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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF FOURTH ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

Abstract. In this paper, Oscillatory and asymptotic behaviour of solutions of a class of nonlinear fourth order neutral differential equations with positive and negative coefficients of the form

$$(H) \qquad (r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0$$

and

$$(NH) \qquad (r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t)$$

are studied under the assumption

$$\int_0^\infty \frac{t}{r(t)} dt = \infty$$

for various ranges of $p(t)$. Using Schauder's fixed point theorem, sufficient conditions are obtained for the existence of bounded positive solutions of (NH).

Key words: functional differential equations, neutral, nonlinear, oscillation, positive and negative coefficients, asymptotic behavior.

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1. Introduction

Consider the fourth order nonlinear neutral delay differential equations of the form

$$(H) \qquad (r(t)(y(t) + p(t)y(t - \tau))'')'' + q(t)G(y(t - \alpha))$$
$$- h(t)H(y(t - \beta)) = 0$$

and its associated forced equations

$$(NH) \qquad (r(t)(y(t) + p(t)y(t - \tau))'')'' + q(t)G(y(t - \alpha))$$
$$- h(t)H(y(t - \beta)) = f(t),$$

where $r, q \in C([0, \infty), (0, \infty))$, $p \in C([0, \infty), \mathbb{R})$, $h \in C([0, \infty), [0, \infty))$, $f \in C([0, \infty), \mathbb{R})$, G and $H \in C(\mathbb{R}, \mathbb{R})$ with $uG(u) > 0$, $vH(v) > 0$, for $u, v \neq 0$, H is bounded, G is non decreasing, $\tau > 0$, $\alpha > 0$ and $\beta > 0$.

The objective of this work is to study oscillatory and asymptotic behavior of the functional differential equations (H) and (NH) under the assumption

$$(H_0) \qquad \qquad \int_0^\infty \frac{t}{r(t)} dt = \infty.$$

Because (H) and (NH) are highly nonlinear, it is interesting to study the both equations under (H_0). If $h(t) \equiv 0$, then (H) and (NH) reduce to

$$(1) \qquad (r(t)(y(t) + p(t)y(t - \tau))'')'' + q(t)G(y(t - \alpha)) = 0$$

and

$$(2) \qquad (r(t)(y(t) + p(t)y(t - \tau))'')'' + q(t)G(y(t - \alpha)) = f(t)$$

respectively.

In [9], Parhi and Tripathy have studied the oscillatory and asymptotic behaviour of solutions of (1) and (2) under the assumption (H_0). Their work showed that, if $q(t) < 0$, then it would be possible to obtain analogous results for oscillation and asymptotic behaviour of solutions of (1) and (2). The problem remains open as to what happens if $q(t)$ changes sign. In particular, if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$, then (1) and (2) can be viewed as

$$(3) \qquad (r(t)(y(t) + p(t)y(t - \tau))'')'' + q^+(t)G(y(t - \alpha))$$
$$- q^-(t)G(y(t - \alpha)) = 0$$

and

$$(4) \qquad (r(t)(y(t) + p(t)y(t - \tau))'')'' + q^+(t)G(y(t - \alpha))$$
$$- q^-(t)G(y(t - \alpha)) = f(t)$$

respectively. Clearly, (3) and (4) are particular case of (H) and (NH) respectively. Hence to enclose our prediction, the present work is devoted to study the more general equations of the type (H) and (NH) rather than (3) and (4). On the other hand, (1) and (2) are special cases of (H) and (NH) respectively and hence study of (H) and (NH) are more illustrative in view of (H_0).

Keeping in view of the above fact, the motivation of the present work has come from the work of Parhi and Tripathy [9]. Since last decade, the study of the behaviour of the solutions of functional differential and difference equations with positive and negative coefficients of first, second and higher order is a major area of research. Most of the work dealt with the existence of positive solutions of the functional equations. However, much attention has not given to oscillation results. This fact is well understood due to the technical difficulties arising in the analysis. We refer the reader to some of the works [1, 3-8, 10]. To the best of our knowledge there are no papers to date on forth order nonlinear differential equations with positive and negative coefficients. The results in this papers are new and generalize the earlier work of [9].

By a *solution* of (H) (or (NH)) we understand a function $y \in C([-\rho, \infty),$ $\mathbb{R})$ such that $(y(t)+p(t)y(t-\tau))$ is twice continuously differentiable, $(r(t)(y(t)$ $+p(t)y(t-\tau)))$ is twice continuously differentiable and (H) (or (NH)) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \alpha, \beta\}$, and $\sup\{|y(t)|; t \geq t_0\} > 0$ for every $t \geq t_0$. A solution of (H) (or (NH)) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

The organization of the paper is as follows. Section 2 deals with the oscillatory and asymptotic behaviour of solutions of (H) under the assumption (H_0) for all ranges of $p(t)$. Section 3, deals with the oscillatory and asymptotic behaviour of solutions of (NH). Using Schauder's fixed point theorem, sufficient conditions have been obtained for the existence of bounded positive solutions of (NH). Finally, Section 4 illustrate the examples to establish the validity of the results obtained in the earlier sections.

2. Oscillation Properties of (H)

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions or bounded solutions of (H) under the assumption (H_0). We need the following lemmas for our use in the sequel.

Lemma 1 ([9], Lemma 2.1). *Let (H_0) hold. Let u be a twice continuously differentiable function on $[0, \infty)$ such that $r(t)u''(t)$ is twice continuously differentiable and $(r(t)u''(t))'' \leq 0$ for large t. If $u(t) > 0$ ultimately, then one of the cases (a) or (b) holds for large t, and if $u(t) < 0$ ultimately, then one of the cases (b), (c), (d) or (e) holds for large t, where*

(a) $u'(t) > 0$, $u''(t) > 0$ and $(r(t)u''(t))' > 0$,
(b) $u'(t) > 0$, $u''(t) < 0$ and $(r(t)u''(t))' > 0$,
(c) $u'(t) < 0$, $u''(t) < 0$ and $(r(t)u''(t))' > 0$,
(d) $u'(t) < 0$, $u''(t) < 0$ and $(r(t)u''(t))' < 0$,
(e) $u'(t) < 0$, $u''(t) > 0$ and $(r(t)u''(t))' > 0$.

Lemma 2 ([9], Lemma 2.2). *Let the conditions of Lemma 1 hold. If $u(t) > 0$ ultimately, then $u(t) > R_T(t)(r(t)u''(t))'$ for $t \geq T \geq 0$, where $R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)} ds$.*

Remark 1. Notice that $R_T(t)$ is increasing function.

Lemma 3 ([3]). *Let F, G, $P : [t_0, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $F(t) = G(t) + P(t)G(t-c)$, for $t \geq t_0 + \max\{0, c\}$. Assume that there exists numbers $P_1, P_2, P_3, P_4 \in \mathbb{R}$ such that $P(t)$ is in one of the following ranges:*
   (i) $P_1 \leq P(t) \leq 0$,
   (ii) $0 \leq P(t) \leq P_2 < 1$,
   (iii) $1 < P_3 \leq P(t) \leq P_4$.
Suppose that $G(t) > 0$ for $t \geq t_0$, $\liminf_{t \to \infty} G(t) = 0$ and that $\lim_{t \to \infty} F(t) = L \in \mathbb{R}$ exists. Then $L = 0$.

Lemma 4 ([3]). *If $q \in C([0, \infty), [0, \infty))$ and*

$$\liminf_{t \to \infty} \int_{t-\tau}^t q(s)ds > \frac{1}{e},$$

then $x'(t) + q(t)x(t - \tau) \leq 0$, $t \geq 0$ cannot have an eventually positive solution.

The results in our paper will make use of the following conditions on the functions in equations (H) and (NH):

(H_1) $\int_0^\infty \frac{s}{r(s)} \int_s^\infty th(t)dt ds < \infty$;

(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u+v)$, $u > 0$, $v > 0$;

(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$ and $H(-u) = -H(u)$ for $u \in \mathbb{R}$;

(H_4) G is sublinear and $\int_0^c \frac{du}{G(u)} < \infty$ for all $c > 0$;

(H_5) $\int_\tau^\infty Q(t)dt = \infty$, $Q(t) = \min\{q(t), q(t-\tau)\}$ for $t \geq \tau$.

Theorem 1. *Assume that conditions $(H_0) - (H_5)$ hold, $\tau \leq \alpha$, and p_1, p_2 and p_3 are positive real numbers. If (i) $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$ holds, then every solution of (H) either oscillatory or converges to zero as $t \to \infty$.*

Proof. Assume that (H) has a nonoscillatory solution on $[t_0, \infty)$, $t_0 \geq 0$ and let it be $y(t)$. Hence, $y(t) > 0$ or < 0 for $t \geq t_0$. Suppose that $y(t) > 0$ for $t \geq t_0$. Define the functions

$$(5) \qquad z(t) = y(t) + p(t)y(t - \tau),$$

$$(6) \qquad k(t) = \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (\theta - s)h(\theta)H(y(\theta - \beta))d\theta ds.$$

Notice that condition (H_1) and the fact that H is bounded function implies that $k(t)$ exists for all t. Now if

$$(7) \qquad w(t) = z(t) - k(t) = y(t) + p(t)y(t - \tau) - k(t),$$

then a calculation shows

$$(8) \qquad (r(t)w''(t))'' = -q(t)G(y(t - \alpha)) \leq 0, \quad (\not\equiv 0)$$

for $t \geq t_0 + \rho$. Clearly, $w(t), w'(t), (r(t)w''(t)), (r(t)w''(t))'$ are monotonic functions on $[t_1, \infty), t_1 \geq t_0 + \rho$. In view of Lemma 1, we have two cases to consider, namely $w(t) > 0$ or $w(t) < 0$ for $t \geq t_1$. Suppose the former holds. By the Lemma 1, any one of the cases (a) or (b) holds. Using (H_2) and (H_3), gives

$$(9) \qquad \begin{aligned} 0 &= (r(t)w''(t))'' + q(t)G(y(t - \alpha)) \\ &\quad + G(p_1)(r(t - \tau)w''(t - \tau))'' \\ &\quad + G(p_1)q(t - \tau)G(y(t - \tau - \alpha)) \\ &\geq (r(t)w''(t))'' + G(p_1)(r(t - \tau)w''(t - \tau))'' \\ &\quad + \lambda Q(t)G(y(t - \alpha) + p_1 y(t - \alpha - \tau)) \\ &\geq (r(t)w''(t))'' + G(p_1)(r(t - \tau)w''(t - \tau))'' \\ &\quad + \lambda Q(t)G(z(t - \alpha)) \end{aligned}$$

for $t \geq t_2 > t_1$. From (6), it follows that $k(t) > 0$ and $k'(t) < 0$, and so $w(t) > 0$ for $t \geq t_1$ implies $w(t) < z(t)$ for $t \geq t_2$. Therefore, (9) yields

$$(10) \quad (r(t)w''(t))'' + G(p_1)(r(t - \tau)w''(t - \tau))'' + \lambda Q(t)G(w(t - \alpha)) \leq 0,$$

for $t \geq t_2$, that is

$$\begin{aligned} 0 &\geq (r(t)w''(t))'' + G(p_1)(r(t - \tau)w''(t - \tau))'' \\ &\quad + \lambda Q(t)G(R_T(t - \alpha)(r(t - \alpha)w''(t - \alpha))') \end{aligned}$$

due to Lemma 2, for $t \geq T + \rho > t_2$. Hence

$$
\begin{aligned}
0 \geq\ & (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' \\
& + \lambda Q(t)G(R_T(t-\alpha))G((r(t-\alpha)w''(t-\alpha))'),
\end{aligned}
$$

that is,

$$
\begin{aligned}
\lambda Q(t)G(R_T(t-\alpha)) \leq\ & -[G((r(t-\alpha)w''(t-\alpha))')]^{-1}(r(t)w''(t))'' \\
& - G(p_1)[G((r(t-\alpha)w''(t-\alpha))')]^{-1} \\
& \times (r(t-\tau)w''(t-\tau))'' \\
\leq\ & -[G((r(t)w''(t))')]^{-1}(r(t)w''(t))'' \\
& - G(p_1)[G((r(t-\tau)w''(t-\tau))')]^{-1} \\
& \times (r(t-\tau)w''(t-\tau))''.
\end{aligned}
$$

Since $\lim_{t\to\infty}(r(t)w''(t))' < \infty$, then using (H_4) the above inequality becomes

$$
\int_{T+\rho}^{\infty} Q(t)G(R_T(t-\alpha))dt < \infty,
$$

which contradicts (H_5) since $R_T(t)$ is monotonic increasing function.

Next, we suppose that $w(t) < 0$ for $t \geq t_1$. Then $z(t) - k(t) < 0$ implies $y(t) \leq z(t) = y(t) + p(t)y(t-\tau) < k(t)$. Thus, $y(t)$ is bounded since $k(t)$ is bounded and monotonic. By the Lemma 1, any one of the cases (b), (c), (d) or (e) holds.

Consider the case (b). Since $\lim_{t\to\infty} k(t)$ exists, $\lim_{t\to\infty} w(t)$ exists, and so $\lim_{t\to\infty} z(t)$ exists. Furthermore, $\lim_{t\to\infty}(r(t)w''(t))'$ exists, and an integration of (8) implies

$$
\int_{t_1}^{\infty} Q(t)G(y(t-\alpha))dt < \infty.
$$

Hence, it is easy to verify that $\liminf_{t\to\infty} y(t) = 0$ due to (H_5). It then follows from Lemma 3 that $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$ since $z(t) \geq y(t)$.

To see that cases (c) and (d) are not possible, first note that $w(t) < 0$, $y(t)$ is bounded, $\lim_{t\to\infty} k(t)$ exists and hence $\lim_{t\to\infty} w(t)$ exists. On the otherhand, integrating successively, $w''(t) < 0$ from t_1 to $t \geq t_1$, yields $\lim_{t\to\infty} w(t) = -\infty$, which is a contradiction.

Consider the case (e). In this case $r(t)w''(t)$ is nondecreasing on $[t_1, \infty)$. Hence for $t \geq t_1$, $r(t)w''(t) \geq r(t_1)w''(t_1)$, that is,

$$
(11) \qquad\qquad tw''(t) \geq \frac{t}{r(t)}r(t_1)w''(t_1).
$$

Integrating (11) from t_1 to t, we obtain

$$tw'(t) \geq w(t) - w(t_1) + t_1 w'(t_1) + r(t_1)w''(t_1) \int_{t_1}^t \frac{s}{r(s)} ds,$$

that is, $tw'(t) > 0$ for large t due to (H_0), a contradiction.

Finally, we suppose that $y(t) < 0$ for $t \geq t_0$. From (H_3), we note that $G(-u) = -G(u)$, $u \in \mathbb{R}$ and $H(-u) = -H(u)$, $u \in \mathbb{R}$. Indeed, $G(1)G(1) = G(1)$ and $G(-1)G(-1) = G(1)$ implies that $G(-1) = -1$ and $G(1) = 1$. Hence putting $x(t) = -y(t)$ for $t \geq t_0$, we obtain $x(t) > 0$ and

$$(r(t)(x(t) + p(t)x(t - \tau))'')'' + q(t)G(x(t - \alpha)) - h(t)H(x(t - \beta)) = 0.$$

Proceeding as above, we can show that every solution of (H) either oscillates or converges to zero as $t \to \infty$. This completes the proof of the theorem. ∎

The following corollary is immediate.

Corollary 1. *Under the conditions of Theorem 1, every unbounded solution of (H) oscillates.*

Theorem 2. *Let $0 \leq p(t) \leq p_1 < 1$. Assume that conditions (H_0), (H_1), (H_3), and*

(H_6) $\liminf\limits_{|x| \to 0} \frac{G(x)}{x} \geq \gamma > 0$;

(H_7) $\liminf\limits_{t \to \infty} \int_{t-\alpha}^t G(R_T(s - \alpha))q(s)ds > (e\gamma G(1 - p_1))^{-1}$;

and

(H_8) $\int_0^\infty q(t)dt = \infty$

hold, then every solution of (H) either oscillates or converges to zero as $t \to \infty$.

Remark 2. (H_7) implies that

(H_9) $$\int_{T+\alpha}^\infty G(R_T(s - \alpha))q(s)ds = \infty.$$

Indeed, if

$$\int_{T+\alpha}^\infty G(R_T(s - \alpha))q(s)ds = b < \infty,$$

then for $t > T + 2\alpha$,

$$\int_{t-\alpha}^t G(R_T(s - \alpha))q(s)ds = \left(\int_{T+\alpha}^t - \int_{T+\alpha}^{t-\alpha} \right) G(R_T(s - \alpha))q(s)ds.$$

implies that

$$\liminf_{t\to\infty} \int_{t-\alpha}^{t} G(R_T(s-\alpha))q(s)ds \le b-b = 0,$$

which contradicts (H_7).

Proof. (Theorem 2.) Let $y(t)$ be a nonoscillatory solution of (H) such that $y(t) > 0$ for $t \ge t_0$. The case $y(t) < 0$ for $t \ge t_0$ is similar. Using (5), (6) and (7) we obtain (8). In view of Lemma 1, we have two cases to consider, namely $w(t) > 0$ and $w(t) < 0$ for for $t \ge t_1 > t_0 + \rho$. Let $w(t) > 0$ on $[t_1, \infty)$. Then any one of the cases (a) or (b) of Lemma 1 holds. In each case, $w(t)$ is nondecreasing. We note that $k(t) > 0$ and $k'(t) < 0$. Hence

$$0 < w'(t) = z'(t) - k'(t),$$

implies that, $z'(t) > 0$ or $z'(t) < 0$ for $t \ge t_2 > t_1$. If $z'(t) > 0$, then $z(t)$ is nondecreasing and

$$
\begin{aligned}
(1 - p(t))z(t) &< z(t) - p(t)z(t-\tau) \\
&= y(t) - p(t)p(t-\tau)y(t-2\tau) < y(t)
\end{aligned}
$$

for $t \ge t_2$, that is,

$$y(t) > (1 - p_1)z(t) > (1 - p_1)w(t).$$

Thus (8) yields

$$G((1 - p_1)w(t-\alpha))q(t) \le -(r(t)w''(t))''.$$

By Lemma 2 and (H_3), the above inequality becomes

$$
\begin{aligned}
(12) \qquad G(1 - p_1)q(t)G(R_T(t-\alpha))G((r(t-\alpha)w''(t-\alpha))') \\
\le -(r(t)w''(t))''
\end{aligned}
$$

for $t \ge T + \alpha > t_2$. Let $\lim_{t\to\infty}(r(t)w''(t))' = c$, $c \in [0, \infty)$. If $0 < c < \infty$, then there exists $c_1 > 0$ such that $(r(t)w''(t))' > c_1$ for $t \ge t_3 > T + \alpha$. From (12), it follows that

$$G(1 - p_1)q(t)G(R_T(t-\alpha))G(c_1) \le -(r(t)w''(t))''$$

for $t \ge t_4 > t_3 + \alpha$. Integrating the above inequality from t_4 to ∞, we get

$$\int_{t_4}^{\infty} q(t)G(R_T(t-\alpha))dt < \infty,$$

a contradiction to (H_9). Hence $c = 0$. Consequently, (H_6) implies that $G((r(t)w''(t))') \geq \gamma(r(t)w''(t))'$ for $t \geq t_3$. Therefore, (12) yields

$$(r(t)w''(t))'' + \gamma G(1 - p_1)q(t)G(R_T(t - \alpha))(r(t - \alpha)w''(t - \alpha))' \leq 0,$$

for $t \geq t_3 + \alpha$. From Lemma 4, it follows that

$$u'(t) + \gamma G(1 - p_1)q(t)G(R_T(t - \alpha))u(t - \alpha) \leq 0$$

admits a positive solution $(r(t)w''(t))'$, which is a contradiction due to (H_7).

If $z'(t) < 0$, then $\lim_{t \to \infty} z(t)$ exists. Using the same type of reasoning as in Theorem 1, it is easy to verify that $\liminf_{t \to \infty} y(t) = 0$ by using (H_8). It then follows from Lemma 3 that $\lim_{t \to \infty} z(t) = 0$. Thus, $\lim_{t \to \infty} y(t) = 0$.

The remaining part of the proof follows from the proof of the Theorem 1. Hence the proof of the theorem is completed. ∎

Corollary 2. *Under the conditions of Theorem 2, every unbounded solutions of (H) oscillates.*

Theorem 3. *Assume that conditions $(H_0) - (H_3)$, (H_5), $\tau \leq \alpha$ hold, and*

(H_{10}) $\frac{G(x_1)}{x_1^\sigma} \geq \frac{G(x_2)}{x_2^\sigma}$ *for* $x_1 \geq x_2 > 0$ *and* $\sigma \geq 1$.

If (i) $0 \leq p(t) \leq p_1 < 1$ or (ii) $1 < p_2 \leq p(t) \leq p_3 < \infty$ holds, then every solution of (H) is either oscillatory or tends to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 1, we obtain

$$(13) \qquad (r(t)w''(t))'' + G(p_1)(r(t - \tau)w''(t - \tau))''$$
$$+ \lambda Q(t)G(z(t - \alpha)) \leq 0$$

for $t \geq t_2$. In view of (8) and Lemma 1, $w(t)$ is nondecreasing, there exists $k > 0$ and $t_3 > 0$ such that $w(t) > k$ for $t \geq t_3$. Hence use of (H_{10}) along with Lemma 2, we obtain

$$G(w(t - \alpha)) = (G(w(t - \alpha))/w^\sigma(t - \alpha))w^\sigma(t - \alpha)$$
$$\geq (G(k)/k^\sigma)(w^\sigma(t - \alpha))$$
$$> (G(k)/k^\sigma)R_T^\sigma(t - \alpha)((r(t - \alpha)w''(t - \alpha))')^\sigma$$

for $t \geq T + \alpha > t_3 + \alpha$. Thus (13) yields,

$$\lambda(G(k)/k^\sigma)R_T^\sigma(t - \alpha)Q(t)((r(t - \alpha)w''(t - \alpha))')^\sigma$$
$$< \lambda Q(t)G(w(t - \alpha)) \leq \lambda Q(t)G(z(t - \alpha))$$
$$\leq -(r(t)w''(t))'' - G(p_1)(r(t - \tau)w''(t - \tau))'',$$

that is,

$$
\begin{aligned}
\lambda(G(k)/k^\sigma)R_T^\sigma(t-\alpha)Q(t) < &-[(r(t-\alpha)w''(t-\alpha))']^{-\sigma}[(r(t)w''(t))'' \\
&+ G(p_1)(r(t-\tau)w''(t-\tau))''] \\
< &-((r(t)w''(t))')^{-\sigma}(r(t)w''(t))'' \\
&- G(p_1)((r(t-\tau)w''(t-\tau))')^{-\sigma}(r(t-\tau)w''(t-\tau))''
\end{aligned}
$$

for $t \geq T + \alpha$. Since $\lim_{t\to\infty}(r(t)w''(t))'$ exists and $R_T(t)$ is nondecreasing, then proceeding as in the proof of Theorem 1 we obtain

$$
\int_{T+\alpha}^{\infty} R_T^\sigma(t-\alpha)Q(t)dt < \infty,
$$

which contradict (H_5). The proof in case $w(t) < 0$ is same as in Theorem 1. Thus the theorem is proved. ∎

Corollary 3. *Under the conditions of Theorem 3, every unbounded solution of (H) oscillates.*

In our next theorem we are able to replace conditions (H_3) and (H_4) in Theorem 1 with conditions (H_{11}) and (H_{12}) below.

Theorem 4. *Assume that conditions $(H_0) - (H_2)$, (H_5), $\tau \leq \alpha$ hold, and*

(H_{11}) $G(u)G(v) \geq G(uv)$ *for* $u > 0$, $v > 0$;

(H_{12}) $G(-u) = -G(u)$, $H(-u) = -H(u)$, $u \in \mathbb{R}$.

If (i) $0 \leq p(t) < p_1 < 1$ or (ii) $1 < p_2 \leq p(t) \leq p_3 < \infty$ holds, then every solution of (H) either oscillates or converges to zero as $t \to \infty$.

Proof. Proceeding as in the proof of the Theorem 3, in case $w(t) > 0$ we again have (13) for $t \geq t_2$. Since $w(t)$ is nondecreasing, then there exist $k > 0$ and $t_3 > t_2$ such that $w(t) > k$ for $t \geq t_3$, that is, $z(t) \geq w(t) > k$ for $t \geq t_3$. Consequently, inequality (13) yields

$$
\lambda G(k) \int_{t_3}^{\infty} Q(t)dt < \infty,
$$

a contradiction to (H_5). The rest of the proof is similar to the Theorem 1. This completes the proof of the theorem. ∎

Corollary 4. *Under the conditions of Theorem 4, every unbounded solutions of (H) oscillates.*

Remark 3. In Theorems 1 and Corollary 1, G is sublinear only, whereas in Theorem 3 and Corollary 3, G is superlinear. But in Theorem 4, G could be linear, sublinear or superlinear.

Next, we consider the case where $p(t)$ is negative. Here p_4, p_5 and p_6 are negative and real numbers.

Theorem 5. *Let* $-1 < p_4 \leq p(t) \leq 0$ *and conditions* (H_0), (H_1), (H_3), (H_4), (H_8) *hold, then every solution of* (H) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H). Because of (H_3), without loss of generality we may suppose that $y(t) > 0$ for $t \geq t_0 > 0$. Setting as in (5), (6) and (7) we obtain (8) for $t \geq t_0 + \rho$. By Lemma 1, $w(t)$ is monotonic on $[t_1, \infty)$, $t_1 \geq t_0 + \rho$. If $w(t) > 0$ for $t \geq t_1$, then any one of the cases (a) or (b) of Lemma 1 holds. Consequently, $w(t) > R_T(t)(r(t)w''(t))'$ for $t \geq t_2 > t_1$ by Lemma 2. Moreover, $w(t) \leq y(t)$ since $p(t) \leq 0$ implies that $y(t) > R_T(t)(r(t)w''(t))'$ for $t \geq t_2$ and hence (8) becomes

$$\int_{t_2+\alpha}^{\infty} q(t)G(R_T(t-\alpha))dt < \infty,$$

which contradict (H_8) since G, and R_T are increasing functions. Hence, $w(t) < 0$ for $t \geq t_1$, and so any one of the cases (b), (c), (d) or (e) of Lemma 1 holds.

We claim that $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = \max\{y(t) : t_1 \leq t \leq \eta_n\}$. We may choose n large enough such that $\eta_n - \tau > t_1$. Hence

$$0 \geq w(\eta_n) \geq y(\eta_n) + p_4 y(\eta_n - \tau) - k(\eta_n) \geq (1 + p_4)y(\eta_n) - k(\eta_n).$$

Since $k(\eta_n)$ is bounded and $1 + p_4 > 0$, then $w(\eta_n) > 0$ for large n which is a contradiction. Thus, our claim holds.

The proof of the cases (c), (d), and (e) cannot hold are similar to the corresponding cases in the proof of Theorem 1. If (b) holds, then as in proof of Theorem 1 we obtain $\liminf_{t \to \infty} y(t) = 0$. Hence, $\lim_{t \to \infty} z(t) = 0$ by Lemma 3. Consequently,

$$\begin{aligned}
0 = \limsup_{t \to \infty} z(t) &\geq \limsup_{t \to \infty}(y(t) + p_4 y(t - \tau)) \\
&\geq \limsup_{t \to \infty} y(t) + \liminf_{t \to \infty}(p_4 y(t - \tau)) \\
&= \limsup_{t \to \infty} y(t) + p_4 \limsup_{t \to \infty} y(t - \tau) \\
&= (1 + p_4)\limsup_{t \to \infty} y(t).
\end{aligned}$$

Since $1 + p_4 > 0$, $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. ∎

Corollary 5. *Under the conditions of Theorem 5, every unbounded solution of (H) oscillates.*

Theorem 6. *Assume that conditions (H_0), (H_1), (H_3), (H_4), and (H_8) hold. If $-\infty < p_5 \leq p(t) < p_6 < -1$, then every bounded solution of (H) either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be bounded nonoscillatory solution of (H), on $[t_0, \infty)$, $t_0 \geq 0$. With (5), (6), and (7) as above, we obtain (8) for $t \geq t_0 + \rho$. Hence from Lemma 1, $w(t)$ is monotonic on $[t_1, \infty)$, $t_1 \geq t_0 + \rho$. If $w(t) > 0$ for $t \geq t_1$, then one of the cases (a) or (b) of Lemma 1 holds. Consequently, $w(t) > R_T(r(t)w''(t))'$ for $t \geq T > t_1$ by Lemma 2. Moreover, $w(t) > y(t)$. Choose $t_2 \in [T, \infty)$ such that $t - \alpha \geq T$ for all $t \in [t_2, \infty)$. Then, $y(t-\alpha) > R_T(t-\alpha)(r(t-\alpha)w''(t-\alpha))'$ for $t \geq t_2$, and (8) becomes

$$\int_{t_2}^{\infty} q(t)G(R_T(t-\alpha))dt < \infty,$$

which contradicts (H_8) since G, R_T are inceasing. Hence, $w(t) < 0$ for $t \geq t_1$, so one of the cases (b), (c), (d) or (e) of Lemma 1 holds.

In case (b), since $w(t) < 0$, $w'(t) > 0$, and $\lim_{t\to\infty} k(t)$ exists, we have $\lim_{t\to\infty} z(t)$ exists. Furthermore, $\lim_{t\to\infty}(r(t)w''(t))'$ exists. Integrating (8) from t_2 to t, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(t-\alpha))dt < \infty,$$

which implies that $\liminf_{t\to\infty} y(t) = 0 = \liminf_{t\to\infty} y(t-\alpha)$ due to (H_8). Hence, $\lim_{t\to\infty} z(t) = 0$ by Lemma 3. Therefore,

$$\begin{aligned}
0 = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(t-\alpha)) \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(t)y(t-\alpha)) \\
&= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(t-\alpha) \\
&= (1 + p_6)\limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1 + p_6) < 0$, we have $\limsup_{t\to\infty} y(t) \leq 0$, so $\lim_{t\to\infty} y(t) = 0$.

Cases (c) and (d) are not possible since $w(t) < 0$ for $t \geq t_1$, $y(t)$ is bounded, and $\lim_{t\to\infty} k(t)$ exists.

If Case (e) holds, we have $r(t)w''(t)$ is nondecreasing on $[t_1, \infty)$. Hence $t > t_2 \geq t_1$, $r(t)w''(t) \geq r(t_2)w''((t_2) > 0$, so

$$tw''(t) \geq r(t_2)w''(t)\frac{t}{r(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$tw'(t) \geq w(t) - w(t_2) + t_2w'(t_2) + r(t_2)w''(t_2) \int_{t_2}^{t} \frac{s}{r(s)}ds,$$

that is, $tw'(t) > 0$ for large t due to (H_0), is a contradiction. This completes the proof of theorem. ∎

3. Oscillation properties of (NH)

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (NH) with suitable forcing functions. Our attention is restricted to the forcing functions which are eventually change sign. We have the following hypotheses regarding $f(t)$.

(H_{13}) There exists $F \in C^2([0, \infty), \mathbb{R})$ such that $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty, rF'' \in C^2([0,\infty), \mathbb{R})$ and $(rF'')'' = f$.

(H_{14}) There exists $F \in C^2([0, \infty), \mathbb{R})$ such that $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = \infty, rF'' \in C^2([0,\infty), \mathbb{R})$ and $(rF'')'' = f$.

Theorem 7. *Let* $0 \leq p(t) \leq p_1 < \infty$. *Assume that* $(H_0) - (H_2)$, (H_{11}), (H_{12}), *and* (H_{14}) *hold. If*

$$(H_{15}) \qquad \limsup_{t\to\infty} \int_{\alpha}^{t} Q(s)G(F(s-\alpha))ds = +\infty \quad and$$

$$\liminf_{t\to\infty} \int_{\alpha}^{t} Q(s)G(F(s-\alpha))ds = -\infty,$$

then equation (NH) *is oscillatory.*

Proof. Let $y(t)$ be a non oscillatory solution of (NH) such that $y(t) > 0$ for $t \geq t_0 > 0$. Defining $z(t)$, $k(t)$, $w(t)$ as in (5), (6) and (7), respectively, equation (NH) becomes

$$(14) \qquad (r(t)w''(t))'' + q(t)G(y(t-\alpha)) = f(t).$$

Let

$$(15) \qquad v(t) = w(t) - F(t) = z(t) - k(t) - F(t).$$

Then, for $t \geq t_0 + \rho$, equation (NH) becomes

$$(16) \qquad (r(t)v''(t))'' = -q(t)G(y(t-\alpha)) \leq 0.$$

Thus, $v(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. Suppose $v(t) > 0$ for $t \geq t_1$ so that Case (a) or (b) of Lemma 1 holds. Then $z(t) > k(t) + F(t) > F(t)$ for $t \geq t_1$. Applying (H_2), and (H_{11}) yields

$$
\begin{aligned}
(17) \quad 0 &= (r(t)v''(t))'' + q(t)G(y(t-\alpha)) + G(p_1)(r(t-\tau)v''(t-\tau))'' \\
&\quad + G(p_1)q(t-\tau)G(y(t-\alpha-\tau)) \\
&\geq (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' \\
&\quad + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\
&\geq (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' \\
&\quad + \lambda Q(t)G(z(t-\alpha)) \\
&\geq (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' \\
&\quad + \lambda Q(t)G(F(t-\alpha))
\end{aligned}
$$

for $t \geq t_2 > t_1$. Integrating the inequality (17) from $t_2 + \alpha$ to t and taking $\limsup$ as $t \to \infty$, we get

$$\limsup_{t \to \infty} \int_{t_2+\alpha}^{t} Q(s)G(F(s-\alpha))ds < \infty,$$

which is a contradiction to (H_{15}).

Therefore, $v(t) < 0$ for $t \geq t_1$. Thus any one of the cases (b), (c), (d) or (e) of Lemma 1 holds. Since $-\infty \leq \lim_{t \to \infty} v(t) \leq 0$, then for each these cases $z(t) = v(t) + k(t) + F(t)$ implies that

$$
\begin{aligned}
\liminf_{t \to \infty} z(t) &= \liminf_{t \to \infty}[k(t) + v(t) + F(t)] \\
&\leq \limsup_{t \to \infty} k(t) + \liminf_{t \to \infty}[v(t) + F(t)] \\
&\leq \lim_{t \to \infty} k(t) + \limsup_{t \to \infty} v(t) + \liminf_{t \to \infty} F(t) \to -\infty,
\end{aligned}
$$

that is, $z(t) < 0$ for large t, a contradiction. This completes the proof of the theorem. ∎

Theorem 8. *Let $-1 < p(t) \leq 0$. Suppose that (H_0), (H_1), (H_{12}), and (H_{14}) hold. If*

$$(H_{16}) \qquad \limsup_{t \to \infty} \int_{\alpha}^{t} q(s)G(F(s-\alpha))ds = +\infty \quad and$$

$$\liminf_{t \to \infty} \int_{\alpha}^{t} q(s)G(F(s-\alpha))ds = -\infty,$$

then every bounded solution of (NH) oscillates.

Proof. Proceeding as in the proof of the Theorem 7, we obtain (16) for $t \geq t_1 \geq t_0 + \rho$. Thus, $v(t)$ is monotonic, so $v(t) > 0$ or $v(t) < 0$ for large t. If $v(t) > 0$ for $t \geq t_1$, then either case (a) or case (b) of Lemma 1 holds for $t \geq t_1$. Since $v(t)$ is monotonic, $z(t) > z(t) - k(t) > F(t)$ implies that $z(t) > F(t)$, so $y(t) > z(t) > F(t)$ for $t \geq t_1$. Choose $t_2 \in [t_1, \infty)$ such that $t - \alpha \geq t_2$ for all $t \in [t_2, \infty)$. Hence, for $t \geq t_2$, $y(t-\alpha) > z(t-\alpha) > F(t-\alpha)$. From (16), we have

$$q(t)G(F(t-\alpha)) \leq q(t)G(y(t-\alpha)) = -(r(t)v''(t))''$$

for $t \geq t_2$. An integration yields a contradiction to (H_{16}).

Now assume $v(t) < 0$ for $t \geq t_1$. Thus, $z(t) - k(t) < F(t)$, and condition (H_{14}) then implies that $\liminf_{t \to \infty} z(t) = -\infty$. This contradicts the fact that $y(t)$ is bounded and completes the proof of theorem. ∎

Theorem 9. *Assume that* $(H_0) - (H_2)$, $(H_{11}) - (H_{13})$, (H_{15}) *hold. If* $0 \leq p(t) \leq p_1 < \infty$ *holds, then every unbounded solution of* (NH) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (NH) such that $y(t) > 0$ for $t \geq t_0$. Using (5) - (7) and (14), we obtain, inequality (16) for $t \geq t_0 + \rho$. Thus, $v(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. First assume $v(t) > 0$ for all $t \geq t_1$. Proceeding as in the proof of the Theorem 7, we obtain contradiction. Hence, $v(t) < 0$ for $t \geq t_1$. From Lemma 1, it follows that any one of the cases (b), (c), (d) or (e) holds. In case (b), $\lim_{t \to \infty} v(t)$ exists and hence $z(t) = v(t) + k(t) + F(t)$, implies that

(18) $$y(t) \leq v(t) + k(t) + F(t).$$

That is, $y(t)$ is bounded, which is a contradiction. For each of the cases (c), (d) or (e), $v(t)$ is a nonincreasing function on $[t_1, \infty)$, so let $\lim_{t \to \infty} v(t) = l$, $l \in [-\infty, 0)$. If $l = -\infty$, then (18) yields

$$
\begin{aligned}
\liminf_{t \to \infty} y(t) &\leq \limsup_{t \to \infty} v(t) + \liminf_{t \to \infty} (k(t) + F(t)) \\
&\leq \limsup_{t \to \infty} v(t) + \limsup_{t \to \infty} k(t) + \liminf_{t \to \infty} F(t) \\
&= \lim_{t \to \infty} v(t) + \lim_{t \to \infty} k(t) + \liminf_{t \to \infty} F(t)
\end{aligned}
$$

that is, $\liminf_{t \to \infty} y(t) = -\infty$, which is a contradiction.

If $-\infty < l < 0$, then in cases (c) and (d), $v'(t)$ is decreasing. Successive integrations of $v''(t)$ again show that $\lim_{t \to \infty} v(t) = -\infty$. If Case (e) holds, $y(t) \leq v(t) + k(t) + F(t) \leq k(t) + F(t)$, which contradicts the unboundedness of $y(t)$. This completes the proof of the theorem. ∎

Our final theorem in this paper gives sufficient conditions for equation (NH) to have a bounded positive solution.

Theorem 10. *Let $0 \leq p(t) \leq p_1 < 1$, and (H_1) and (H_{13}) hold with*

$$-\frac{1}{8}(1 - p_1) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1 - p_1).$$

Furthermore, assume that G and H are Lipschitzian on the intervals of the form $[b, c]$, $0 < b < c < \infty$. If

$$\int_0^\infty \frac{s}{r(s)} \int_s^\infty tq(t)dtds < \infty,$$

then (NH) admits a positive bounded solution.

Proof. It is possible to choose $t_0 > 0$ large enough such that for $t \geq t_0 > 0$,

$$\int_{t_0}^\infty \frac{t}{r(t)} \int_t^\infty sh(s)dsdt < \frac{1 - p_1}{4L}$$

and

$$\int_{t_0}^\infty \frac{t}{r(t)} \int_t^\infty sq(s)dsdt < \frac{1 - p_1}{4L},$$

where $L = \max\{L_1, L_2, G(1), H(1)\}$ and L_1, L_2 are Lipschitz constants of G and H on $[\frac{1}{8}(1 - p_1), 1]$ respectively. Let $X = BC([t_0, \infty), \mathbb{R})$. Then X is a Banach Space with respect to supremum norm defined by

$$||x|| = \sup_{t \geq t_0}\{|x(t)|\}.$$

Let

$$S = \left\{x \in X : \frac{1}{8}(1 - p_1) \leq x(t) \leq 1, \ t \geq t_0\right\}.$$

Hence S is a complete metric space. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0 + \rho), & t \in [t_0, t_0 + \rho] \\ -p(t)y(t - \tau) + \frac{1}{2}(1 + p_1) + F(t) + k(t), & \\ \quad -\int_t^\infty(\frac{s-t}{r(s)} \int_s^\infty (u - s)q(u)G(y(u - \alpha))du)ds, & t \geq t_0 + \rho \end{cases}$$

Indeed,

$$k(t) = \int_t^\infty \frac{s - t}{r(s)} \int_s^\infty (u - s)h(u)H(y(u - \beta))duds$$

$$\leq H(1) \int_t^\infty \frac{s}{r(s)} \int_s^\infty uh(u)duds < \frac{1}{4}(1 - p_1)$$

implies that

$$Ty(t) < \frac{1 + p_1}{2} + \frac{1 - p_1}{4} + \frac{1 - p_1}{4} = 1.$$

On the other hand,

$$\int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)q(u)G(y(u-\alpha))du\,ds < \frac{1-p_1}{4}$$

implies that

$$Ty(t) > -p_1 + \frac{1}{2}(1+p_1) - \frac{1}{8}(1-p_1) - \frac{1}{4}(1-p_1) = \frac{1}{8}(1-p_1).$$

Hence $Ty \in S$, that is, $T : S \to S$.

Next, we show that T is continuous. Let $y_k(t) \in S$ such that $\lim_{k\to\infty} ||y_k(t) - y(t)|| = 0$ for all $t \geq t_0$. Because S is closed, $y(t) \in S$. Indeed,

$$
\begin{aligned}
|(Ty_k) - (Ty)| &\leq p(t)|y_k(t-\tau) - y(t-\tau)| \\
&+ |\int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)q(u)[G(y_k(u-\alpha)) - G(y(u-\alpha))]du\,ds| \\
&+ |\int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)h(u)[H(y_k(u-\beta)) - H(y(u-\beta))]du\,ds| \\
&\leq p_1|y_k(t-\tau) - y(t-\tau)| \\
&+ \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)q(u)|G(y_k(u-\alpha)) - G(y(u-\alpha))|du\,ds \\
&+ \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)h(u)|H(y_k(u-\beta)) - H(y(u-\beta))|du\,ds \\
&\leq p_1||y_k - y|| + L_1||y_k - y|| \int_t^\infty \frac{s}{r(s)} \int_s^\infty uq(u)du\,ds \\
&+ L_2||y_k - y|| \int_t^\infty \frac{s}{r(s)} \int_s^\infty uh(u)du\,ds,
\end{aligned}
$$

implies that

$$||(Ty_k) - (Ty)|| \leq ||y_k - y|| \left[p_1 + \frac{1-p_1}{4} + \frac{1-p_1}{4} \right] \to 0$$

as $k \to \infty$. Hence T is continuous.

In order to apply Schauder's fixed point theorem [3] we need to show that Ty is precompact. Let $y \in S$. For $t_2 \geq t_1$,

$$
\begin{aligned}
(Ty)(t_2) - (Ty)(t_1) &= p(t_2)y(t_2-\tau) - p(t_1)y(t_1-\tau) \\
&+ k(t_2) - k(t_1) + F(t_2) - F(t_1) \\
&+ \int_{t_1}^\infty \frac{s-t_1}{r(s)} \int_s^\infty (u-s)q(u)G(y(u-\alpha))du\,ds \\
&- \int_{t_2}^\infty \frac{s-t_2}{r(s)} \int_s^\infty (u-s)q(u)G(y(u-\alpha))du\,ds,
\end{aligned}
$$

that is,

$$
\begin{aligned}
|(Ty)(t_2) - (Ty)(t_1)| \leq\ & |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| + |F(t_2) - F(t_1)| \\
& + |\int_{t_2}^{\infty} \frac{s - t_2}{r(s)} \int_{s}^{\infty} (u - s)h(u)H(y(u - \beta))duds \\
& - \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)h(u)H(y(u - \beta))duds| \\
& + |\int_{t_2}^{\infty} \frac{s - t_2}{r(s)} \int_{s}^{\infty} (u - s)q(u)G(y(u - \alpha))duds \\
& - \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)q(u)G(y(u - \alpha))duds| \\
\leq\ & |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| + |F(t_2) - F(t_1)| \\
& + |\int_{t_2}^{\infty} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)h(u)H(y(u - \beta))duds \\
& - \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)h(u)H(y(u - \beta))duds| \\
& + |\int_{t_2}^{\infty} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)q(u)G(y(u - \alpha))duds \\
& - \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)q(u)G(y(u - \alpha))duds| \\
=\ & |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| + |F(t_2) - F(t_1)| \\
& + |\int_{t_1}^{t_2} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)h(u)H(y(u - \beta))duds| \\
& + |\int_{t_1}^{t_2} \frac{s - t_1}{r(s)} \int_{s}^{\infty} (u - s)q(u)G(y(u - \alpha))duds| \\
& \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}
$$

Thus, Ty is precompact. By Schauder's fixed point theorem T has a fixed point, that is, $Ty = y$. Consequently, $y(t)$ is a solution of (NH) on $[\frac{1}{8}(1 - p_1), 1]$. This completes the proof of the theorem. ∎

Remark 4. Theorems similar to Theorem 10 can be proved in other ranges of $p(t)$.

4. Examples

Example 1.

$$
(19) \quad (y(t) + (e^{-4t} + 1)y(t - 2\pi))^{(iv)} + 4e^{4\pi}y(t - 4\pi) - 100e^{-4t + \theta_1 - 2\pi}
$$
$$
\times (1 + e^{2t - 2\theta_1} \sin^2(t - \theta_1))\frac{y(t - \theta_1)}{1 + y^2(t - \theta_1)} = -4e^{t - 2\pi}\sin t, \quad t \geq 74,
$$

where $r(t) = 1$, $p(t) = e^{-4t} + 1$, $q(t) = 4e^{4\pi}$, $h(t) = 100e^{-4t+\theta_1-2\pi}(1 + e^{2t-2\theta_1}\sin^2(t-\theta_1))$, $G(u) = u$, $H(u) = \frac{u}{1+u^2}$ and $f(t) = -4e^{t-2\pi}\sin t$. Indeed, if we choose $F(t) = e^{t-2\pi}\sin t$, then $(r(t)F''(t))'' = f(t)$.

Clearly, $(H_0) - (H_2)$, (H_{11}), (H_{12}), (H_{14}) and (H_{15}) are satisfied. Hence Theorem 7 can be applied to (19), that is, every unbounded solution of (19) oscillates. Indeed, $y(t) = e^t \sin t$ is such a solution of (19).

Example 2. Consider

$$(20) \quad (y(t) + e^{-t-2\pi}y(t-2\pi))^{(iv)} + (4 + 7e^{-t})e^{-2\pi}y(t-2\pi)$$

$$- 24e^{-t-\frac{\pi}{2}}(1 + e^{-2t+\pi}\cos^2 t)\frac{y(t-\frac{\pi}{2})}{1+y^2(t-\frac{\pi}{2})} = 0, \quad t \geq 7.$$

Clearly, $(H_0) - (H_2)$, (H_5), (H_{11}) and (H_{12}) are satisfied. Hence by Theorem 4 every solution of (20) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = e^{-t}\sin t$ is such a solution of (20).

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