

A. BUNYAWAT AND S. SUANTAI

**STRONG CONVERGENCE THEOREMS OF ONE-STEP  
ITERATIVE SCHEME FOR A COUNTABLE FAMILY  
OF MULTIVALUED MAPPINGS IN A BANACH SPACE**

ABSTRACT. In this paper, we introduce a new one-step iterative scheme for finding a common fixed point of a countable family of multivalued mappings in a real uniformly convex Banach space. By using the best approximation operators, a necessary and sufficient condition for strong convergence of the proposed method is given. Moreover, we establish some strong convergence theorems of the proposed iterative scheme under some control conditions.

KEY WORDS: multivalued mappings, common fixed point, best approximation operators.

*AMS Mathematics Subject Classification:* 47H09, 47H10.

**1. Introduction**

An element  $p \in K$  is called a *fixed point* of a single valued mapping  $T$  if  $p = Tp$  and of a multivalued mapping  $T$  if  $p \in Tp$ . The set of fixed points of  $T$  is denoted by  $F(T)$ .

Let  $X$  be a uniformly convex real Banach space and  $K$  be a nonempty closed convex subset of  $X$  and  $CB(K)$  be a family of nonempty closed bounded subsets of  $K$  and  $P(K)$  be a nonempty proximal bounded subsets of  $K$ . The *Hausdorff metric* on  $CB(X)$  is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$

for all  $A, B \in CB(X)$ .

Let  $X$  be a real Banach space. A subset  $K$  of  $X$  is called *proximal* if for each  $x \in X$ , there exists an element  $k \in K$  such that

$$d(x, k) = d(x, K),$$

where  $d(x, K) = \inf\{\|x - y\| : y \in K\}$  is the distance from the point  $x$  to the set  $K$ .

A single valued mapping  $T : K \rightarrow K$  is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in K$ . A multivalued mapping  $T : K \rightarrow CB(K)$  is said to be *nonexpansive* if  $H(Tx, Ty) \leq \|x - y\|$  for all  $x, y \in K$ .

A map  $T : K \rightarrow CB(K)$  is called *hemicompact* if, for any sequence  $\{x_n\}$  in  $K$  such that  $d(x_n, Tx_n) \rightarrow 0$  as  $n \rightarrow \infty$ , there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow p \in K$ . We note that if  $K$  is compact, then every multivalued mapping  $T$  is hemicompact.

A map  $T : K \rightarrow CB(K)$  is said to satisfy *Condition (I)* if there is a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$  for  $r \in (0, \infty)$  such that

$$d(x, Tx) \geq f(d(x, F(T)))$$

for all  $x \in K$ .

A family  $\{T_i : K \rightarrow CB(K), i \in \mathbb{N}\}$  is said to satisfy *Condition (II)* if there is a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$  for  $r \in (0, \infty)$  such that

$$d(x, T_i x) \geq f(d(x, \bigcap_{i=1}^{\infty} F(T_i)))$$

for all  $i \in \mathbb{N}$  and  $x \in K$ .

In 1969, Nadler [7] proved some fixed point theorems for multivalued contraction mappings of the nonempty closed and bounded subsets of a complete metric space and generalized a result of Edelstein for compact multivalued local contractions. Moreover, he gave a counterexample to a theorem about  $(\varepsilon, \lambda)$ -uniformly locally expansive mappings.

Hussain and Khan [6], in 2003, introduced the best approximation operator  $P_T$  to find fixed points of \*-nonexpansive multivalued mapping and proved strong convergence of its iterates on a closed convex unbounded subset of a Hilbert space.

In 2009, Shahzad and Zegeye [9] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multivalued maps in Banach spaces. They also relaxed compactness of the domain of  $T$  and constructed an iteration scheme which removes the restriction of  $T$ , namely,  $Tp = \{p\}$  for any  $p \in F(T)$ . Then, Abkar and Eslamian [2] generalized and modified the iteration of Shahzad and Zegeye [9] from two step of quasi-nonexpansive multivalued maps to multi-step of finite family of multivalued maps and removed the restriction  $Tp = \{p\}$  by used nonexpansiveness of  $P_{T_i}$ . They also proved strong convergence theorem of a common fixed point of  $\{T_i\}_{i=1}^m$  in a complete CAT(0) space.

In 2011, Cholamjiak and Suantai [4] introduced two new iterative procedures with errors for two multi-valued maps and proved strong convergence theorems of the proposed iterations in uniformly convex Banach spaces. Let

$T_1, T_2$  be two multivalued maps from  $D$  into  $P(D)$  and  $P_{T_i}x = \{y \in T_i x : \|x - y\| = d(x, T_i x)\}, i = 1, 2$ . Let  $\{x_n\}$  be the sequence defined by  $x_1 \in D$ ,

$$\begin{aligned} y_n &= \alpha'_n z'_n + \beta'_n x_n + (1 - \alpha'_n - \beta'_n)u_n, \quad n \geq 1, \\ x_{n+1} &= \alpha_n z_n + \beta_n y_n + (1 - \alpha_n - \beta_n)v_n, \quad n \geq 1, \end{aligned}$$

where  $z'_n \in P_{T_1}x_n$  and  $z_n \in P_{T_2}y_n$ .

Later, Chulamjiak et al. [3] introduced a modified Mann iteration as follow:  $x_1 \in D$ ,

$$x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)P_{T_n}x_n, \quad \forall n \geq 1.$$

They obtained weak and strong convergence theorems for a countable family of multi-valued mappings by using the best approximation operator in a Banach space and applied the main results to the problem of finding a common fixed point of a countable family of nonexpansive multi-valued mappings. In this work, they also gave some examples of multi-valued mappings  $T$  such that  $P_T$  are nonexpansive.

Recently, Abbas et al. [1] introduced a new one-step iterative process to approximate common fixed points of two multivalued nonexpansive mappings in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic boundary conditions. Let  $S, T : K \rightarrow CB(K)$  be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n, \quad n \in \mathbb{N}, \end{cases}$$

where  $y_n \in T x_n$  and  $z_n \in S x_n$  such that  $\|y_n - p\| \leq d(p, S x_n)$  and  $\|z_n - p\| \leq d(p, T x_n)$  whenever  $p$  is a fixed point of any one of the mappings  $S$  and  $T$ , and  $\{a_n\}, \{b_n\}, \{c_n\}$  are sequences of numbers in  $(0, 1)$  satisfying  $a_n + b_n + c_n = 1$ .

Later, Eslamian and Abkar [5] generalized and modified the iteration of Abbas et al. [1] from two mapping to the infinite family mappings  $\{T_i : i \in \mathbb{N}\}$  of multivalued mapping as follow:  $x_0 \in E$ ,

$$x_{n+1} = a_{n,0}x_n + a_{n,1}z_{n,1} + a_{n,2}z_{n,2} + \dots + a_{n,m}z_{n,m}, \quad n \geq 0,$$

where  $z_{n,i} \in P_{T_i}(x_n)$  and  $\{a_{n,k}\}$  are sequence of numbers in  $[0, 1]$  such that for every natural number  $n$ ,  $\sum_{k=0}^m a_{n,k} = 1$ . They proved strong convergence theorem of this iterative scheme to a common fixed point of  $\{T_i\}$  such that each  $P_{T_i}$  satisfies the condition (C).

In this paper, we introduced a new iteration for a countable family of multivalued mapping  $\{T_i\}$  in a uniformly convex Banach space. Let  $P(K)$

be nonempty proximal bounded subsets of  $K \subset X$  and  $\{T_i : K \rightarrow P(K)\}$  be a family of given multivalued mappings and

$$P_{T_i}(x) = \{y \in T_i(x) : \|x - y\| = d(x, T_i(x))\}.$$

For  $x_1 \in K$ , we define

$$(1) \quad x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^n \alpha_{n,i}x_{n,i},$$

where the sequences  $\{\alpha_{n,i}\} \subset [0, 1)$  satisfying  $\sum_{i=0}^n \alpha_{n,i} = 1$  and  $x_{n,i} \in P_{T_i}x_n$  for  $i = 1, 2, \dots, n$ . The main purpose of this paper is to prove strong convergence of the iterative scheme (1) to a common fixed point of  $\{T_i\}$ .

To prove our main results, the following lemma is needed.

**Lemma 1** ([8]). *Suppose that  $X$  is a uniformly convex Banach space and  $0 < p \leq t_n \leq q < 1$  for all positive integers  $n$ . Also suppose that  $\{x_n\}$  and  $\{y_n\}$  are two sequences of  $X$  such that  $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$ ,  $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$  holds for some  $r \geq 0$ . Then  $\limsup_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

## 2. Main results

We first prove that the sequence  $\{x_n\}$  generated by (1) is an approximative fixed point sequence of each  $T_i$  ( $i = 1, 2, \dots, n$ ).

**Lemma 2.** *Let  $K$  be a nonempty closed convex subset of an uniformly convex Banach space  $X$ . For  $i = 1, 2, \dots, n$ , let  $\{T_i\}$  be a sequence of multivalued mappings from  $K$  into  $P(K)$  with  $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$  such that all  $P_{T_i}$  are nonexpansive. Let  $\{x_n\}$  be a sequence defined by (1). Then*

- (i)  $\|x_{n+1} - p\| \leq \|x_n - p\|$  for all  $p \in F$ ,
- (ii)  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F$ .

**Proof.** Let  $p \in F$ . Then  $P_{T_i}p = \{p\}$ . Hence

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^n \alpha_{n,i}\|x_{n,i} - p\| \\ &= \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^n \alpha_{n,i}d(x_{n,i}, P_{T_i}p) \\ &\leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^n \alpha_{n,i}H(P_{T_i}x_n, P_{T_i}p) \\ &\leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^n \alpha_{n,i}\|x_n - p\| = \|x_n - p\|. \end{aligned}$$

So (i) is satisfied, hence (ii) is obtained from (i). ■

**Theorem 1.** *Let  $K$  be a nonempty closed convex subset of an uniformly convex Banach space  $X$ . For  $i = 1, 2, \dots, n$ , let  $\{T_i\}$  be a sequence of multi-valued mappings from  $K$  into  $P(K)$  with  $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$  such that all  $P_{T_i}$  are nonexpansive. Let  $\{x_n\}$  and  $\{x_{n,i}\}, i \in \mathbb{N}$  be the sequences defined by (1). If  $\lim_{n \rightarrow \infty} \alpha_{n,i}$  and  $\lim_{n \rightarrow \infty} \alpha_{n,n}$  exist and lie in  $[0, 1)$  for all  $i \in \mathbb{N} \cup \{0\}$ , then*

- (i)  $\lim_{n \rightarrow \infty} \|x_n - x_{n,i}\| = 0$  for all  $i \in \mathbb{N}$  and
- (ii)  $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$  for all  $i \in \mathbb{N}$ .

**Proof.** Suppose that  $\lim_{n \rightarrow \infty} \|x_n - p\| = c$  for some  $c \geq 0$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_{n+1} - p\| &= \lim_{n \rightarrow \infty} \|\alpha_{n,0}(x_n - p) + \alpha_{n,1}(x_{n,1} - p) \\ &\quad + \alpha_{n,2}(x_{n,2} - p) + \dots + \alpha_{n,n}(x_{n,n} - p)\| \\ &= \lim_{n \rightarrow \infty} \left\| (1 - \alpha_{n,0}) \left[ \frac{\alpha_{n,1}}{1 - \alpha_{n,0}}(x_{n,1} - p) + \frac{\alpha_{n,2}}{1 - \alpha_{n,0}}(x_{n,2} - p) \right. \right. \\ &\quad \left. \left. + \dots + \frac{\alpha_{n,n}}{1 - \alpha_{n,0}}(x_{n,n} - p) \right] + \alpha_{n,0}(x_n - p) \right\| = c. \end{aligned}$$

Since all  $P_{T_i}$  are nonexpansive and  $F \neq \emptyset$ , we have  $\|x_{n,i} - p\| = d(x_{n,i}, P_{T_i} p) \leq H(P_{T_i} x_n, P_{T_i} p) \leq \|x_n - p\|$  for each  $p \in F$  and  $i \in \mathbb{N}$ .

Taking  $\limsup$  on both sides, we get  $\limsup_{n \rightarrow \infty} \|x_{n,i} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = c$  for all  $i \in \mathbb{N}$ . Next,

$$\begin{aligned} \limsup_{n \rightarrow \infty} &\left\| \frac{\alpha_{n,1}}{1 - \alpha_{n,0}}(x_{n,1} - p) + \frac{\alpha_{n,2}}{1 - \alpha_{n,0}}(x_{n,2} - p) \right. \\ &\quad \left. + \dots + \frac{\alpha_{n,n}}{1 - \alpha_{n,0}}(x_{n,n} - p) \right\| \\ &\leq \limsup_{n \rightarrow \infty} \left[ \frac{\alpha_{n,1}}{1 - \alpha_{n,0}} \|x_{n,1} - p\| + \frac{\alpha_{n,2}}{1 - \alpha_{n,0}} \|x_{n,2} - p\| \right. \\ &\quad \left. + \dots + \frac{\alpha_{n,n}}{1 - \alpha_{n,0}} \|x_{n,n} - p\| \right] \\ &\leq \limsup_{n \rightarrow \infty} \left[ \frac{\alpha_{n,1}}{1 - \alpha_{n,0}} \|x_n - p\| + \frac{\alpha_{n,2}}{1 - \alpha_{n,0}} \|x_n - p\| \right. \\ &\quad \left. + \dots + \frac{\alpha_{n,n}}{1 - \alpha_{n,0}} \|x_n - p\| \right] \\ &= \limsup_{n \rightarrow \infty} \left( \frac{\alpha_{n,1} + \alpha_{n,2} + \dots + \alpha_{n,n}}{1 - \alpha_{n,0}} \right) \|x_n - p\| \\ &= \limsup_{n \rightarrow \infty} \|x_n - p\| = c. \end{aligned}$$

It follows from Lemma 1 that

$$\lim_{n \rightarrow \infty} \left\| \frac{\alpha_{n,1}}{1 - \alpha_{n,0}}(x_{n,1} - p) + \frac{\alpha_{n,2}}{1 - \alpha_{n,0}}(x_{n,2} - p) + \dots + \frac{\alpha_{n,n}}{1 - \alpha_{n,0}}(x_{n,n} - p) - (x_n - p) \right\| = 0.$$

This yields

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left( \frac{1}{1 - \alpha_{n,0}} \right) \|\alpha_{n,1}x_{n,1} + \alpha_{n,2}x_{n,2} \\ &\quad + \dots + \alpha_{n,n}x_{n,n} + \alpha_{n,0}x_n - x_n\| \\ &= \lim_{n \rightarrow \infty} \left( \frac{1}{1 - \alpha_{n,0}} \right) \|x_{n+1} - x_n\|. \end{aligned}$$

This implies by our control condition that,  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

By using the argument as above, we get  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_{n,i}\| = 0$  for all  $i \in \mathbb{N}$ . Since  $\|x_n - x_{n,i}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n,i}\|$ , we obtain  $\lim_{n \rightarrow \infty} \|x_n - x_{n,i}\| = 0$  for all  $i \in \mathbb{N}$ . Since  $d(x_n, T_i x_n) = d(x_n, P_{T_i} x_n) \leq \|x_n - x_{n,i}\|$ , it follows that  $d(x_n, T_i x_n) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $i \in \mathbb{N}$ .  $\blacksquare$

**Theorem 2.** *Let  $X$  be an uniformly convex Banach space and  $K$  be a nonempty closed convex subset of  $X$ . For  $i = 1, 2, \dots, n$ , let  $\{T_i\}$  be a sequence of multivalued mappings from  $K$  into  $P(K)$  with  $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$  such that all  $P_{T_i}$  are nonexpansive. Let  $\{x_n\}$  be a sequence defined by (1) with the conditions that  $\lim_{n \rightarrow \infty} \alpha_{n,i}$  and  $\lim_{n \rightarrow \infty} \alpha_{n,n}$  exist and lie in  $[0, 1)$  for all  $i \in \mathbb{N} \cup \{0\}$ . Then  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_i\}$  if and only if  $\lim_{n \rightarrow \infty} \inf d(x_n, F) = 0$ .*

**Proof.** The necessity is obvious. Conversely, assume that  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ . From Lemma 2(i), we get  $\|x_{n+1} - p\| \leq \|x_n - p\|$  for all  $p \in F$ . Hence  $d(x_{n+1}, F) \leq d(x_n, F)$ . Thus  $\lim_{n \rightarrow \infty} d(x_n, F)$  exists. By hypothesis, we have  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ .

Next, we will show that  $\{x_n\}$  is a Cauchy sequence in  $K$ .

Let  $\epsilon$  be arbitrary. Since  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ , there exists  $n_0$  such that for all  $n \geq n_0$ ,  $d(x_n, F) < \frac{\epsilon}{2}$ . Then there exist a  $p^* \in F$  such that  $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$ . For  $n \geq n_0$  and  $m \in \mathbb{N}$ , by Lemma 2(i), we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\|x_{n_0} - p^*\| < \epsilon.$$

Thus,  $\{x_n\}$  is a Cauchy sequence in  $K$ . Hence  $\lim_{n \rightarrow \infty} x_n = q$  for some  $q \in K$ . This implies by Theorem 1(i) that for each  $i \in \mathbb{N}$ ,

$$\begin{aligned} d(x_n, T_i q) &\leq d(x_n, x_{n,i}) + d(x_{n,i}, T_i q) \\ &= d(x_n, x_{n,i}) + d(x_{n,i}, P_{T_i} q) \end{aligned}$$

$$\begin{aligned} &\leq d(x_n, x_{n,i}) + H(P_{T_i}x_n, P_{T_i}q) \\ &\leq d(x_n, x_{n,i}) + d(x_n, q) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that  $d(q, T_iq) = 0$ . Hence  $q \in T_iq$  for all  $i \in \mathbb{N}$ . Consequently,  $q \in F$ . ■

**Theorem 3.** *Let  $X$  be an uniformly convex Banach space and  $K$  be a closed convex subset of  $X$ . For  $i = 1, 2, \dots, n$ , let  $\{T_i\}$  be a sequence of multivalued and continuous mappings from  $K$  into  $P(K)$  with  $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$  such that all  $P_{T_i}$  are nonexpansive. Let  $\{x_n\}$  be a sequence defined by (1) with the conditions that  $\lim_{n \rightarrow \infty} \alpha_{n,i}$  and  $\lim_{n \rightarrow \infty} \alpha_{n,n}$  exist and lie in  $[0, 1)$  for all  $i \in \mathbb{N} \cup \{0\}$ . Assume that one of  $T_i$  is hemicompact. Then  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_i\}$ .*

**Proof.** Suppose that  $T_{i_0}$  is hemicompact for some  $i_0 \in \mathbb{N}$ . Then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = q \in K$ . By Theorem 1(ii), we have  $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$  for all  $i \in \mathbb{N}$ . From continuity of  $T_i$ , we get  $d(x_{n_k}, T_i x_{n_k}) \rightarrow d(q, T_i q)$ . This implies that  $d(q, T_i q) = 0$  and  $q \in F$ . Since  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists, it follows that  $\{x_n\}$  converges strongly to  $q$ . ■

**Corollary 1.** *Let  $X$  be an uniformly convex Banach space and  $K$  be a closed convex subset of  $X$ . For  $i = 1, 2, \dots, n$ , let  $\{T_i\}$  be a sequence of multivalued mappings from  $K$  into  $P(K)$  with  $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$  such that all  $P_{T_i}$  are nonexpansive. Assume that  $\{T_i\}$  satisfying condition (II). Let  $\{x_n\}$  be a sequence defined by (1) with the conditions that  $\lim_{n \rightarrow \infty} \alpha_{n,i}$  and  $\lim_{n \rightarrow \infty} \alpha_{n,n}$  exist and lie in  $[0, 1)$  for all  $i \in \mathbb{N} \cup \{0\}$ . Then  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_i\}$ .*

**Proof.** Since  $\{T_i\}$  satisfies condition (II), we have  $d(x_n, T_i(x_n)) \geq f(d(x_n, F))$  where  $f$  is a nondecreasing function defined by  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$  for  $r \in (0, \infty)$ . By Theorem 1(ii), we have  $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$  for all  $i \in \mathbb{N}$ . It follows that  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ . By Theorem 2, we get the result. ■

**Remark 1.** The following sequences are the examples of the control sequences in Theorem 1-3 and Corollary 1:

1.  $\alpha_{n,i} = \frac{1}{n}$  for  $i = 0, 1, \dots, n$  and  $\alpha_{n,i} = 0$  otherwise for all  $n \in \mathbb{N}$ .  
We see that,  $\lim_{n \rightarrow \infty} \alpha_{n,i} = 0$  for  $i = 1, 2, \dots, n$  and  $\lim_{n \rightarrow \infty} \alpha_{n,n} = 0$ .
2.  $\alpha_{n,i} = \frac{1}{2}$  for  $i = n - 1, n$  and  $\alpha_{n,i} = 0$  otherwise for all  $n \in \mathbb{N}$ .  
We see that,  $\lim_{n \rightarrow \infty} \alpha_{n,i} = 0$  for  $i = 1, 2, \dots, n$  and  $\lim_{n \rightarrow \infty} \alpha_{n,n} = \frac{1}{2}$ .

**Acknowledgements.** This research is supported by Chiang Mai University, Chiang Mai, Thailand.

## References

- [1] ABBAS M., KHAN S.H., KHAN A.R., AGARWAL R.P., Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme, *Applied Mathematics Letters*, 24(2011), 97-102.
- [2] ABKAR A., ESLAMIAN M., Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in CAT(0) spaces, *Nonlinear Analysis*, 75(2012), 1895-1903.
- [3] CHOLAMJIAK P., CHOLAMJIAK W., CHO Y.J., SUANTAI S., Weak and strong convergence to common fixed points of a countable family of multi-valued mappings in Banach spaces, *Thai Journal of Mathematics*, 9(2011) 505-520.
- [4] CHOLAMJIAK W., SUANTAI S., Approximation of common fixed points of two quasi-nonexpansive multi-valued maps in Banach spaces, *Computers and Mathematics with Applications*, 61(2011), 941-949.
- [5] ESLAMIAN M., ABKAR A., One-step iterative process for a finite family of multivalued mappings, *Math. Comp. Model.*, 54(2011), 105-111.
- [6] HUSSAIN N., KHAN A.R., Applications of the best approximation operator to \*-nonexpansive maps in Hilbert spaces, *Numer. Funct. Anal. Optim.*, 24(2003), 327-338.
- [7] NADLER, JR. S.B., Multi-valued contraction mappings, *Pacific J. Mathematics*, 30(1969), 475-488.
- [8] SCHU J., Weak and strong convergence to fixed points of asymptotically non-expansive mappings, *Bull. Austral. Math. Soc.*, 43(1991), 153-159.
- [9] SHAHZAD N., ZEGEYE H., On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, *Nonlinear Analysis*, 71(2009), 838-844.

AUNYARAT BUNYAWAT  
DEPARTMENT OF MATHEMATICS  
FACULTY OF SCIENCE  
CHIANG MAI UNIVERSITY  
CHIANG MAI 50200, THAILAND  
*e-mail:* aunyarat@mwit.ac.th

SUTHEP SUANTAI  
DEPARTMENT OF MATHEMATICS  
FACULTY OF SCIENCE  
CHIANG MAI UNIVERSITY  
CHIANG MAI 50200, THAILAND  
*e-mail:* suthep.s@cmu.ac.th

*Received on 13.01.2013 and, in revised form, on 19.09.2013.*