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Ali Olgun and Fatma Taşdelen

ON THE RATES OF CONVERGENCE OF CERTAIN BIVARIATE LINEAR POSITIVE OPERATORS

Abstract. In this paper, we present a sequence of linear positive bivariate operators and investigate the approximation properties of them. Next we study the rates of converge of this approximation by means modulus of continuity and functions from Lipschitz class. After we give a Voronovskaya type theorem for n Morever, we give an r th order generalization of these operators. Finally, we investigate approximation properties of this generalization and observe the rates of convergence for them.

Key words: linear positive operator, modulus of continuity, order of approximation, Voronovskaya type theorem, Kantorovich operators, Beta operators.

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1. Introduction

Let $f \in C([0,1])$. The well known Beta operators are defined as follows:

$$B_n(f;x) := \int_0^1 \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx, n(1-x))} f(t)\, dt,$$

where $n \in \mathbb{N} = \{1, 2...\}$, $x \in (0,1)$ and B is the familiar Beta function and we set $B_n(f,k) := f(k),\ k = 0,1$. Some approximation properties of Beta operators were studied in [5], [6], [10] and references there in.

Let $x \in [0,1]$ and $f \in C([0,1])$. The Bernstein-Kantorovich operators are defined by

$$K_n(f;x) := n \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s)\, ds,\ n \in \mathbb{N}.$$

Also some approximation properties of K_n can be wieved in [1].

Now, we construct the tensor product of B_m and K_n, $B_{mn} := B_m^{(1)} \circ K_n^{(2)}$, here " $\circ$ " denotes the composition and $B_m^{(1)}$ and $K_n^{(2)}$ are parametric extensions of B_m and K_n. Clearly B_{mn}, $m, n \in \mathbb{N}$ are linear positive operators.

$$B_{mn}(f;x,y) := n\sum_{k=0}^{n}\binom{n}{k}y^k(1-y)^{n-k}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_0^1 f(\theta,s)\frac{\theta^{mx-1}(1-\theta)^{m(1-x)-1}}{B(mx,m(1-x))}d\theta ds,$$

where $f \in C(I^2)$, i.e, continuous real-valued functions defined on I^2, $I^2 := [0,1] \times [0,1]$.

In this paper, we study the approximation properties of the operators B_{mn} and obtain the rates of convergence by means of modulus of continuity and functions from Lipschitz class. For second order differentiable functions, Voronovskaya [11] was the first to prove a theorem for Bernstein polynomials known as Voronovskaya Theorem. Later on, it was studied by many authors for some other linear positive operators (e.g. [3], [4], [8]). For B_{mn}, we will also prove a Voronovskaya type theorem for an arbitrary continuous function by a function having all continuous partial derivatives up to order two in B_{mn}. Moreover, we state an r th oder generalization of B_{mn}. It is known that r th order generalization of linear positive operators of functions with one variable were introduced in [7], [9]. Finally, we study the approximation properties of this generalization and establish the rates of convergence.

2. Approximation Properties of B_{mn}

In this section, we give some approximation properties of B_{mn} on I^2.

Lemma 1. *For all $m, n \in N$, we have*

$$\begin{aligned} B_{mn}(1;x,y) &= 1, \\ B_{mn}(\theta;x,y) &= x, \\ B_{mn}(s;x,y) &= y + \frac{1}{2n}, \\ B_{mn}(\theta^2+s^2;x,y) &= \frac{mx^2+x}{m+1} + y^2 + \frac{y(1-y)}{n} + \frac{y}{n} + \frac{1}{3n^2}. \end{aligned}$$

Using Lemma 1 we have the following theorem for the convergence of the operators B_{mn}.

Theorem 1. *Let $f(x,y) \in C(I^2)$, $(x,y) \in I^2$, then $B_{mn}(f;x,y)$ converges to $f(x,y)$ uniformly on I^2 i. e.:*

$$\lim_{m,n\to\infty} \|B_{mn}(f;x,y) - f(x,y)\|_{C(I^2)} = 0.$$

Proof is clear from Lemma 1 and the well known Volkov's theorem [10]. Now, we give the following Lemmas which we shall use.

Lemma 2. *For all $m, n \in N$, we have*

$$\begin{aligned}
B_{mn}(\theta s; x, y) &= x\left(y + \frac{1}{2n}\right), \\
B_{mn}\left(\theta^2; x, y\right) &= \frac{mx^2 + x}{m+1}, \\
B_{mn}\left(\theta^3; x, y\right) &= \frac{(mx+2)(mx+1)mx}{(m+2)(m+1)m}, \\
B_{mn}\left(\theta^4; x, y\right) &= \frac{(mx+3)(mx+2)(mx+1)mx}{(m+3)(m+2)(m+1)m}, \\
B_{mn}\left(s^2; x, y\right) &= y^2 + \frac{y(1-y)}{n} + \frac{y}{n} + \frac{1}{3n^2}, \\
B_{mn}\left(s^3; x, y\right) &= \frac{4n(n-2)(n-1)y^3 + 18n(n-1)y^2 + 12ny + 1}{4n^3}, \\
B_{mn}\left(s^4; x, y\right) &= \frac{(n-3)(n-2)(n-1)y^4 + 8(n-2)(n-1)y^3}{n^3} \\
&\quad + \frac{15(n-1)y^2 + 5y}{n^3} + \frac{1}{5n^4}.
\end{aligned}$$

Lemma 3. *For operators B_{mn} we have*

$$\begin{aligned}
B_{mn}\left((s-y)^2; x, y\right) &= \frac{y(1-y)}{n} + \frac{1}{3n^2}, \\
B_{mn}\left((\theta-x)^2; x, y\right) &= \frac{x(1-x)}{m+1}, \\
B_{mn}((\theta-x)(s-y); x, y) &= 0, \\
B_{mn}\left((\theta-x)^4; x, y\right) &= \frac{3(m-6)x^4 + 6(6-m)x^3 + 3(m-8)x^2 + 6x}{(m+3)(m+2)(m+1)}, \\
B_{mn}\left((s-y)^4; x, y\right) &= \frac{3(n-2)y^4 + (16-6n)y^3 + 5(n-3)y^2 + 4y}{n^3} \\
&\quad + \frac{1}{5n^4}.
\end{aligned}$$

In the sequel, we take the operators B_{mn}, as follows:

$$B_{mn}(f; x, y) = n\sum_{k=0}^{n} P_n(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_0^1 \psi_{m,x}(\theta) f(\theta, s)\, d\theta ds, \tag{1}$$

where

$$P_n(y) = \binom{n}{k} y^k (1-y)^{n-k} \tag{2}$$

and

$$\psi_{m,x}(\theta) = \frac{\theta^{mx-1}(1-\theta)^{m(1-x)-1}}{B(mx, m(1-x))} \tag{3}$$

for the sake of shortness.

3. Rates of convergence

In this section, we study the rates of convergence in Theorem 1 by means of total modulus of continuity and elements of Lipschitz class.

Let $f \in C(I^2)$. The total modulus of continuity of f, denoted by $w(f;\delta)$, is defined by

$$w(f;\delta) = \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\le\delta} |f(x_1,y_1) - f(x_2,y_2)|.$$

Morever, partial modulus of continuity with respect to x and y are given by

$$w^{(1)}(f;\delta) = \max_{0\le y\le 1}\max_{|x_1-x_2|\le\delta} |f(x_1,y) - f(x_2,y)|,$$

and

$$w^{(2)}(f;\delta) = \max_{0\le x\le 1}\max_{|y_1-y_2|\le\delta} |f(x,y_1) - f(x,y_2)|,$$

respectively. It is known that a necessary and sufficient condition for a function f to be in $C(I^2)$ is

$$\lim_{\delta\to 0} w(f;\delta) = 0.$$

We shall use the following property of the total modulus of continuity:

$$w(f;\lambda\delta) \le (1+[\lambda])\, w(f;\delta)$$

for any $\lambda \in \mathbb{R}$, here $[\lambda]$ is the greatest integer that does not exceed λ (the same properties also hold for partial modulus of continuity), and

$$|f(\theta,s) - f(x,y)| \le w(f;\delta)\left(1 + \frac{\sqrt{(\theta-x)^2+(s-y)^2}}{\delta}\right). \tag{4}$$

The next result gives the rates of convergence of the sequence $\{B_{mn}(f;x,y)\}$, $f \in C(I^2)$, in Theorem 1 by means of the total modulus of continuity.

Theorem 2. *For all $f \in C(I^2)$, we have*

$$\|B_{mn}(f;x,y) - f(x,y)\|_{C(I^2)} \le \frac{3}{2} w(f;\delta_{mn}), \tag{5}$$

where $\delta_{mn} = \sqrt{\frac{1}{m+1} + \frac{3n+4}{3n^2}}$.

Proof. Let $f \in C\left(I^2\right)$. Since B_{mn} are linear and monotone from (4) then we get that

$$
\begin{aligned}
(6)\qquad \left|B_{mn}\left(f;x,y\right)-f\left(x,y\right)\right| &\leq B_{mn}\left(\left|f\left(\theta,s\right)-f\left(x,y\right)\right|;x,y\right)\\
&\leq w\left(f;\delta_{mn}\right)B_{mn}\left(1+\frac{\sqrt{\left(\theta-x\right)^2+\left(s-y\right)^2}}{\delta_{mn}};x,y\right)\\
&= w\left(f;\delta_{mn}\right)\Bigg\{1+\frac{n}{\delta_{mn}}\sum_{k=0}^{n}P_n\left(y\right)\\
&\quad\times\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_0^1\psi_{m,x}\left(\theta\right)\sqrt{\left(\theta-x\right)^2+\left(s-y\right)^2}d\theta ds\Bigg\}.
\end{aligned}
$$

Applying Hölder's inequality to the inner integral in (6), then (6) turns into the following from,

$$
\begin{aligned}
B_{mn}\left|\left(f;x,y\right)-f\left(x,y\right)\right| &\leq w\left(f;\delta_{mn}\right)\Bigg\{1+\frac{n}{\delta_{mn}}\sum_{k=0}^{n}P_n\left(y\right)\\
\times\int_{\frac{k}{n}}^{\frac{k+1}{n}}&\left\{\int_0^1\left[\left(\theta-x\right)^2+\left(s-y\right)^2\right]\psi_{m,x}\left(\theta\right)d\theta\right\}^{\frac{1}{2}}\left\{\int_0^1\psi_{m,x}\left(\theta\right)d\theta\right\}^{\frac{1}{2}}ds\Bigg\}.
\end{aligned}
$$

Again applying Hölder's inequality to the second (outer) integral, then the last inequality takes the from

$$
\begin{aligned}
\left|B_{mn}\left(f;x,y\right)-f\left(x,y\right)\right| &\leq w\left(f;\delta_{mn}\right)\Bigg\{1+\frac{n}{\delta_{mn}}\sum_{k=0}^{n}P_n\left(y\right)\\
\times&\left[\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_0^1\left[\left(\theta-x\right)^2+\left(s-y\right)^2\right]\psi_{m,x}\left(\theta\right)d\theta ds\right]^{\frac{1}{2}}\left[\int_{\frac{k}{n}}^{\frac{k+1}{n}}ds\right]^{\frac{1}{2}}\Bigg\}.
\end{aligned}
$$

Finally, applying the Cauchy-Schwarz-Bunyakowsky inequality to the summation in the last inequality, then we obtain the following:

$$
\begin{aligned}
\left|B_{mn}\left(f;x,y\right)-f\left(x,y\right)\right| &\leq w\left(f;\delta_{mn}\right)\Bigg\{1+\frac{n}{\delta_{mn}}\\
\times&\left[\frac{1}{n}\sum_{k=0}^{n}P_n\left(y\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_0^1\left[\left(\theta-x\right)^2+\left(s-y\right)^2\right]\psi_{m,x}\left(\theta\right)d\theta ds\right]^{\frac{1}{2}}\Bigg\},
\end{aligned}
$$

here $P_n(y)$ and $\psi_{m,x}(\theta)$ are defined by (2) and (3), respectively. Using Lemma 1 and Lemma 2 and taking maximum over I^2, the desired result is obtained in (5). ■

We have the following theorem for the partial modulus of continuities.

Theorem 3. *Let f be continuous on I^2 and bounded on R^2, then we have*

$$|B_{mn}(f;x,y) - f(x,y)| \leq \frac{3}{2}\left\{w^{(1)}(f;\delta_m) + w^{(2)}(f;\delta_n)\right\},$$

where $\delta_m = \frac{1}{\sqrt{m+1}}$, $\delta_n = \sqrt{\frac{3n+4}{3n^2}}$ and $w^{(1)}$, $w^{(2)}$ are the partial modulus of continuity with respect to x and y, respectively.

Proof. For proof, we can use similarly way in Theorem 2. *Let $f(x,y)$ be continuous on I^2* and bounded on R^2, then we get that following:

$$\begin{aligned}
|B_{mn}(f;x,y) - f(x,y)| &\leq B_{mn}(|f(\theta,s) - f(x,y)|;x,y) \\
&= B_{mn}(|f(\theta,s) - f(\theta,y) + f(\theta,y) - f(x,y)|;x,y) \\
&\leq B_{mn}(|f(\theta,s) - f(\theta,y)|;x,y) + B_{mn}(|f(\theta,y) - f(x,y)|;x,y) \\
&\leq w^{(2)}(f;\delta_n)B_{mn}\left(1 + \frac{|s-y|}{\delta_n};x,y\right) \\
&\quad + w^{(1)}(f;\delta_m)B_{mn}\left(1 + \frac{|\theta-x|}{\delta_m};x,y\right) \\
&= \frac{3}{2}\left\{w^{(1)}(f;\delta_m) + w^{(2)}(f;\delta_n)\right\}.
\end{aligned}$$

Thus, the proof of Theorem 3 is finished. ■

Now, we will investigate the rates of convergence of B_{mn} by means of the Lipschitz class $Lip_M(\gamma)$ for $0 < \gamma \leq 1$. Recapulate that $Lip_M(\gamma)$ is given by

$$|f(x_1,y_1) - f(x_2,y_2)| \leq M\left[(x_1-x_2)^2 + (y_1-y_2)^2\right]^{\frac{\gamma}{2}}, \tag{7}$$

here (x_1,y_1), $(x_2,y_2) \in I^2$ and $M > 0$.

Theorem 4. *Let $f \in Lip_M(\gamma)$, $0 < \gamma \leq 1$, then we have*

$$\|B_{mn}(f;x,y) - f(x,y)\|_{C(I^2)} \leq \frac{M}{2^\gamma}\delta_{mn}^\gamma, \tag{8}$$

where δ_{mn} is given by $\delta_{mn} = \sqrt{\frac{1}{m+1} + \frac{3n+4}{3n^2}} = \sqrt{\delta_m^2 + \delta_n^2}$, which is the same in Theorem 2, and δ_m, δ_n are the same as in Theorem 3.

Proof. Let $f \in Lip_M(\gamma)$, $0 < \gamma \leq 1$. Using linearicity and monotonicity of B_{mn}, we get that

$$|B_{mn}(f;x,y) - f(x,y)| \leq B_{mn}(|f(\theta,s) - f(x,y)|;x,y) \tag{9}$$
$$\leq Mn\sum_{k=0}^{n} P_n(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_0^1 \psi_{m,x}(\theta)\left[(\theta-x)^2 + (s-y)^2\right]^{\frac{\gamma}{2}} d\theta ds$$

by (7). Applying Hölder's inequality to the inner integral in (9), we obtain that

$$|B_{mn}(f;x,y) - f(x,y)| \leq Mn\sum_{k=0}^{n} P_n(y)$$
$$\times \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left\{\int_0^1 \psi_{m,x}(\theta)\left[(\theta-x)^2 + (s-y)^2\right] d\theta\right\}^{\frac{\gamma}{2}} \left\{\int_0^1 \psi_{m,x}(\theta)\,d\theta\right\}^{\frac{2-\gamma}{2}} ds$$
$$= Mn\sum_{k=0}^{n} P_n(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left\{\int_0^1 \psi_{m,x}(\theta)\left[(\theta-x)^2 + (s-y)^2\right] d\theta\right\}^{\frac{\gamma}{2}} ds.$$

Again, to the outher integral is applied by Hölder's inequality, then the last inequality takes the following form:

$$|B_{mn}(f;x,y) - f(x,y)|$$
$$\leq Mn\sum_{k=0}^{n} P_n(y) \left\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_0^1 \psi_{m,x}(\theta)\left[(\theta-x)^2 + (s-y)^2\right] d\theta ds\right\}^{\frac{\gamma}{2}} \left\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} ds\right\}^{\frac{2-\gamma}{2}}$$
$$= Mn\sum_{k=0}^{n} P_n(y) \left\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_0^1 \psi_{m,x}(\theta)\left[(\theta-x)^2 + (s-y)^2\right] d\theta ds\right\}^{\frac{\gamma}{2}} \left\{\frac{1}{n}\right\}^{\frac{2-\gamma}{2}}.$$

Now, applying Hölder's inequality for the sum in the last inequality we reach to the following:

$$|B_{mn}(f;x,y) - f(x,y)| \leq M\left\{n\sum_{k=0}^{n} P_n(y)\,\frac{1}{n}\right\}^{\frac{2-\gamma}{2}}$$
$$\times \left\{n\sum_{k=0}^{n} P_n(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_0^1 \psi_{m,x}(\theta)\left[(\theta-x)^2 + (s-y)^2\right] d\theta ds\right\}^{\frac{\gamma}{2}}.$$

Using Lemma 1, Lemma 2 and taking maximum over I^2, (8) is obtained easily, which completes the proof. ∎

4. A Voronovskaya type theorem

In this section, we will give a Voronovskaya type Theorem in $C^2\left(I^2\right)$ for operators B_{mn} for $m=n$.

Let C^2 denote the space of all functions f having all continuous partial derivatives up to order 2 exist, i.e.:

$$C^2 := \left\{f \in C : f_{x^i y^j} \text{ exist for } 0 \le i,j \le 2 \right. \\ \left. \text{and } 0 \le i+j \le 2 \text{ is continuous}\right\},$$

where $f_{x^i y^j} := \frac{\partial^2 f(x,y)}{\partial x^i \partial y^j}$.

Firstly, we need the following lemma.

Lemma 4. *Let $(x,y) \in I^2$. Then, we get*

$$\lim_{m\to\infty} m^2 B_{mn}\left((\theta-x)^4;x,y\right) = 3x^4-6x^3+3x^2, \tag{10}$$

$$\lim_{n\to\infty} n^2 B_{mn}\left((s-y)^4;x,y\right) = 3y^4-6y^3+5y^2. \tag{11}$$

Proof. By Lemma 3 and the linearity of B_{mn}, we may write that

$$\begin{aligned} & m^2 B_{mn}\left((\theta-x)^4;x,y\right) \\ &\quad = m^2\left[\frac{3(m-6)x^4+6(6-m)x^3+3(m-8)x^2+6x}{(m+3)(m+2)(m+1)}\right] \\ & n^2 B_{mn}\left((s-y)^4;x,y\right) \\ &\quad = n^2\left[\frac{3(n-2)y^4+(16-6n)y^3+5(n-3)y^2+4y}{n^3}+\frac{1}{5n^4}\right]. \end{aligned}$$

Taking limit as $m\to\infty$, $n\to\infty$ respectively, then, the proof is complete. ∎

Theorem 5. *For every $f \in C^2\left(I^2\right)$ and $(x,y) \in I^2$, we have*

$$\begin{aligned} & \lim_{n\to\infty} n\left\{B_{nn}(f;x,y)-f(x,y)\right\} \\ &\qquad = \frac{1}{2}\left\{f_y + x(1-x)f_{xx} + y(1-y)f_{yy}\right\}. \end{aligned} \tag{12}$$

Proof. Let $(x,y) \in I^2$ and $f_x, f_y, f_{xx}, f_{xy}, f_{yy} \in C^2(I)$. Define the function Φ as follows:

$$\Phi_{(x,y)}(\theta,s) = \begin{cases} \frac{f(\theta,s)-f(x,y)-(\theta-x)f_x-(s-y)f_y}{\sqrt{(\theta-x)^4+(s-y)^4}} \\ \quad - \frac{\frac{1}{2}\left\{(\theta-x)^2 f_{xx}+2(\theta-x)(s-y)f_{xy}+(s-y)^2 f_{yy}\right\}}{\sqrt{(\theta-x)^4+(s-y)^4}}, & (\theta,s) \neq (x,y) \\ 0, & (\theta,s) = (x,y). \end{cases}$$

Then, by assumption we have $\Phi_{(x,y)}(x,y) = 0$ and the function $\Phi_{(x,y)}(\cdot,\cdot) \in C^2(I^2)$. Hence, by the Taylor formula for $f \in C^2(I^2)$, we get

$$\begin{aligned} (13)\quad f(\theta,s) &= f(x,y) + (\theta-x) f_x + (s-y) f_y \\ &+ \frac{1}{2}\left\{(\theta-x)^2 f_{xx} + 2(\theta-x)(s-y) f_{xy} + (s-y)^2 f_{yy}\right\} \\ &+ \Phi_{(x,y)}(\theta,s)\sqrt{(\theta-x)^4+(s-y)^4}. \end{aligned}$$

We apply the linear operators nB_{nn} to (13) and using Lemma 2, we have

$$\begin{aligned} (14)\qquad & n\left\{B_{nn}(f;x,y) - f(x,y)\right\} \\ &= \frac{1}{2}\left\{f_y + \frac{nx(1-x)}{n+1} f_{xx} + \left[y(1-y) + \frac{1}{3n}\right] f_{yy}\right\} \\ &+ nB_{nn}\left(\Phi_{(x,y)}(\theta,s)\sqrt{(\theta-x)^4+(s-y)^4}; x,y\right). \end{aligned}$$

If we apply the Cauchy-Schwarz inequality for the second term on the right-hand side of (14) then we conclude that

$$\begin{aligned} (15)\quad & n\left|B_{nn}\left(\Phi_{(x,y)}(\theta,s)\sqrt{(\theta-x)^4+(s-y)^4}; x,y\right)\right| \\ &\le \left[B_{nn}\left(\Phi^2_{(x,y)}(\theta,s); x,y\right)\right]^{\frac{1}{2}} \\ &\quad \times \left[n^2 B_{nn}\left((\theta-x)^4 + (s-y)^4; x,y\right)\right]^{\frac{1}{2}} \\ &= \left[B_{nn}\left(\Phi^2_{(x,y)}(\theta,s); x,y\right)\right]^{\frac{1}{2}} \\ &\quad \times \left[n^2 B_{nn}\left((\theta-x)^4; x,y\right) + n^2 B_{nn}\left((s-y)^4; x,y\right)\right]^{\frac{1}{2}}. \end{aligned}$$

Let $\varphi_{(x,y)}(\theta,s) = \Phi^2_{(x,y)}(\theta,s)$. In this case, observe that $\varphi_{(x,y)}(x,y) = 0$ and $\varphi_{(x,y)}(\cdot,\cdot) \in C^2(I^2)$. From Theorem 1,

$$\begin{aligned} (16)\qquad \lim_{n\to\infty} B_{nn}\left(\Phi^2_{(x,y)}(\theta,s); x,y\right) &= \lim_{n\to\infty} B_{nn}\left(\varphi_{(x,y)}(\theta,s); x,y\right) \\ &= \varphi_{(x,y)}(x,y) = 0. \end{aligned}$$

Using (16) and Lemma 4, we have from (15)

$$\lim_{n\to\infty} nB_{nn}\left(\Phi_{(x,y)}(\theta,s)\sqrt{(\theta-x)^4+(s-y)^4};x,y\right)=0. \tag{17}$$

On the other hand, observe that

$$\lim_{n\to\infty}\frac{nx(1-x)}{n+1}=x(1-x), \quad \lim_{n\to\infty}\frac{1}{3n}=0. \tag{18}$$

Then, taking limit as $n\to\infty$ in (14) and using (17) and (18) we have (12). So the proof is completed. ■

5. A Generalization of order r of B_{mn}

Let $C^r(I^2)$, $r\in\mathbb{N}\cup\{0\}$, denote the space of all functions f having all continuous partial derivatives up to order r exist at $(x,y)\in I^2$. Let $B_{mn}^{[r]}$ denote the following generalization of B_{mn}.

$$B_{mn}^{[r]}(f;x,y)=n\sum_{k=0}^{n}P_n(y)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_0^1\psi_{m,x}(\theta)P_{r,(\theta,s)}(x-\theta,y-s)\,d\theta ds, \tag{19}$$

where

$$P_{r,(\theta,s)}(x-\theta,y-s)=\sum_{h=0}^{r}\sum_{i+j=h}\frac{1}{h!}\binom{h}{j}f_{x^iy^j}(\theta,s)(x-\theta)^i(y-s)^j, \tag{20}$$

and by the subscripts appeared on $f_{x^iy^j}$, we denote to write the partial derivatives of f, i.e.: $f_{x^iy^j}:=\frac{\partial^r f(x,y)}{\partial x^i\partial y^j}$; $r=i+j$. Now let us write

$$(x-\theta,y-s)=u(\alpha,\beta), \tag{21}$$

where (α,β) is a unit vector, $u>0$ and let us write

$$F(u)=f(\theta+u\alpha,s+u\beta)=f(\theta+(x-\theta),s+(y-s)). \tag{22}$$

Clearly Taylor's formula for $F(u)$ at $u=0$ turns into Taylor's formula for $f(x,y)$ at (θ,s). Morever, $r-th$ derivative takes the form (see [2])

$$F^{[r]}(u)=\sum_{i+j=r}\binom{r}{j}f_{x^iy^j}(\theta+u\alpha,s+u\beta)\alpha^i\beta^j,\ r\in\mathbb{N}. \tag{23}$$

Theorem 6. *Let* $f \in C^r\left(I^2\right)$ *and* $F^{[r]}(u) \in Lip_M(\gamma)$, *then the following inequality*

$$\left\| B_{mn}^{[r]}(f;x,y) - f(x,y) \right\|_{C(I^2)} \leq \frac{\gamma M B(\gamma,r)}{(\gamma+r)(r-1)!} \left\| B_{mn}^{[r]}\left(|(x,y)-(\theta,s)|^{r+\gamma}; x,y \right) \right\|_{C(I^2)} \tag{24}$$

holds, where $F^{[r]}(u)$ *is given by (23),* $B(\gamma,r)$ *is the familiar Beta function,* $0 < \gamma \leq 1$ *and* $M > 0$.

Proof. From (19) and (20), we have

$$f(x,y) - B_{mn}^{[r]}(f;x,y) = n \sum_{k=0}^{n} P_n(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_0^1 \psi_{m,x}(\theta) \left\{ f(x,y) - P_{r,(\theta,s)}(x-\theta, y-s) \right\} d\theta ds. \tag{25}$$

Using the integral form of the remainder term appeared in the previous formula, we obtain that

$$\begin{aligned} f(x,y) - P_{r,(\theta,s)}(x-\theta,y-s) &= \frac{1}{(r-1)!} \int_0^1 \sum_{i+j=h} \frac{1}{h!} \binom{h}{j} (x-\theta)^i (y-s)^j \\ &\quad \times f_{x^i y^j}(\theta + t(x-\theta), s + t(y-s))(1-t)^{r-1} dt. \end{aligned} \tag{26}$$

(26) turns into the following form:

$$F(u) - \sum_{h=0}^{r} \frac{1}{h!} F^{(h)}(0) u^h = \frac{u^r}{(r-1)!} \int_0^1 \left[F^{(r)}(tu) - F^{(r)}(0) \right] (1-t)^{r-1} dt \tag{27}$$

by using (21)-(23).

Taking (21), (26) and (27) into account and considering the fact that $F^{[r]}(u) \in Lip_M(\gamma)$, then it follows that

$$\left| f(x,y) - P_{r,(\theta,s)}(x-\theta,y-s) \right| = \left| F(u) - \sum_{h=0}^{r} \frac{1}{h!} F^{(h)}(0) u^h \right| \tag{28}$$

$$\leq \frac{|u|^r}{(r-1)!}\int_0^1 \left[F^{(r)}(tu)-F^{(r)}(0)\right](1-t)^{r-1}\,dt$$
$$\leq \frac{|u|^{r+\gamma}}{(r-1)!}MB(\gamma+1,r)$$
$$\leq \frac{M}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)\,|u|^{r+\gamma}$$
$$\leq \frac{M}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)\,|x-\theta,y-s|^{r+\gamma}.$$

Hence combining (25) and (28) we obtain (24), which completes the proof. ∎

Now, define the function $g\in C\left(I^2\right)$ as

$$g(\theta,s)=|(x,y)-(\theta,s)|^{r+\gamma}.$$

It is clear that $g(x,y)=0$. From Theorem 1 we get that

$$\|B_{mn}(g;x,y)\|_{C(I^2)}\to 0 \quad \text{as} \quad m,n\to\infty.$$

From (24), we arrive at the following approximation:

$$\left\|B_{mn}^{[r]}(f;x,y)-f(x,y)\right\|_{C(I^2)}\to 0 \ \text{ as } \ m,n\to\infty.$$

Taking Theorem 2 and Theorem 4 into consideration the following results can be obtained from Theorem 6.

Corollary 1. *Under the conditions of Theorem 6, it follows that*

$$\left\|B_{mn}^{[r]}(f;x,y)-f(x,y)\right\|_{C(I^2)}\leq \frac{M}{(r-1)!}\frac{\gamma}{(\gamma+r)}B(\gamma,r)\,\frac{3}{2}w(g;\delta_{mn}),$$

here $\delta_{mn}=\sqrt{\frac{1}{m+1}+\frac{3n+4}{3n^2}}$ *which is the same as in Theorem 2.*

Corollary 2. *Under the conditions of Theorem 6 and assuming that* $g(x,y)\in Lip_{(\sqrt{2})^r}(\gamma)$ *in Theorem 4, it follows that*

$$\left\|B_{mn}^{[r]}(f;x,y)-f(x,y)\right\|_{C(I^2)}\leq \frac{M}{(r-1)!}\frac{\gamma}{(\gamma+r)}B(\gamma,r)\,\frac{\left(\sqrt{2}\right)^r}{2^\gamma}\delta_{mn}^\gamma,$$

where $\delta_{mn}=\sqrt{\frac{1}{m+1}+\frac{3n+4}{3n^2}}$ *which is the same as in Theorem 2.*

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Ali Olgun
Department of Mathematics
Faculty of Science and Arts
Kirikkale University
71450, Yahşihan /Kirikkale, Turkey
e-mail: aliolgun71@gmail.com

Fatma Taşdelen
Department of Mathematics
Faculty of Science and
Ankara University
06100, Tandogan/Ankara, Turkey
e-mail: tasdelen@science.ankara.edu.tr

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