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a-LOCAL FUNCTION AND ITS PROPERTIES IN IDEAL TOPOLOGICAL SPACES

Abstract. In this paper, we introduce the notation of a-local function and study its properties in ideal topological space. We construct a topology τ^{a^*} for X by using a-open set and an $\mathcal{I}$ on X. We defined a-compatible of τ with ideal and show that τ is a-compatible with $\mathcal{I}$ then $\tau^{a^*} = \beta(\mathcal{I}, \tau)$, where $\beta(\mathcal{I}, \tau) = \{V\text{-}I : V \in \tau^a(x),\ I \in \mathcal{I}\}$ is a basis of τ^{a^*} Also, The relationships other local functions such as local function [12, 6] and semi-local function [7] are introduced.

Key words: ideal topological space, a-local function, a-open set.

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1. Introduction

The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathaswamy [13]. Jankovic and Hamlett [6] investigated further properties of ideal space. In this paper, we introduce the notation of a-local function and study its properties in ideal topological space. We construct a topology τ^{a^*} for X by using a-open set and an $\mathcal{I}$ on X. We defined a-compatible of τ with ideal and show that τ is a-compatible with $\mathcal{I}$ then $\tau^{a^*} = \beta(\mathcal{I}, \tau)$, where $\beta(\mathcal{I}, \tau) = \{V\text{-}I : V \in \tau^a(x),\ I \in \mathcal{I}\}$ is a basis of τ^{a^*} (Theorem 4). Also, The relationships other local functions such as local function [12, 6] and semi-local function [7] are introduced.

2. Preliminaries

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [10] if $A = int(cl(A))$ $(resp. A = cl(int(A)))$. A is called δ-open [11] if for each $x \in A$, there exist a regular open set G such that $x \in G \subset A$. The complement of δ-open set is called δ-closed. A point $x \in X$ is called a

δ-cluster point of A if $int(cl(U)) \cap A \neq \emptyset$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $cl_\delta(A)$ [11]. The set δ-interior of A [11] is the union of all regular open sets of X contained in A and its denoted by $int_\delta(A)$. A is δ-open if $int_\delta(A) = A$. δ-open sets forms a topology τ^δ. The collection of all δ-open sets in X is denoted by $\delta O(X)$. A subset A of a space (X, τ) is said to be semi-open [9] if $A \subset cl(int(A))$. The complement of semi-open is said to be semi-closed. The collection of all semi-open sets in X is denoted by $SO(X)$. The semi-closure of A in (X, τ) is defined by the intersection of all semi-closed sets containing A and is denoted by $scl(A)$ [1].

A subset A of a space (X, τ) is said to be a-open (resp. a-closed) [2, 3] if $A \subset int(cl(int_\delta(A)))$ (resp. $cl(int(cl_\delta(A))) \subset A$. For a topological space (X, τ), the family of all a-open sets of X forms a topology [2, 3], denoted by τ^a, for X. The collection of all a-open sets containing x in X is denoted by $\tau^a(x)$. Let A be a subset of a space X. The intersection of all a-closed sets containing A is called a-closure of A [3] and is denoted by $aCl(A)$. The a-interior of A, denoted by $aInt(A)$, is defined by the union of all a-open sets contained in A [3].

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions:

(1) $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$;

(2) $A \in \mathcal{I}$ *and* $B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function [12, 6] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}, \text{ for every U} \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $Cl^*(x) = A \cup A^*(\mathcal{I}, \tau)$. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X. The hypothesis $X = X^*$ [5] is equivalent to hypothesis $\tau \cap \mathcal{I} = \emptyset$. For every ideal topological space there exits a topology $\tau^*(\mathcal{I})$ finer than τ generated by $\beta(\mathcal{I}, \tau) = \{U\text{-}A \mid U \in \tau \text{ and } A \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau)$ is not always topology [6]. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be a subset of of X. Then $A_*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in SO(X, x)\}$ is called semi local function of A with respect to $\mathcal{I}$ and τ [7]. Let $(X, \mathcal{I}, \tau)$ ba an ideal topological space. We say that the topology τ is *compatible* with the $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following hold for every $A \subset X$, if for every $x \in A$ there exists a $U \in \tau$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ [6].

Lemma 1 ([4]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, and A, B subsets of X. Then the following properties hold:*

(1) *If $A \subseteq B$, then $A^* \subseteq B^*$;*

(2) *If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$;*

(3) *$A^* = cl(A^*) \subset cl(A)$;*

(4) *$(A \cup B)^* = A^* \cup B^*$;*

(5) *$(A \cap B)^* \subset A^* \cup B^*$.*

3. a-local function

Definition 1. *Let $(X, \tau, \mathcal{I})$ be an ideal in topological space and A be a subset of X. Then $A^{a^*}(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$, for every $U \in \tau^a(x)\}$ is called a-local function of A with respect to $\mathcal{I}$ and τ. We denote simply A^{a^*} for $A^{a^*}(\mathcal{I}, \tau)$.*

Remark 1. The notation of the local function, semi local function are independent with a-local function notation as the following example.

Example 1. Let $X = \{x, y, w, z\}$ with a topology $\tau = \{\emptyset, X, \{x, y\}\}$ and $\mathcal{I} = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Take $A = \{w, z\}$. Then $A^* = \{\emptyset\}$, $A_* = \{z\}$, $A^{a^*} = X$.

Remark 2. (1) The minimal ideal is $\{\emptyset\}$ in any ideal topological space $(X, \tau, \mathcal{I})$ and the maximal ideal is $P(X)$. It can be deduce that $A^{a^*}(\{\emptyset\})$ $= aCl(A) \neq cl(A)$ and $A^{a^*}(P(X)) = \emptyset$ for every $A \subset X$.

(2) If $A \in \mathcal{I}$, then $A^{a^*} = \emptyset$.

(3) Neither $A \subset A^{a^*}$ nor $A^{a^*} \subset A$ in general.

Theorem 1. *Let $(X, \tau, \mathcal{I})$ an ideal in topological space and A, B subsets of X. Then for a-local functions the following properties hold:*

(1) $(\emptyset)^{a^*} = \emptyset$,

(2) *If $A \subset B$, then $A^{a^*} \subset B^{a^*}$,*

(3) *For another ideal $J \supset I$ on X, $A^{a^*}(J) \subset A^{a^*}(\mathcal{I})$,*

(4) *$A^{a^*} \subset aCl(A)$,*

(5) *$A^{a^*}(\mathcal{I}) = aCl(A^{a^*}) \subset aCl(A)$ (i.e A^{a^*} is an a-closed subset of $aCl(A)$),*

(6) $(A^{a^*})^{a^*} \subset A^{a^*}$,

(7) $(A \cup B)^{a^*} = A^{a^*} \cup B^{a^*}$,

(8) A^{a^*}-$B^{a^*} = (A$-$B)^{a^*}$-$B^{a^*} \subset (A$-$B)^{a^*}$,

(9) *If* $U \in \tau^a$, *then* $U \cap A^{a^*} = U \cap (U \cap A)^{a^*} \subset (U \cap A)^{a^*}$,

(10) *If* $U \in I$, *then* $(A$-$U)^{a^*} \subset A^{a^*} = (A \cup U)^{a^*}$,

Proof. (1) This prove is obvious.

(2) Let $x \in A^{a^*}$, then $U \cap A \notin \mathcal{I}$ for every $U \in \tau^a(x)$. Therefore $U \cap B \notin \mathcal{I}$ for each $U \in \tau^a(x)$. Since $A \subset B$ implies that $U \cap A \subset U \cap B$. If $U \cap B \in \mathcal{I}$ then, $U \cap A \in \mathcal{I}$. Hence $x \in B^{a^*}$ and $A^{a^*} \subset B^{a^*}$.

(3) Let $x \in A^{a^*}(J)$. Then for every $\tau^a(x)$, $U \cap A \notin J$. This implies that $U \cap A \notin \mathcal{I}$, so $x \in A^{a^*}(I)$. Hence $A^{a^*}(J) \subset A^{a^*}(\mathcal{I})$.

(4) Let $x \in A^{a^*}$. Then for every a-open set containing x, $U_x \cap A \notin \mathcal{I}$. This implies that $U_x \cap A \neq \emptyset$. Hence $x \in a\text{-}cl(A)$.

(5) $A^{a^*} \subset aCl(A^{a^*})$ hold in general. Let $x \in aCl(A^{a^*})$. Then $A^{a^*} \cap U \neq \emptyset$ for every $U \in \tau^a(x)$. Therefore, there exist some $y \in A^{a^*} \cap U$ and $U \in \tau^a(x)$ since $y \in A^{a^*}$, $A \cap U \notin \mathcal{I}$ and hence $x \in A^{a^*}$. Thus $aCl(A^{a^*}) \subset A^{a^*}$. Now, Let $aCl(A^{a^*}) = A^{a^*}$, Then $A \cap U \notin \mathcal{I}$ for every $U \in \tau^a(x)$. This implies that $A \cap U \neq \emptyset$ for every $U \in \tau^a(x)$ and so, $x \in aCl(X, x)$. Consequently, $A^{a^*} = aCl(A^{a^*}) \subset aCl(A)$ and A^{a^*} is an a-closed.

(6) Let $x \in (A^{a^*})^{a^*}$. Then, for every $U \in \tau^a(x)$, $A^{a^*} \cap U \notin \mathcal{I}$ and hence $A^{a^*} \cap U \neq \emptyset$ for every $U \in \tau^a(x)$. Thus we have $A \cap U \notin \mathcal{I}$ and $x \in A^{a^*}$.

(7) $A \subset A \cup B$, and $B \subset A \cup B$ and $A^{a^*} \cup B^{a^*} \subset (A \cup B)^{a^*}$ by (1). Conversely, let $x \in (A \cup B)^{a^*}$. Then for every $U \cap (A \cup B) \notin \mathcal{I} = (U \cap A) \cup (U \cap B) \notin \mathcal{I}$. Therefore, $(U \cap A) \notin \mathcal{I}$ or $(U \cap B) \notin \mathcal{I}$. This implies that $x \in A^{a^*}$ or $x \in B^{a^*}$, that is, $x \in A^{a^*} \cup B^{a^*}$. So we obtain the equality.

(8) Since A-$B \subset A$, by (1), $(A$-$B)^{a^*} \subset A^{a^*}$ and hence $(A$-$B)^{a^*}$-$B^{a^*} \subset A^{a^*}$-B^{a^*}. Conversely $A \subset (A$-$B) \cup B$, by (7), $A^{a^*} \subset (A$-$B)^{a^*} \cup B^{a^*}$ and hence A^{a^*}-$B^{a^*} \subset (A$-$B)^{a^*} \cup B^{a^*})$-B^{a^*}. Therefore, A^{a^*}-$B^{a^*} \subset (A$-$B)^{a^*}$-$(B^{a^*} \cup B^{a^*})$ and so, A^{a^*}-$B^{a^*} \subset (A$-$B)^{a^*}$-B^{a^*}.

(9) Assume $U \in \tau^a(x)$ and $x \in U \cap A^{a^*}$. Then $x \in U$ and $x \in A^{a^*}$. For $V \in \tau^a(x)$, $U \cap V \in \tau^a(x)$ [3]. Thus $V \cap (U \cap A) = (U \cap V) \cap A \notin \mathcal{I}$. So $x \in (U \cap A)^{a^*}$. Therefore $U \cap A^{a^*} \subset (U \cap A)^{a^*}$. Also $U \cap A^{a^*} \subset U \cap (U \cap A)^{a^*}$, since $A \cap U \subset A$. Then by (1), $(A \cap U)^{a^*} \subset A^{a^*}$ and $U \cap (A \cap U)^{a^*} \subset U \cap A^{a^*}$. So we get the result.

(10) By (7) and Remark 2(2) $(A \cup U)^{a^*} = A^{a^*} \cup U^{a^*} = A^{a^*} \cup \emptyset = A^{a^*}$, since A-$U \subset A$ by (1), $(A$-$U)^{a^*} \subset (A)^{a^*}$. So, we get the result. ■

Theorem 2. *Let (X,τ) a topological space, $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on X and let A be a subset of X. Then the following properties hold:*

(1) *If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $A^{a^*}(\mathcal{I}_2) \subset A^{a^*}(\mathcal{I}_1)$;*

(2) $A^{a^*}(\mathcal{I}_1 \cap \mathcal{I}_2) = A^{a^*}(\mathcal{I}_1) \cup A^{a^*}(\mathcal{I}_2)$.

Proof. (1) Let $\mathcal{I}_1 \subset \mathcal{I}_2$ and $x \in A^{a^*}(\mathcal{I}_2)$. Then $A \cap U \notin \mathcal{I}_2$ for every $U \in \tau^a(x)$ and hence $A \cap U \notin \mathcal{I}_1$,Then $x \in A^{a^*}(\mathcal{I}_1)$. This shows that $A^{a^*}(\mathcal{I}_2) \subset A^{a^*}(\mathcal{I}_1)$.

(2) Since $\mathcal{I}_1 \cap \mathcal{I}_2 \subset \mathcal{I}_1$ and $\mathcal{I}_1 \cap \mathcal{I}_2 \subset \mathcal{I}_2$, by Theorem 2 (1) we have. $A^{a^*}(\mathcal{I}_1) \subset A^{a^*}(\mathcal{I}_1 \cap \mathcal{I}_2)$ and $A^{a^*}(\mathcal{I}_2) \subset A^{a^*}(\mathcal{I}_1 \cap \mathcal{I}_2)$. Hence we have $A^{a^*}(\mathcal{I}_1) \cup A^{a^*}(\mathcal{I}_2) \subset A^{a^*}(\mathcal{I}_1 \cap \mathcal{I}_2)$. Conversely let $x \in A^{a^*}(\mathcal{I}_1 \cap \mathcal{I}_2)$. Then for every $U \in \tau^a(x)$, $U \cap A \notin \mathcal{I}_1 \cap \mathcal{I}_2$ hence $U \cap A \notin \mathcal{I}_1$ or $U \cap A \notin \mathcal{I}_2$. This shows that $x \in A^{a^*}(\mathcal{I}_1)$ or $x \in A^{a^*}(\mathcal{I}_2)$ and $x \in A^{a^*}(\mathcal{I}_1) \cup A^{a^*}(\mathcal{I}_2)$. So, we get the result. ■

Lemma 2. *Let $(X,\tau,\mathcal{I})$ be an ideal ideal topological space. If $U \in \tau^a(x)$, then $U \cap A^{a^*} = U \cap (U \cap A)^{a^*} \subseteq (U \cap A)^{a^*}$ for any subset A of X.*

Proof. Suppose that $U \in \tau^a(x)$ and $x \in U \cap A^{a^*}$. Then $x \in U$ and $x \in A^{a^*}$. Let V be any a-open set containing x. Then $V \cap U \in \tau^a(x)$ and $V \cap (U \cap A) = (V \cap U) \cap A \notin \mathcal{I}$. This shows that $x \in (U \cap A)^{a^*}$ and hence we obtain $U \cap A^{a^*} \subseteq (U \cap A)^{a^*}$. Moreover, $U \cap A^{a^*} \subseteq U \cap (U \cap A)^{a^*}$ and by Theorem 1 $(U \cap A)^{a^*} \subseteq A^{a^*}$ and $U \cap (U \cap A)^{a^*} \subseteq U \cap A^{a^*}$. Therefore, $U \cap A^{a^*} = U \cap (U \cap A)^{a^*}$. ■

4. The open sets of τ^{a^*}

In this section we have investigated τ^{a^*} finer than τ^a in the term of the closure operator $aCl^*(A) = A \cup A^{a^*}$. A basis $\beta(\mathcal{I},\tau)$ for τ^{a^*} can be described as follows: A subset A of an ideal space $(X,\mathcal{I},\tau)$ is said to be τ^{a^*}-closed if $A^{a^*} \subset A$. Thus we have $U \in \tau^{a^*}$ if and only if X-U is τ^{a^*} - closed which implies $(X\text{-}U)^{a^*} \subset (X\text{-}U)$ and hence $U \subset X\text{-}(X\text{-}U)^{a^*}$. Thus if $x \in U$, $x \notin (X\text{-}U)^{a^*}$, then there exist $V \in \tau^a(x)$ such that $V \cap (X\text{-}U) \in \mathcal{I}$. Hence, let $I = V \cap (X\text{-}U)$ and we have $x \in V\text{-}I \subset U$ where $V \in \tau^a(x)$ and $I \in \mathcal{I}$. So the basis for τ^{a^*} is $\beta(\mathcal{I},\tau) = \{V\text{-}I : V \in \tau^a(x), I \in \mathcal{I}\}$ and β is not, in general, a topology. See Theorem 4.

Theorem 3. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space, $aCl^*(A) = A^{a^*} \cup A$ and A, B be subsets of X. Then*

(1) $aCl^*(\emptyset) = \emptyset$.

(2) $A \subseteq aCl^*(A)$.

(3) $aCl^*(A \cup B) = aCl^*(A) \cup aCl^*(B)$.

(4) $aCl^*(A) = aCl^*(aCl^*(A))$.

Proof. By Theorem 1, we obtain
(1) $aCl^*(\emptyset) = (\emptyset)^{a^*} \cup \emptyset = \emptyset$.

(2) $A \subseteq A \cup A^{a^*} = aCl^*(A)$.

(3) $aCl^*(A \cup B) = (A \cup B)^{a^*} \cup (A \cup B) = (A^{a^*} \cup B^{a^*}) \cup (A \cup B) = aCl^*(A) \cup aCl^*(B)$.

(4) $aCl^*(aCl^*(A)) = aCl^*(A^{a^*} \cup A) = (A^{a^*} \cup A)^{a^*} \cup (A^{a^*} \cup A) = ((A^{a^*})^{a^*} \cup A^{a^*}) \cup (A^{a^*} \cup A) = A^{a^*} \cup A = aCl^*(A)$. ■

Lemma 3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X. Then $A^{a^*} - B^{a^*} = (A - B)^{a^*} - B^{a^*}$.*

Proof. We have by Theorem 1 $A^{a^*} = [(A - B) \cup (A \cap B)]^{a^*} = (A - B)^{a^*} \cup (A \cap B)^{a^*} \subseteq (A - B)^{a^*} \cup B^{a^*}$. Thus $A^{a^*} - B^{a^*} \subseteq (A - B)^{a^*} - B^{a^*}$. By Theorem 1, $(A - B)^{a^*} \subseteq A^{a^*}$ and hence $(A - B)^{a^*} - B^{a^*} \subseteq A^{a^*} - B^{a^*}$. Hence $A^{a^*} - B^{a^*} = (A - B)^{a^*} - B^{a^*}$. ■

Lemma 4. *Let $(X, \tau, \mathcal{I})$ be an ideal ideal topological space and A, B be subsets of X. Then*

(1) *If $A \subseteq B$, then $aCl^*(A) \subseteq aCl^*(B)$.*

(2) *$aCl^*(A \cap B) \subseteq aCl^*(A) \cap aCl^*(B)$.*

(3) *If U is a-open, then $U \cap aCl^*(A) \subseteq aCl^*(U \cap A)$.*

Proof. (1) Since $A \subseteq B$, by Theorem 1 we have $aCl^*(A) = A \cup A^{a^*} \subseteq B \cup B^{a^*} = aCl^{a^*}(B)$.

(2) This is obvious by (1).

(3) Since U is a-open, by Lemma 2 we have $U \cap aCl^{a^*}(A) = U \cap (A \cup A^{a^*}) = (U \cap A) \cup (U \cap A^{a^*}) \subseteq (U \cap A) \cup (U \cap A)^{a^*} = aCl^*(U \cap A)$. ■

Theorem 4. *Let $(X, \mathcal{I}, \tau)$ be an ideal topological space. Then $\beta(\mathcal{I}, \tau)$ is a basis for τ^{a^*}.*

Proof. Since $\emptyset \in \mathcal{I}$, Then V-$\emptyset = V \in \tau^a(x)$ and $\tau^a(x) \subset \beta$ from which it follows that $X = \cup\beta$ (recall that a - open sets is forms a topology). Also $\beta_1, \beta_2 \in \beta$, and $I_1, I_2 \in \mathcal{I}$, we have $\beta_1 = V_1$-I_1 and $\beta_2 = V_2$-I_2, where $V_1, V_2 \in \tau^a(x)$. Then $\beta_1 \cap \beta_2 = (V_1$-$I_1) \cap (V_2$-$I_2) = (V_1 \cap (X$-$I_1)) \cap (V_2 \cap (X$-$I_2)) = (V_1 \cap V_2)$-$(I_1 \cup I_2) \in \beta$, where $V_1 \cap V_2 \in \tau^a(x)$, $I_1 \cup I_2 \in \mathcal{I}$. ■

Remark 3. The topology τ^{a^*} finer than τ^a. See the following example.

Example 2. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$. Set $A = \{a, c\}$. Then $A \in \tau^{a^*}$, but A it is not a-open. So $A \notin \tau^a(x)$.

Example 3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\ , \{a, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$. Set $A = \{a, c, d\}$. Then $A \in \tau^{a^*}$, but $A \notin \tau^a(x)$.

The following examples show that $\beta(\mathcal{I}, \tau)$ is not a topology in general.

Example 4. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$ be ideal in X, where $Int_\delta(A) = \{c, d\}$ is the union of all regular open set of X contained in A and $\{\emptyset, X, \{c, d\}\} \in \tau^a$. Consider the collection of subsets of X defined as $\beta(\mathcal{I}, \tau) = \{V\text{-}I : V \in \tau^a(x),\ I \in \mathcal{I}\}$ $= \{\emptyset, X, \{c\}, \{d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Thus $\beta(\mathcal{I}, \tau)$ is not open under union of any collection of open sets ($i.e$ $\{c\} \cup \{d\} \notin \beta(\mathcal{I}, \tau)$) and hence it is not a topology.

5. a-compatible topology with an ideal

Definition 2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then τ is said to be a-compatible with respect to $\mathcal{I}$, denoted by $\tau \sim^a \mathcal{I}$ if and only if, for every $x \in A$ there exist $U \in \tau^a(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.*

Theorem 5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A subset of X. Then the following are equivalent:*

(1) *$\tau \sim^a \mathcal{I}$,*

(2) *If a subset A of X has a cover a- open sets of whose intersection with A is in $\mathcal{I}$, then A is in $\mathcal{I}$, in other words $A^{a^*} = \emptyset$, then $A \in \mathcal{I}$,*

(3) *For every $A \subset X$, if $A \cap A^{a^*} = \emptyset$, $A \in \mathcal{I}$,*

(4) *For every $A \subset X$, A-$A^{a^*} \in \mathcal{I}$,*

(5) *For every $A \subset X$, if A contains no nonempty subset B with $B \subset B^{a^*}$, then $A \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2) The proof is obvious.

(2) $\Rightarrow$ (3) Let $A \subset X$ and let $x \in A$. Then $x \notin A^{a^*}$ and there exists $U_x \in \tau^a(x)$ such that $U_x \cap A \in \mathcal{I}$. Since A has a cover $A \subset \cup\{U_x : x \in A\}$ and $U_x \in \tau^a(x)$ by (2), $A \in \mathcal{I}$.

(3) $\Rightarrow$ (4) For any $A \subset X$, since $A \cap A^{a^*} = \emptyset$, then A-$A^{a^*} \subset A$ and by Theorem 1(1) $(A\text{-}A^{a^*})^{a^*} \subset A^{a^*}$ and $(A\text{-}A^{a^*})^{a^*} \cap (A - A^{a^*}) \subset (A\text{-}A^{a^*}) \cap A^{a^*} = \emptyset$. Then by (3) we have $A\text{-}A^{a^*} \in \mathcal{I}$.

(4) $\Rightarrow$ (5) By (4), for any $A \subset X$, $A\text{-}A^{a^*} \in \mathcal{I}$. $A = (A\text{-}A^{a^*}) \cup (A \cap A^{a^*})$ and by Theorem 1(4), $A^{a^*} = (A\text{-}A^{a^*})^{a^*} \cup (A \cap A^{a^*})^{a^*} = (A \cap A^{a^*})^{a^*}$. Therefore, we have $A^{a^*} \cap A = (A \cap A^{a^*})^{a^*} \cap A$, then $A^{a^*} \cap A \subset (A \cap A^{a^*})^{a^*}$, and $(A \cap A^{a^*}) \subset A$. By assumption $(A \cap A^{a^*}) = \emptyset$. So A contains no nonempty subset. Hence $A\text{-}A^{a^*} = A$ by (4) $A \in \mathcal{I}$.

(5) $\Rightarrow$ (1) Let $A \subset X$ and assume that for every $x \in A$, there exists $U \in \tau^a(x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap A^{a^*} = \emptyset$. Since $(A\text{-}A^{a^*})^{a^*} \cap (A\text{-}A^{a^*}) \subset (A\text{-}A^{a^*}) \cap A^{a^*} = \emptyset$. So, $A\text{-}A^{a^*}$ contains no nonempty subset B with $B \subset B^{a^*}$. By (5), $A\text{-}A^{a^*} \in \mathcal{I}$ and hence $A = A \cap (X\text{-}A^{a^*}) = A\text{-}A^{a^*} \in \mathcal{I}$. ■

Theorem 6. *Let $(X, \tau, \mathcal{I})$ be an ideal ideal topological space. If τ is a-compatible with $\mathcal{I}$, then the following properties are equivalent:*

(1) *For every $A \subseteq X$, $A \cap A^{a^*} = \emptyset$ implies that $A^{a^*} = \emptyset$;*

(2) *For every $A \subseteq X$, $(A - A^{a^*})^{a^*} = \emptyset$;*

(3) *For every $A \subseteq X$, $(A \cap A^{a^*})^{a^*} = A^{a^*}$.*

Proof. First, we show that (1) holds if τ is a-compatible with $\mathcal{I}$. Let A be any subset of X and $A \cap A^{a^*} = \emptyset$. By Theorem 5, $A \in \mathcal{I}$ and by Remark 2(3) $A^{a^*} = \emptyset$.

(1) $\Rightarrow$ (2) Assume that for every $A \subseteq X$, $A \cap A^{a^*} = \emptyset$ implies that $A^{a^*} = \emptyset$. Let $B = A - A^{a^*}$, then

$$\begin{aligned} B \cap B^{a^*} &= (A - A^{a^*}) \cap (A - A^{a^*})^{a^*} \\ &= (A \cap (X - A^{a^*})) \cap (A \cap (X - A^{a^*}))^{a^*} \\ &\subseteq [A \cap (X - A^{a^*})] \cap [A^{a^*} \cap (X - A^{a^*})^{a^*}] = \emptyset. \end{aligned}$$

By (1), we have $B^{a^*} = \emptyset$. Hence $(A - A^{a^*})^{a^*} = \emptyset$.

(2) $\Rightarrow$ (3) Assume for every $A \subseteq X$, $(A - A^{a^*})^{a^*} = \emptyset$.

$$\begin{aligned} A &= (A - A^{a^*}) \cup (A \cap A^{a^*}) \\ A^{a^*} &= [(A - A^{a^*}) \cup (A \cap A^{a^*})]^{a^*} \\ &= (A - A^{a^*})^{a^*} \cup (A \cap A^{a^*})^{a^*} \\ &= (A \cap A^{a^*})^{a^*}. \end{aligned}$$

(3) $\Rightarrow$ (1) Assume for every $A \subseteq X$, $A \cap A^{a^*} = \emptyset$ and $(A \cap A^{a^*})^{a^*} = A^{a^*}$. This implies that $\emptyset = \emptyset^{a^*} = A^{a^*}$. ■

Theorem 7. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*

(1) $\tau^a \cap \mathcal{I} = \emptyset$;

(2) *If $I \in \mathcal{I}$, then $aInt(I) = \emptyset$;*

(3) *For every $G \in \tau^a$, $G \subseteq G^{a^*}$;*

(4) $X = X^{a^*}$.

Proof. (1) $\Rightarrow$ (2) Let $\tau^a \cap \mathcal{I} = \emptyset$ and $I \in \mathcal{I}$. Suppose that $x \in aInt(I)$. Then there exists $U \in \tau^a$ such that $x \in U \subseteq I$. Since $I \in \mathcal{I}$ and hence $\emptyset \neq \{x\} \subseteq U \in \tau^a \cap \mathcal{I}$. This is contrary that $\tau^a \cap \mathcal{I} = \emptyset$. Therefore, $aInt(I) = \emptyset$.

(2) $\Rightarrow$ (3) Let $x \in G$. Assume $x \notin G^{a^*}$ then there exists $U_x \in \tau^a(x)$ such that $G \cap U_x \in \mathcal{I}$. By (2), $x \in G \cap U_x = aInt(G \cap U_x) = \emptyset$. Hence $x \in G^{a^*}$ and $G \subseteq G^{a^*}$.

(3) $\Rightarrow$ (4) Since X is a-open, then $X = X_*$.

(4) $\Rightarrow$ (1) $X = X^{a^*} = \{x \in X : U \cap X = U \notin \mathcal{I}$ for each a-open set U containing $x\}$. Hence $\tau^a \cap \mathcal{I} = \emptyset$. ■

Theorem 8. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and τ be a-compatible with $\mathcal{I}$. Then for every $G \in \tau^a$ and any subset A of X, $(G \cap A)^{a^*} = (G \cap A^{a^*})^{a^*} = aCl(G \cap A^{a^*})$.*

Proof. (1) Let $G \in \tau^a$. Then by Lemma 2, $G \cap A^{a^*} = G \cap (G \cap A)^{a^*} \subseteq (G \cap A)^{a^*}$ and hence $(G \cap A^{a^*})^{a^*} \subseteq ((G \cap A)^{a^*})^{a^*} \subseteq (G \cap A)^{a^*}$ by Theorem 1. (2) Now by using Theorem 1 and Theorem 6, we obtain $(G \cap (A - A^{a^*}))^{a^*} \subseteq G^{a^*} \cap (A - A^{a^*})^{a^*} = G^{a^*} \cap \emptyset = \emptyset$. Moreover, $(G \cap A)^{a^*} - (G \cap A^{a^*})^{a^*} \subseteq ((G \cap A) - (G \cap A^{a^*}))^{a^*} = (G \cap (A - A^{a^*}))^{a^*} = \emptyset$, which implies that $(G \cap A)^{a^*} \subseteq (G \cap A^{a^*})^{a^*}$. By (1) and (2), we obtain $(G \cap A)^{a^*} = (G \cap A^{a^*})^{a^*}$.

By Theorem 1, $(G \cap A)^{a^*} = (G \cap A^{a^*})^{a^*} \subseteq aCl(G \cap A^{a^*})$. Also, in view of Lemma 2, we have $G \cap A^{a^*} \subseteq (G \cap A)^{a^*}$ and hence $aCl(G \cap A^{a^*}) \subseteq aCl((G \cap A)^{a^*}) = (G \cap A)^{a^*}$. Consequently, we obtain $(G \cap A^{a^*})^{a^*} = (G \cap A)^{a^*} = aCl(G \cap A^{a^*})$. ■

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