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A UNIQUENESS RESULT ON MEROMORPHIC FUNCTIONS SHARING TWO SETS II

Abstract. We employ the notion of weighted sharing of sets to deal with the well known question of Gross and obtain a uniqueness result on meromorphic functions sharing two sets which will improve an earlier result of Lahiri [15] and a recent one of Banerjee [2].

Key words: meromorphic functions, uniqueness, weighted sharing, shared set.

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1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We shall use the standard notations of value distribution theory :

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\ \ \ldots$$

(see [10]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h)=o(T(r,h)) \quad as\ r\longrightarrow\infty,\ \ r\notin E.$$

For any constant a, we define

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

If for some $a\in\mathbb{C}\cup\{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$, where each a-point of f is counted according to its multiplicity. Denote by $\overline{E}_f(S)$ the reduced form of $E_f(S)$. If $E_f(S)=E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM.

The following question is due to F. Gross [8] which is a very interesting question.

Question A. *Can one find two finite sets S_j $(j=1,2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?*

In [8] Gross wrote *If the answer to* Question A *is affirmative it would be interesting to know how large both sets would have to be ?*

Corresponding to the Gross' question the following question [20] is a natural one.

Question B. *Can one find two finite sets S_j $(j=1,2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?*

The shared set problems relative to a meromorphic function has been studied by many authors. {see [1]-[7], [9], [11], [15], [17]-[18], [20]-[27]}.

In [6] Fang and Lahiri exhibited the following range set S with smaller cardinalities than that obtained by the previous authors, where some restrictions on the poles of f and g are imposed.

Theorem A ([6]). *Let $S=\{z: z^n+az^{n-1}+b=0\}$ where $n(\geq 7)$ be an integer and a and b be two nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple root. If f and g are two non-constant meromorphic functions having no simple poles such that $E_f(S)=E_g(S)$ and $E_f(\{\infty\})=E_g(\{\infty\})$ then $f\equiv g$.*

In 2001 an idea of gradation of sharing of values and sets known as weighted sharing has been introduced in {[13], [14]} which measure how close a shared value is to being shared CM or to being shared IM. Below we are giving the notion.

Definition 1 ([13, 14]). *Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2 ([13]). *Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a\in S} E_k(a; f)$.*

With the notion of weighted sharing of sets Lahiri [15] improved Theorem A as follows.

Theorem B ([15]). *Let S be defined as in Theorem A and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty, f) + \Theta(\infty, g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

Suppose that the polynomial (see [9]) $P(w)$ is defined by

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \tag{1}$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. We consider

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{2}$$

where α_1 and α_2 are two distinct roots of

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

So we observe that

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \tag{3}$$

We have from (1)

$$\begin{aligned} P'(w) &= naw^{n-1} - 2n(n-1)w + 2n(n-2)b \\ &= \frac{n}{w}[aw^n - 2(n-1)w^2 + 2(n-2)bw]. \end{aligned} \tag{4}$$

Noting that $P'(0) \neq 0$, we can get from (4) and $P'(w) = 0$ that

$$aw^n - 2(n-1)w^2 + 2(n-2)bw = 0.$$

Now at each root of $P^{'}(w)=0$ we get

$$\begin{aligned} P(w) &= aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \\ &= 2(n-1)w^2 - 2(n-2)bw - n(n-1)w^2 \\ &\quad + 2n(n-2)bw - (n-1)(n-2)b^2 \\ &= -(n-1)(n-2)(w-b)^2 \end{aligned}$$

So only $w=b$ can make $P(b)=P^{'}(b)=0$. But $P^{'}(b)=nb(ab^{n-2}-2)\neq 0$, which implies that a zero of $P^{'}(w)$ is not a zero of $P(w)$. In other words each zero of $P(w)$ is simple.

Recently the present first author has ascertained the fact that replacing the range set in *Theorem B* by the zero sets of $P(w)$ better results can be achieved. Below we are stating Banerjee's [2] result.

Theorem C. ([2]). *Let $S=\{w \mid P(w)=0\}$, where $P(w)$ is given by (1) and $n\geq 6$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ and $\Theta_f^{*}+\Theta_g^{*}+\min\{\Theta(b,f),\Theta(b,g)\}>8-n$, where $\Theta_f^{*}=2\ \Theta(0,f)+\Theta(\infty,f)$, $\Theta(b,f)$ and Θ_g^{*} is defined similarly. Then $f\equiv g$*

Now it is quite natural to ask the following questions:

(i) What can be said about the relationship between f and g, if $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ in *Theorem C* is replaced with $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, where k is a non-negative integer?

(ii) Can the deficiency condition in *Theorem C* be further relaxed?

In this paper we pay our attention to give some affirmative answers to the above questions. In this direction, we carry out our investigations and give some partial solutions of the above questions imposing some restrictions. In this regards we improve the *Theorem C.*

The following theorem is the main result of the paper.

Theorem 1. *Let $S=\{w \mid P(w)=0\}$, where $P(w)$ is given as (1), where $n\ (\geq 6)$ is an integer. Let $c,d\in\mathbb{C}$ be such that $c,d\notin S\cup\{0,b\}$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,m)=E_g(S,m)$ and $E_f(\{\infty\},k)=E_g(\{\infty\},k)$ and that f and g have respectively c-point and d-point of multiplicity $\geq p+1$ where p, k are non-negative integers or infinity such that $p^{*}+\frac{1}{k+1}\leq 1$, where $p^{*}=1$, if $p=0$ and $p^{*}=\frac{2}{p+1}$, if $p\geq 1$. If*

(i) $m\geq 2$ and

$\Theta_f+\Theta_g+\min\{\delta_f,\delta_g\}+p^{*}\min\{\delta(c,f),\delta(d,g)\}>7+p^{*}+\frac{1}{k+1}-n$

(ii) or if $m=1$ and

$\Theta_f+\Theta_g+\frac{1}{2}\min\{\Theta(0,f)+\Theta(b,f)+\Theta(\infty,f)+\delta_f,\Theta(0,g)+\Theta(b,g)+\Theta(\infty,g)+\delta_g\}+\min\{\delta_f,\delta_g\}+p^{*}\min\{\delta(c,f),\delta(d,g)\}>8+p^{*}+\frac{1}{k+1}-n$

(iii) *or if* $m = 0$ *and*
$\Theta_f + \Theta_g + \Theta(0; f) + \Theta(b, f) + \Theta(\infty, f) + 2\delta_f + \Theta(0, g) + \Theta(b, g) + \Theta(\infty, g) + 2\delta_g + \min\{\Theta(0, f) + \Theta(b, f) + \Theta(\infty, f), \Theta(0, g) + \Theta(b, g) + \Theta(\infty, g)\} + p^* \min\{\delta(c, f), \delta(d, g)\} > 13 + p^* + \frac{1}{k+1} - n$
then $f \equiv g$, *where* $\Theta_f = 2\Theta(0, f) + 2\Theta(b, f) + \Theta(\infty, f) + \frac{1}{2(k+1)}\delta_{(k+1}(\infty, f)$ *and* $\delta_f = \sum_{w \in S} \delta(w, f)$, Θ_g *and* δ_g *can be similarly defined.*

Corollary 1. *Let* S *be given as in* Theorem 1 *where* n (≥ 6) *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S, 2) = E_g(S, 2)$ *and* $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ *and* $\Theta_f + \Theta_g + \min\{\delta_f, \delta_g\} + p^* \min\{\delta(c, f), \delta(d, g)\} > 7 + p^* - n$, *where* $p^* \leq 1$ *then* $f \equiv g$, *where* $\Theta_f = 2\Theta(0; f) + 2\Theta(b; f) + \Theta(\infty; f)$ *and* Θ_g *have the same meaning.*

It is assumed that the readers are familiar with the standard definitions and notations of value distribution theory which can be found,e.g.,in [10]. We are still going to explain the following two notations used in the paper.

Definition 3 ([12])**.** *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting function of simple* a*-points of* f. *For a positive integer* m *we denote by* $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ *the counting function of those* a*-points of* f *whose multiplicities are not greater (not less) than* m *where each* a*-point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ *are defined similarly, where in counting the* a*-points of* f *we ignore the multiplicities.*

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ *are defined analogously.*

Definition 4. *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(1, 0)$. *Let* z_0 *be a* 1*-point of* f *with multiplicity* p, *a* 1*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, 1; f)$ *the reduced counting function of those* 1*-points of* f *and* g *where* $p > q$, *by* $N_E^{1)}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q = 1$, *by* $\overline{N}_E^{(2}(r, 1; f)$ *the reduced counting function of those* 1*-points of* f *and* g *where* $p = q \geq 2$. *In the same way we can define* $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$. *In a similar manner we can define* $\overline{N}_L(r, a; f)$ *and* $\overline{N}_L(r, a; g)$ *for* $a \in \mathbb{C} \cup \{\infty\}$. *When* f *and* g *share* $(1, m)$, $m \geq 1$ *then* $N_E^{1)}(r, 1; f) = N(r, 1; f \mid= 1)$.

Definition 5 ([13, 14])**.** *Let* f, g *share* $(a, 0)$. *We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H the following function.

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Let f and g be two non-constant meromorphic functions and

$$F=R(f),\qquad G=R(g), \tag{5}$$

where $R(w)$ is given as (2). From (2) and (5) it is clear that

$$T(r,f)=\frac{1}{n}T(r,F)+S(r,f),\quad T(r,g)=\frac{1}{n}T(r,G)+S(r,g) \tag{6}$$

Lemma 1 (Lemma 2.18 in [3])**.** *Let F, G be given as (5) and $H\not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k). Then*

$$\begin{aligned} N_E^{1)}(r,1;F) \;\le\; & \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;f)\\ & +\overline{N}(r,b;f)+\overline{N}_*(r,\infty;f,g)\\ & +\overline{N}(r,0;g)+\overline{N}(r.b;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f' which are not the zeros of $f(f-b)$ and $F-1$. Of course $\overline{N}_0(r,0;g')$ is defined similarly.

Lemma 2. *Let f be a non-constant meromorphic function and a_i, $i=1,2,\ldots,k$ be finite distinct complex numbers, where $k\ge 2$. Then*

$$N(r,0;f')\le T(r,f)+\overline{N}(r,\infty;f)-\sum_{i=1}^{k}m(r,a_i;f)+S(r,f)$$

Proof. Let $F=\sum\limits_{i=1}^{k}\frac{1}{f-a_i}$. Then $\sum\limits_{i}^{k}m(r,a_i;f)=m(r,F)+O(1)$. Note that

$$\begin{aligned} m(r,F) \;\le\; & m(r,0;f')+m(r,\sum_{i=1}^{k}\frac{f'}{f-a_i})\\ =\; & T(r,f')-N(r,0;f')+S(r,f).\end{aligned}$$

Also we observe that

$$\begin{aligned} T(r,f^{'}) &= m(r,f^{'}) + N(r,f^{'}) \\ &\leq m(r,f) + m(r,\frac{f^{'}}{f}) + N(r,f) + \overline{N}(r,f) \\ &= T(r,f) + \overline{N}(r,f) + S(r,f). \end{aligned}$$

Hence the Lemma follows. ■

Lemma 3 ([19])**.** *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 4. *Let f, g be two non-constant meromorphic functions sharing $(\infty, 0)$ and suppose α_1 and α_2 are two distinct roots of the equation $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. Then*

$$\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}\,\frac{g^n}{(g-\alpha_1)(g-\alpha_2)} \not\equiv \frac{n^2(n-1)^2}{a^2},$$

where n (≥ 3) is an integer.

Proof. We omit the proof since the proof can be found out in the proof of Theorem 3 [9] (second half of page 26). ■

Lemma 5. *Let F, G be given as (5), where $n \geq 6$ is an integer. If $F \equiv G$, then $f \equiv g$.*

Proof. We omit the proof since the proof can be found out in [9] (page 27). ■

Lemma 6. *Let F, G be given as (5). Also let S be given as in* Theorem 1*, where $n \geq 3$ is an integer. If $E_f(S,0) = E_g(S,0)$ then $S(r,f) = S(r,g)$.*

Proof. Since $E_f(S,0) = E_g(S,0)$, it follows that F and G share $(1,0)$. We denote the distinct elements of S by w_j, $j = 1, 2, \ldots n$. Since F, G share $(1,0)$ from the second fundamental theorem we have

$$\begin{aligned} (n-2)T(r,g) &\leq \sum_{j=1}^{n} \overline{N}\left(r, w_j; g\right) + S(r,g) \\ &= \sum_{j=1}^{n} \overline{N}\left(r, w_j; f\right) + S(r,g) \\ &\leq nT(r,f) + S(r,g). \end{aligned}$$

Similarly we can deduce

$$(n-2)T(r,f) \le nT(r,g) + S(r,f).$$

The last inequalities imply $T(r,f) = O\left(T(r,g)\right)$ and $T(r,g) = O\left(T(r,f)\right)$ and so we have $S(r,f) = S(r,g)$. ∎

3. Proof of the theorem

Proof. [Proof of Theorem 1] Let F, G be given as (5). Since $E_f(S,m) = E_g(S,m)$, it follows that F, G share $(1,m)$.

Case 1. Suppose that $H \not\equiv 0$.

Subcase 1.1. $m \ge 1$. While $m \ge 2$, using Lemma 2 with $k = n+2$, $a_1 = 0$, $a_2 = b$ and a_i, $i = 3, \ldots, n+2$, where $a_3, \ldots, a_{n+2}$ are the distinct zeros of $P(w)$ we note that

$$\begin{aligned}
(7)\quad & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_*(r,1;F,G) \\
&\le \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\ge 2) + \overline{N}(r,1;G \mid\ge 3) \\
&\le \overline{N}_0(r,0;g^{'}) + \sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g \mid= 2) + 2\overline{N}(r,\omega_j;g \mid\ge 3)\} \\
&\le N(r,0;g^{'} \mid g \ne 0,b) \\
&\le N(r,0;g^{'}) - N(r,0;g) + \overline{N}(r,0;g) - N(r,b;g) + \overline{N}(r,b;g) \\
&\le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,b;g) + T(r,g) - N(r,0;g) \\
&\quad - N(r,b;g) - m(r,0;g) - m(r,b;g) - \sum_{w\in S} m(r,w;g) + S(r,g) \\
&\le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,b;g) - T(r,g) \\
&\quad - \sum_{w\in S} m(r,w;g) + S(r,g).
\end{aligned}$$

From second fundamental theorem we get

$$\begin{aligned}
(8)\quad (n+2)\,T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,c;f) + \overline{N}(r,\infty;f) \\
&\quad + \overline{N}(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\ge 2) - N_c(r,0;f^{'}) + S(r,f),
\end{aligned}$$

where $N_c(r,0;f^{'})$ is the counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-b)(f-c)$ and $F-1$. Also we note that $\overline{N}_0(r,0;f^{'}) =$

$\overline{N}(r,c;f\mid\geq 2)+\overline{N}_c(r,0;f^{'})$. Hence using (7), Lemmas 1 and 2 we get from (8) for $\varepsilon>0$ that

$$\begin{aligned}
(9)\quad &(n+2)\,T(r,f)\\
&\leq 2\overline{N}(r,0;f)+2\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,c;f)+\overline{N}(r,0;g)\\
&\quad+\overline{N}(r,b;g)+\overline{N}(r,1;F\mid\geq 2)-N_c(r,0;f^{'})+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+\ \overline{N}_*(r,\infty;f,g)+S(r,f)\\
&\leq 2\overline{N}(r,0;f)+2\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)\\
&\quad+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;g^{'})+\overline{N}(r,c;f\mid\geq 2)\\
&\quad+\overline{N}(r,c;f)+\frac{1}{2(k+1)}\{N(r,\infty;f\mid\geq k+1)+N(r,\infty;g\mid\geq k+1)\}\\
&\quad+S(r,f)+S(r,g)\\
&\leq 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)\right\}\\
&\quad+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+p^*N(r,c;f)\\
&\quad+\frac{1}{2(k+1)}\{N(r,\infty;f\mid\geq k+1)+N(r,\infty;g\mid\geq k+1)\}\\
&\quad-T(r,g)-\sum_{w\in S}m(r,w;g)+S(r,f)+S(r,g))\\
&\leq\left(5+p^*+\frac{1}{2(k+1)}-\Theta_f-p^*\delta(c;f)+\frac{1}{2}\varepsilon\right)T(r,f)\\
&\quad+\ \left(4+\frac{1}{2(k+1)}-\Theta_g-\delta_g+\frac{1}{2}\varepsilon\right)T(r,g)+S(r,f)+S(r,g)\\
&\leq\left(9+p^*+\frac{1}{k+1}-\Theta_f-\Theta_g-\delta_g-p^*\delta(c;f)+\varepsilon\right)T(r)+S(r),
\end{aligned}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$.

Also from second fundamental theorem for g we get

$$\begin{aligned}
(10)\qquad (n+2)\,T(r,g)&\leq\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,d;g)+\overline{N}(r,\infty;g)\\
&\quad+\overline{N}(r,1;G\mid=1)+\overline{N}(r,1;G\mid\geq 2)-N_d(r,0;g^{'})+S(r,g),
\end{aligned}$$

where $N_d(r,0;g^{'})$ is the counting function of those zeros of $g^{'}$ which are not the zeros of $g(g-b)(g-d)$ and $G-1$. Also we note that $\overline{N}_0(r,0;g^{'})=\overline{N}(r,d;g\mid\geq 2)+\overline{N}_d(r,o;g^{'})$.

In a similar way to (9) we can obtain

$$\begin{aligned}
(11)\qquad &(n+2)\,T(r,g)\\
&\leq\Big(9+p^*+\frac{1}{k+1}-\Theta_f-\Theta_g-\delta_f-p^*\delta(d;g)+\varepsilon\Big)T(r)+S(r).
\end{aligned}$$

Combining (9) and (11) we see that

$$(n+2)\,T(r) \leq \Big(9+p^{*}+\frac{1}{k+1}-\Theta_{f}-\Theta_{g}-\min\{\delta_{f},\delta_{g}\} \\ -p^{*}\min\{\delta(c;f),\delta(d;g)\}+\varepsilon\Big)T(r)+S(r).$$

That is

$$\Big(n-7-p^{*}-\frac{1}{k+1}+\Theta_{f}+\Theta_{g}+p^{*}\min\{\delta(c;f),\delta(d;g)\} \\ +\min\{\delta_{f},\delta_{g}\}-\varepsilon\Big)T(r)\leq S(r) \tag{12}$$

Since $\varepsilon > 0$, (12) leads to a contradiction. While $m = 1$, using Lemma 3, (7) changes to

$$\begin{aligned}
&\overline{N}_{0}(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_{*}(r,1;F,G) \\
&\leq \overline{N}_{0}(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_{L}(r,1;G)+\overline{N}(r,1;F\mid\geq 3) \\
&\leq N(r,0;g^{'}\mid g\neq 0,b)+\frac{1}{2}\sum_{j=1}^{n}\{N(r,\omega_{j};f)-\overline{N}(r,\omega_{j};f)\} \\
&\leq N(r,0;g^{'}\mid g\neq 0,b)+\frac{1}{2}N(r,0;f^{'}\mid f\neq 0,b) \\
&\leq \overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)-T(r,g)-\sum_{w\in S}m(r,w;g) \\
&\quad+\frac{1}{2}\left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)\right\}-\frac{1}{2}T(r,f) \\
&\quad-\frac{1}{2}\sum_{w\in S}m(r,w;f)+S(r,f)+S(r,g)
\end{aligned} \tag{13}$$

So using (13), Lemmas 1 and 2 and using the same argument as in (8) we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned}
&(n+2)\,T(r,f) \\
&\leq \left\{\frac{5}{2}\overline{N}(r,0;f)+\frac{5}{2}\overline{N}(r,b;f)+\frac{3}{2}\overline{N}(r,\infty;f)\right\} \\
&\quad+2\overline{N}(r,0;g)+2\overline{N}(r,b;g)+\overline{N}(r,\infty;g)+p^{*}N(r,c;f) \\
&\quad+\frac{1}{2(k+1)}\{N(r,\infty;f\mid\geq k+1)+N(r,\infty;g\mid\geq k+1)\} \\
&\quad-T(r,g)-\sum_{w\in S}m(r,w;g)-\frac{1}{2}T(r,f) \\
&\quad-\frac{1}{2}\sum_{w\in S}m(r,w;f)+S(r,f)+S(r,g)
\end{aligned} \tag{14}$$

$$\leq \Big(6 + p^* + \frac{1}{2(k+1)} - \Theta_f - \frac{1}{2}\{\Theta(0; f) + \Theta(b; f)$$
$$+ \Theta(\infty; f) + \delta_f\} - p^*\delta(c; f) + \frac{1}{2}\varepsilon\Big) T(r, f)$$
$$+ \left(4 + \frac{1}{2(k+1)} - \Theta_g - \delta_g + \frac{1}{2}\varepsilon\right) T(r, g)$$
$$+ S(r, f) + S(r, g)$$
$$\leq \Big(10 + p^* + \frac{1}{k+1} - \Theta_f - \Theta_g$$
$$- \frac{1}{2}\{\Theta(0; f) + \Theta(b; f) + \Theta(\infty; f) + \delta_f\}$$
$$- p^*\delta(c; f) - \delta_g + \varepsilon\Big) T(r) + S(r).$$

Similarly we can obtain

$$(n+2)\, T(r, g) \leq \Big(10 + p^* + \frac{1}{k+1} - \Theta_f - \Theta_g \tag{15}$$
$$- \frac{1}{2}\{\Theta(0; g) + \Theta(b; g) + \Theta(\infty; g) + \delta_g\}$$
$$- p^*\delta(d; g) - \delta_f + \varepsilon\Big) T(r) + S(r).$$

Combining (14) and (15) we see that

$$\Big(n - 8 - p^* - \frac{1}{k+1} + \Theta_f + \Theta_g + \min\{\delta_f, \delta_g\} \tag{16}$$
$$+ p^* \min\{\delta(c; f), \delta(d; g)\}$$
$$+ \frac{1}{2} \min\{\Theta(0; f) + \Theta(b; f) + \Theta(\infty; f) + \delta_f,$$
$$\Theta(0; g) + \Theta(b; g) + \Theta(\infty; g) + \delta_g\} - \varepsilon\Big)\, T(r) \leq S(r).$$

Since $\varepsilon > 0$ is arbitrary, we see that (16) leads to a contradiction.

Subcase 1.2. $m = 0$. Using Lemma 3 we note that

$$\overline{N}_0(r, 0; g') + \overline{N}_E^{(2}(r, 1; F) + 2\overline{N}_L(r, 1; G) + 2\overline{N}_L(r, 1; F) \tag{17}$$
$$\leq \overline{N}_0(r, 0; g') + \overline{N}_E^{(2}(r, 1; G) + \overline{N}_L(r, 1; G)$$
$$+ \overline{N}_L(r, 1; G) + 2\overline{N}_L(r, 1; F)$$
$$\leq \overline{N}_0(r, 0; g') + \overline{N}(r, 1; G \mid\geq 2) + \overline{N}_L(r, 1; G) + 2\overline{N}_L(r, 1; F)$$
$$\leq N(r, 0; g' \mid g \neq 0, b) + \overline{N}(r, 1; G \mid\geq 2) + 2\overline{N}(r, 1; F \mid\geq 2)$$
$$\leq 2\{N(r, 0; g' \mid g \neq 0, b) + N(r, 0; f' \mid f \neq 0, b)\}$$

$$\begin{aligned}
&\leq 2\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,b;g)\\
&\quad+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,b;f)\}\\
&\quad-2T(r,f)-2T(r,g)-2\sum_{w\in S}m(r,w;f)\\
&\quad-2\sum_{w\in S}m(r,w;g)+S(r,f)+S(r,g).
\end{aligned}$$

Hence using (17), Lemmas 1 and 2 we get from second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned}
(n+2)\,T(r,f)&\leq \overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)\\
&\quad+\overline{N}(r,c;f)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)\\
&\quad+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)-N_c(r,0;f')+S(r,f)\\
&\leq \left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)\right\}+\overline{N}(r,\infty;f)\\
&\quad+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,c;f)+\overline{N}(r,c;f\mid\geq 2)\\
&\quad+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)+\overline{N}_0(r,0;g')\\
&\quad+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+3\overline{N}(r,0;g)\\
&\quad+2\overline{N}(r,\infty;g)+4\overline{N}(r,b;f)+3\overline{N}(r,b;g)+p^*N(r,c;f)\\
&\quad+\frac{1}{2(k+1)}\{N(r,\infty;f\mid\geq k+1)+N(r,\infty;g\mid\geq k+1)\}\\
&\quad-2T(r,f)-2T(r,g)\\
&\quad-2\sum_{w\in S}m(r,w;f)-2\sum_{w\in S}m(r,w;g)+S(r,f)+S(r,g)\\
&\leq \Big(15+p^*+\frac{1}{k+1}-\Theta_f-\Theta_g-\Theta(0;f)-\Theta(b;f)\\
&\quad-\Theta(\infty;f)-2\delta_f-\Theta(0;g)-\Theta(b;g)-\Theta(\infty;g)-2\delta_g\\
&\quad-\Theta(0;f)-\Theta(b;f)-\Theta(\infty;f)-p^*\delta(c;f)+\varepsilon\Big)T(r)+S(r).
\end{aligned}\tag{18}$$

In a similar manner we can obtain

$$\begin{aligned}
(n+2)\,T(r,g)&\leq \Big(15+p^*+\frac{1}{k+1}-\Theta_f-\Theta_g-\Theta(0;f)\\
&\quad-\Theta(b;f)-\Theta(\infty;f)-2\delta_f-\Theta(0;g)-\Theta(b;g)\\
&\quad-\Theta(\infty;g)-2\delta_g-\Theta(0;g)-\Theta(b;g)\\
&\quad-\Theta(\infty;g)-p^*\delta(b;g)+\varepsilon\Big)T(r)+S(r).
\end{aligned}\tag{19}$$

Combining (18) and (19) we see that

$$\begin{aligned}(20)\qquad \Big(n-13-p^{*}-\frac{1}{k+1}+\Theta_f+\Theta_g+\Theta(0;f)+\Theta(b;f)\\ +\Theta(\infty;f)+2\delta_f+\Theta(0;g)+\Theta(b;g)+\Theta(\infty;g)+2\delta_g\\ +\ \min\{\Theta(0;f)+\Theta(b;f)+\Theta(\infty;f),\Theta(0;g)+\Theta(b;g)\\ +\Theta(\infty;g)\}+p^{*}\{\delta(c;f),\delta(d;g)\}-\varepsilon\Big)T(r)\leq S(r).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we see that (20) leads to a contradiction.

Case 2. Suppose that $H \equiv 0$. Now proceeding in the same way as done in [2] we can prove $f \equiv g$. ■

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