| \\ &\leq \|x_n - x_n\| + \eta_n + L\|T_3(PT_3)^{n-2}x_n - x_n\| \\ &\quad \longrightarrow 0 \; as \; n \to \infty. \end{aligned}$$

This completes the prove of (iii).

Theorem 1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive in the intermediate sense with respect to sequences $\{\tau_n\}, \{\ell_n\}, \{\eta_n\}$, respectively such that $F \neq \emptyset$, $\tau_n \to 0$ as $n \to \infty$, $\ell_n \to 0$ as $n \to \infty$, $\eta_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty} \ell_n < \infty$, $\sum_{n=1}^{\infty} \eta_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ be real sequences in $[0,1]$ such that $a_n + \delta_n, b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded sequences in C. Assume that T_1, T_2, T_3 are uniformly L-Lipschitzian. If one of T_i $(i = 1, 2, 3)$ is a completely continuous and one of the following conditions $(C1)$-$(C5)$ is satisfied:*

$(C1)$ $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \delta_n) < 1,$
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C2)$ $0 < \liminf_{n\to\infty} b_n, \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C3)$ $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \sigma_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C4)$ $\liminf_{n\to\infty} b_n > 0,$ *and* $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \delta_n) < 1,$ *and*
$0 < \liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1.$

$(C5)$ $0 < \liminf_{n\to\infty} \alpha_n, \liminf_{n\to\infty} \beta_n, \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n+\rho_n) < 1.$

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *defined as in (11) converge strongly to a common fixed point of* T_1, T_2 *and* T_3.

Proof. Assume that one of the conditions $(C1)$-$(C5)$ is satisfied. Using Lemma 6, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for $i = 1, 2, 3$. Suppose one of T_1, T_2 and T_3 say T_1 is completely continuous. Since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1 x_{n_k}\}$ converges to $q \in C$. But $\|x_{n_k} - q\| \leq \|T_1 x_{n_k} - x_{n_k}\| + \|T_1 x_{n_k} - q\|$, we obtain $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Hence, $\{x_{n_k}\}$ converges to $q \in C$. By continuity of T_i,, we obtain $T_i x_{n_k} \to T_i q$ as $k \to \infty$. Since

$$\|T_i q - q\| \leq \|T_i x_{n_k} - T_i q\| + \|T_i x_{n_k} - x_{n_k}\| + \|x_{n_k} - q\| \to 0 \ as\ k \to \infty,$$

we have $T_i q = q$ $(i = 1, 2, 3)$. Hence, $q \in F$. By Lemma 5 (i), $\lim_{n\to\infty} \|x_n - q\|$ exists. It follows that $\lim_{n\to\infty} \|x_n - q\| = 0$. By Lemma 5, we obtain

$$\|T_1(PT_1)^{n-1} x_n - x_n\| \to 0, \tau_n \to 0, \quad \ell_n \to 0$$

and

$$\|T_2(PT_2)^{n-1} z_n - x_n\| \to 0 \ \text{ as } \ n \to \infty.$$

Hence,

$$\begin{aligned} \|y_n - x_n\| \ &\leq \ b_n \|T_2(PT_2)^{n-1} z_n - x_n\| + c_n \|T_1(PT_1)^{n-1} x_n - x_n\| \\ &\quad + \sigma_n \|v_n - x_n\| \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| \ &\leq \ a_n \|T_1(PT_1)^{n-1} x_n - x_n\| + \delta_n \|u_n - x_n\| \\ &\to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

These imply $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. ∎

Remark 1. Theorem 2.3 extends Theorem 2.3 of Inprasit and Wattanataweekul [7]. Observe that if $\tau_n = 0 \ \forall \ n$, $\ell_n = 0 \ \forall \ n$ and $\eta_n = 0 \ \forall \ n$, then we obtain Theorem 2.3 of [7]. Similarly, Theorem 2.3 improves, extends and unifies the results of Nilsrakoo and Saejung [13], Xu and Noor [27] and Suantai [25].

The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy Condition A [23] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - q\| : q \in F(T)\}$. As Tan and Xu [26] pointed out, the Condition A is weaker than the compactness of C.

The following is a strong convergence result for asymptotically quasi-nonexpansive nonself-mappings in the intermediate sense in a uniformly convex Banach space satisfying Condition A.

Theorem 2. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive in the intermediate sense with respect to sequences $\{\tau_n\}$, $\{\ell_n\}$, $\{\eta_n\}$, respectively such that $F \neq \emptyset$, $\tau_n \to 0$ as $n \to \infty$, $\ell_n \to 0$ as $n \to \infty$, $\eta_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty} \ell_n < \infty$, $\sum_{n=1}^{\infty} \eta_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded sequences in C. Suppose T_1 satisfies Condition A and T_2, T_3 are uniformly L-Lipschitzian and one of the conditions $(C1)$-$(C5)$ in Theorem 2.3 is satisfied. Then the sequence $\{x_n\}$ defined as in (11) converges strongly to a common fixed point of T_1, T_2 and T_3*

Proof. Let $q \in F$. Using Lemma 5, $\lim_{n\to\infty} \|x_n - q\|$ exists. Hence, $\{x_n - q\}$ is bounded. Then there exists a constant H such that $\|x_n - q\| \leq H$ for each $n \geq 1$. This together with (15), we obtain

$$\|x_{n+1} - q\| \leq \|x_n - q\| + D_n, \tag{43}$$

where $D_n = KH + K(1 + \delta_n + \sigma_n + \rho_n + \tau_n^2 + \ell_n^2 + \eta_n^2) < \infty$ for each $n \geq 1$. By Lemma 6, we obtain $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ $(i = 1, 2, 3)$. But T_1 satisfies Condition A, hence we have $\lim_{n\to\infty} d(x_n, F(T_1)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F(T_1)) = 0$ and $\sum_{n=1}^{\infty} D_n < \infty$, for any $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, F(T_1)) < \frac{\epsilon}{4}$ and $\sum_{k=n_0}^{n} D_k < \frac{\epsilon}{2}$ for each $n \geq n_0$. Let $n \in \mathbb{N}$ be such that $n \geq n_0$. Then we can find $q^* \in F$ such that $\|x_n - q^*\| < \frac{\epsilon}{4}$. This implies by (43) that for $m \geq 1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q^*\| + \|x_n - q^*\| \\ &\leq 2\|x_n - q^*\| + \sum_{k=n}^{n+m-1} D_k \\ &= 2\|x_n - q^*\| + \sum_{k=n_0}^{n+m-1} D_k < 2(\frac{\epsilon}{4}) + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{44}$$

This implies that $\{x_n\}$ is a Cauchy sequence and so it is convergent. Suppose $\lim_{n\to\infty} x_n = p$. But $d(x_n, F(T_1)) \to 0$ as $n \to \infty$, it follows that $d(p, F(T_1)) = 0$ and hence $p \in F(T_1)$. We now show that $p \in F(T_2) \cap F(T_3)$. Since T_2, T_3 are uniformly L-Lipschitzian and by using Lemma 6, we have

$$\begin{aligned} \|T_i p - p\| &\leq \|T_i x_n - T_i p\| + \|T_i x_n - x_n\| + \|x_n - p\| \\ &\leq L\|x_n - p\| + \|x_n - x_n\| + \tau_n + \ell_n + \eta_n \\ &\quad + \|x_n - p\| \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{45}$$

Hence, $T_i p = p$ $(i = 2, 3)$. So that $p \in F$. The proof of the theorem is complete. ■

Theorem 3. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive in the intermediate sense with respect to sequences $\{\tau_n\}$, $\{\ell_n\}$, $\{\eta_n\}$, respectively such that $F \neq \emptyset$, $\tau_n \to 0$ as $n \to \infty$, $\ell_n \to 0$ as $n \to \infty$, $\eta_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty} \ell_n < \infty$, $\sum_{n=1}^{\infty} \eta_n < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded sequences in C. Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian and $I - T_i$ $(i = 1, 2, 3)$ is demiclosed at 0. If one of the following conditions $(C1)$-$(C5)$ in Theorem 1 is satisfied, then the sequence $\{x_n\}$ defined as in (11) converges weakly to a common fixed point of T_1, T_2 and T_3.*

Proof. Suppose one of the conditions (C1)-(C5) is satisfied. By Lemma 5 and Lemma 6, we obtain $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ $(i = 1, 2, 3)$. But X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality, we suppose that $x_n \to u$ weakly as $n \to \infty$. By the demiclosedness of $I - T_i$ at 0, we have $u \in F$. Suppose subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. Since $I - T_i$ $(i = 1, 2, 3)$ is demiclosed at 0, we have u and $v \in F$. By Lemma 5, we have $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. Hence, from Lemma 4, we obtain $u = v$. Therefore, $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. ■

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