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GLOBAL BEHAVIOR OF HIGHER ORDER A RATIONAL DIFFERENCE EQUATION

Abstract. The main objective of this paper is to study the global asymptotic stability and the periodic character of the rational difference equation

$$y_{n+1} = \frac{\alpha y_{n-2r-1}}{\beta + \gamma y_{n-2l}^{p} y_{n-2k}^{q}}, \quad n = 0, 1, \dots,$$

where the parameters α, β, γ, p, q are nonnegative real numbers and initial conditions are nonnegative real numbers, l, r, k are nonnegative integers, such that $l \leq k$ and $r \leq k$. Also, we give some numerical simulations to the equation to illustrate our results.

Key words: difference equation, periodic solution, globally asymptotically stable, semicycles.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

In this paper we investigate the global asymptotic stability and the periodic character of the rational difference equation

$$y_{n+1} = \frac{\alpha y_{n-2r-1}}{\beta + \gamma y_{n-2l}^{p} y_{n-2k}^{q}}, \quad n = 0, 1, \dots, \tag{1}$$

where the parameters α, β, γ, p, q and initial conditions are nonnegative real numbers, l, r, k are nonnegative integers, such that $l \leq k$, $r \leq k$ and $\beta + \gamma y_{n-2l}^{p} y_{n-2k}^{q} > 0$. Some numerical simulations to the equation are given to illustrate our results. Here, we recall some notations and results which will be useful in our investigation [14, 15, 16] .

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots x_{n-k}), \quad n = 0, 1, \dots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 (Equilibrium Point). *A point* $\bar{x} \in I$ *is called an equilibrium point of equation* (2) *if*

$$\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x}).$$

That is, $x_n = \bar{x}$ *for* $n \geq 0$, *is a solution of* (2), *or equivalently,* $\bar{x}$ *is a fixed point of* f.

Definition 2 (Periodicity). *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -k$.

Definition 3 (Semicycle). (*i*) *A positive semicycle of* $\{x_n\}_{n=-k}^{\infty}$ *of equation* (2) *consists of a "string" of terms* $\{x_s, x_{s+1}, \ldots, x_m\}$, *all terms greater than or equal to the equilibrium* $\bar{x}$, *with* $s \geq -k$ *and* $m \leq \infty$ *and such that*

$$\text{either } s = -k, \quad \text{or} \quad s > -k \quad \text{and} \quad x_{s-1} < \bar{x},$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

(*ii*) *A negative semicycle of* $\{x_n\}_{n=-k}^{\infty}$ *of equation* (2) *consists of a "string" of terms* $\{x_s, x_{s+1}, \ldots, x_m\}$, *all terms less than the equilibrium* $\bar{x}$, *with* $s \geq -k$ *and* $m \leq \infty$ *and such that*

$$\text{either } s = -k, \quad \text{or} \quad s > -k \quad \text{and} \quad x_{s-1} \geq \bar{x},$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

Definition 4 (Stability). (*i*) *The equilibrium point* $\bar{x}$ *of* (2) *is called* locally stable *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta,$$

we have $|x_n - \bar{x}| < \varepsilon$ *for all* $n \geq -k$.

(*ii*) *The equilibrium point* $\bar{x}$ *of* (2) *is called* locally asymptotically stable *if* $\bar{x}$ *is locally stable solution of* (2), *and there exists* $\gamma > 0$ *such that for all* $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \bar{x}$.

(*iii*) *The equilibrium point* $\bar{x}$ *of* (2) *is called a* global attractor *if for all* $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ *we have* $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) *The equilibrium point* $\bar{x}$ *of* (2) *is called* globally asymptotically stable *if* $\bar{x}$ *is locally stable and* $\bar{x}$ *is also global attractor.*

(v) *The equilibrium point* $\bar{x}$ *of* (2) *is called* unstable *if* $\bar{x}$ *is not locally stable.*

Definition 5. *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of equation* (2) *is called nonoscillatory if there exists* $N \geq -k$ *such that either*

$$x_n > \bar{x}, \quad \textit{for all} \quad n \geq N,$$

or

$$x_n < \bar{x}, \quad \textit{for all} \quad n \geq N,$$

and it is called oscillatory if it is not nonoscillatory.

The linearized equation of equation (2) about the equilibrium $\bar{x}$ is the linear difference equation

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} z_{n-i}, \quad n = 0, 1, \ldots \tag{3}$$

The characteristic equation of equation(2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} \lambda^{k-i} = 0. \tag{4}$$

Theorem 1. (i) *If all roots of equation* (4) *have absolute values less than one, then the equilibrium point* $\bar{x}$ *of equation* (2) *is locally asymptotically stable.*

(ii) *If at least one of the roots of equation* (4) *has absolute value greater than one, then the equilibrium point* $\bar{x}$ *of equation* (2) *is unstable. The equilibrium point* $\bar{x}$ *of equation* (2) *is called a saddle point if equation* (4) *has roots both inside and outside the unit disk.*

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior of the solution of difference equations for example: Hamza et al. [13] studied the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_{n-2l}x_{n-2k}}.$$

In [10], El-Owaidy et al. studied the dynamics of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}.$$

Also in [3], Battaloglu et al. studied the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_{n-(k+1)}^p}.$$

Ahmed [1] studied the global asymptotic behavior and the periodic character of solutions of the third-order rational difference equation

$$x_{n+1} = \frac{b x_{n-1}}{A + B x_n^p x_{n-2}^q}.$$

Other related results on rational difference equations can be found in refs. [2, 4, 5, 6, 7, 8, 9, 11, 12, 13, 17, 18, 19, 20, 21].

2. Main results

The change of variables $y_n = (\frac{\beta}{\gamma})^{\frac{1}{p+q}} x_n$ reduces equation (1) to the difference equation

$$x_{n+1} = \frac{a x_{n-2r-1}}{1 + x_{n-2l}^p x_{n-2k}^q}, \quad n = 0, 1, \ldots \tag{5}$$

where $a = \frac{\alpha}{\beta} > 0$. Note that $\bar{x}_1 = 0$ is always an equilibrium point of equation (5). When $a > 1$, equation (5) also possesses the unique positive equilibrium $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$.

3. Local and global stability

Theorem 2. *Assume that $r < k$ for equation (5), then the following statements are true:*

(*i*) *If $a < 1$, then the equilibrium point $\bar{x}_1 = 0$ of equation (5) is locally asymptotically stable.*

(*ii*) *If $a > 1$, then the equilibrium point $\bar{x}_1 = 0$ of equation (5) is a saddle point.*

(*iii*) *If $a > 1$, then the equilibrium point $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ of equation (5) is unstable.*

Proof. The linearized equation of equation (5) about the equilibrium point $\bar{x}_1 = 0$ is

$$z_{n+1} - a z_{n-2r-1} = 0, \quad n = 0, 1, \ldots,$$

so the associated characteristic equation about $\bar{x}_1$ is

$$\lambda^{2k+1} - a\lambda^{2k-2r-1} = 0,$$

so

$$\lambda^{2k-2r-1}(\lambda^{2r+2} - a) = 0,$$

then $\lambda = 0$, $\lambda = \pm \sqrt[2r+2]{a}$. Then the proof of (i) and (ii) follows from Theorem 1. Now the linearized equation of equation (5) about the equilibrium point $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ is

$$z_{n+1} - z_{n-2r-1} + \frac{p(a-1)}{a} z_{n-2l} + \frac{q(a-1)}{a} z_{n-2k} = 0 \quad n = 0, 1, \ldots,$$

and the associated characteristic equation about $\bar{x}_2$ is

$$\lambda^{2k+1} - \lambda^{2k-2r-1} + \frac{p(a-1)}{a}\lambda^{2k-2l} + \frac{q(a-1)}{a} = 0.$$

Let

$$G(\lambda) = \lambda^{2k+1} - \lambda^{2k-2r-1} + \frac{p(a-1)}{a}\lambda^{2k-2l} + \frac{q(a-1)}{a},$$

then $G(-1) = \frac{(p+q)(a-1)}{a} > 0$, and $\lim_{\lambda \to -\infty} G(\lambda) = -\infty$, so $G(\lambda)$ has at least a real root in $(-\infty, -1)$. Consequently, $\bar{x}_2$ is unstable. This completes the proof. ■

Theorem 3. *Assume that $r < k$ and $a < 1$, then the equilibrium point $\bar{x}_1 = 0$ of equation* (5) *is globally asymptotically stable.*

Proof. It was shown by Theorem 2 that the equilibrium point $\bar{x}_1 = 0$ of equation (5) is locally asymptotically stable when $a < 1$. So, it is suffices to show that

$$\lim_{n\to\infty} x_n = 0.$$

Let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of equation (5). We have

$$x_{n+1} = \frac{a x_{n-2r-1}}{1 + x_{n-2l}^p x_{n-2k}^q} \leq a x_{n-2r-1}.$$

Then it can be written for $\eta = 0, 1, \ldots$

$$\begin{aligned} x_{2\eta(r+1)+1} &\leq a^{\eta+1} x_{-2r-1}, \\ x_{2\eta(r+1)+2} &\leq a^{\eta+1} x_{-2r}, \\ &\vdots \\ x_{2\eta(r+1)+2r+2} &\leq a^{\eta+1} x_0. \end{aligned}$$

If $a < 1$, then $\lim_{\eta\to\infty} a^{\eta+1} = 0$, and $\lim_{n\to\infty} x_n = 0$. This completes the proof. ■

Theorem 4. *Assume that $r = k$ for equation* (5), *then the following statements are true:*

(*i*) *If $a < 1$, then the equilibrium point $\bar{x}_1 = 0$ of equation* (5) *is locally asymptotically stable.*

(*ii*) *If $a > 1$, then the equilibrium point $\bar{x}_1 = 0$ of equation* (5) *is unstable.*

(*iii*) *If $a > 1$, then the equilibrium point $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ of equation* (5) *is a saddle point.*

Proof. The linearized equation of equation (5) about the equilibrium point $\bar{x}_1 = 0$ is

$$z_{n+1} - az_{n-2k-1} = 0, \quad n = 0, 1, \ldots,$$

so the associated characteristic equation about $\bar{x}_1$ is

$$\lambda^{2k+2} - a = 0,$$

then $\lambda = \pm \sqrt[2r+2]{a}$. Then the proof of (*i*) and (*ii*) follows from Theorem 1. Now the linearized equation of equation (5) about the equilibrium point $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ is

$$z_{n+1} + \frac{p(a-1)}{a} z_{n-2l} + \frac{q(a-1)}{a} z_{n-2k} - z_{n-2k-1} = 0, \quad n = 0, 1, \ldots,$$

and the associated characteristic equation about $\bar{x}_2$ is

$$\lambda^{2k+2} + \frac{p(a-1)}{a} \lambda^{2k-2l+1} + \frac{q(a-1)}{a} \lambda - 1 = 0.$$

Let $G(\lambda) = \lambda^{2k+2} + \frac{p(a-1)}{a}\lambda^{2k-2l+1} + \frac{q(a-1)}{a}\lambda - 1$, we have $G(0) = -1 < 0$, $G(1) = \frac{(p+q)(a-1)}{a} > 0$, then there exists a real root in $(0, 1)$. Also $G(-1) = \frac{-(p+q)(a-1)}{a} < 0$, and $\lim_{\lambda \to -\infty} G(\lambda) = \infty$, so $G(\lambda)$ has at least another root in $(-\infty, -1)$. Consequently, $\bar{x}_2$ is a saddle point. This completes the proof. ■

Theorem 5. *Assume that $r = k$ and $a < 1$, then the equilibrium point $\bar{x}_1 = 0$ of equation* (5) *is globally asymptotically stable.*

Proof. As the proof of Theorem 3. ■

4. Semicycle analysis

Theorem 6. *Assume that $r < k$, $a > 1$ and let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of equation* (5) *such that either*

$$(6) \quad x_{-2k}, x_{-2k+2}, \ldots, x_0 > \bar{x}_2 \quad \textit{and} \quad x_{-2k+1}, x_{-2k+3}, \ldots, x_{-1} < \bar{x}_2,$$

or

(7) $\quad x_{-2k}, x_{-2k+2}, \ldots, x_0 < \bar{x}_2 \quad$ *and* $\quad x_{-2k+1}, x_{-2k+3}, \ldots, x_{-1} > \bar{x}_2.$

Then $\{x_n\}_{n=-2k}^{\infty}$ *oscillates about* $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ *with semicycles of length one.*

Proof. Assume that (6) holds. Then,

$$x_1 = \frac{ax_{-2r-1}}{1 + x_{-2l}^p x_{-2k}^q} < \frac{a\bar{x}_2}{1 + \bar{x}_2^{p+q}} = \bar{x}_2,$$

and

$$x_2 = \frac{ax_{-2r}}{1 + x_{-2l+1}^p x_{-2k+1}^q} > \frac{a\bar{x}_2}{1 + \bar{x}_2^{p+q}} = \bar{x}_2.$$

The proof follows by induction. The case where (7) holds is similar and will be omitted. This completes the proof. ■

Theorem 7. *Assume that* $r = k$, $a > 1$ *and let* $\{x_n\}_{n=-2k-1}^{\infty}$ *be a solution of equation* (5) *such that either*

(8) $\quad x_{-2k}, x_{-2k+2}, \ldots, x_0 > \bar{x}_2 \quad$ *and* $\quad x_{-2k-1}, x_{-2k+1}, ..., x_{-1} < \bar{x}_2,$

or

(9) $\quad x_{-2k}, x_{-2k+2}, \ldots, x_0 < \bar{x}_2 \quad$ *and* $\quad x_{-2k-1}, x_{-2k+1}, ..., x_{-1} > \bar{x}_2.$

Then $\{x_n\}_{n=-2k-1}^{\infty}$ *oscillates about* $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ *with semicycles of length one.*

Proof. As the proof of Theorem 6. ■

5. Existence of periodic solutions

Theorem 8. *Assume that* $r = k$ *and* $a = 1$. *Then every solution of equation* (5) *converges to a periodic solution of equation* (5) *with period* $2(k+1)$ *and there exist periodic solutions of equation* (5) *with prime period* $2(k+1)$.

Proof. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of equation (5). For $n \geq 0$, we have

$$0 \leq x_{n+1} = \frac{x_{n-2k-1}}{1 + x_{n-2l}^p x_{n-2k}^q} \leq x_{n-2k-1}, \quad n = 0, 1, \ldots,$$

hence the subsequence $\{x_{2n(k+1)+m}\}_{n=-1}^{\infty}$ are decreasing for each $1 \leq m \leq 2k+2$. Let

$$\lim_{n\to\infty} x_{2n(k+1)+m} = \rho_m, \quad m = 1, 2, \ldots, 2k+2.$$

It is clear that $\{\ldots, \rho_1, \rho_2, \ldots, \rho_{2k+2}, \rho_1, \rho_2, \ldots, \rho_{2k+2}, \ldots\}$ is a $2(k+1)$ periodic solution of equation (5).

Now let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be distinct positive real numbers. It follows that the sequence

$$\ldots, \varphi_0, 0, \varphi_1, 0, \ldots, \varphi_k, 0, \varphi_0, 0, \varphi_1, 0, \ldots, \varphi_k, \ldots$$

is periodic solution of equation (5) with prime period $2(k+1)$. ■

Theorem 9. *Assume that $r < k$ and $a = 1$. Then every solution of equation (5) converges to a periodic solution of equation (5) with period $2(r+1)$ and there exist periodic solutions of equation (5) with prime period $2(r+1)$.*

Proof. As the proof of Theorem 8. ■

6. Boundedness of solutions of equation (5)

Theorem 10. *Assume that $r < k$ and $a = 1$, then every solution of equation (5) is bounded.*

Proof. Let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of equation (5). It follows from equation (5) that

$$x_{n+1} = \frac{x_{n-2r-1}}{1 + x_{n-2l}^{p} x_{n-2k}^{q}} \leq x_{n-2r-1}.$$

Then in view of the proof of Theorem 3, we have for $\eta = 0, 1, \ldots$

$$\begin{aligned} x_{2\eta(r+1)+1} &\leq x_{-2r-1}, \\ x_{2\eta(r+1)+2} &\leq x_{-2r}, \\ &\vdots \\ x_{2\eta(r+1)+2r+2} &< x_0. \end{aligned}$$

so every solution of equation (5) is bounded from above by $A = \max\{x_{-2r-1}, x_{-2r}, \ldots, x_0\}$. ■

Theorem 11. *Assume that $r = k$ and $a > 1$. Then equation (5) possesses unbounded solutions. In particular, every solution of equation (5) which oscillates about the equilibrium $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ with semicycles of length one is unbounded.*

Proof. We will prove that every solution $\{x_n\}_{n=-2k-1}^{\infty}$ of equation (5) which oscillates about $\bar{x}_2 = (a-1)^{\frac{1}{p+q}}$ with semicycles of length one is unbounded (see Theorem 7). Assume that $\{x_n\}_{n=-2k-1}^{\infty}$ is a solution of equation (5) such that

$$x_{-2k}, x_{-2k+2}, \ldots, x_0 > \bar{x}_2 \quad \text{and} \quad x_{-2k-1}, x_{-2k+1}, \ldots, x_{-1} < \bar{x}_2,$$

is satisfied. It follows that for all $i \geq 0$ and $0 \leq j \leq k$, we have

$$x_{2(k+1)(i+1)+2j} > x_{2(k+1)i+2j} \quad \text{and} \quad x_{2(k+1)(i+1)+2j+1} < x_{2(k+1)i+2j+1}.$$

Hence, for each $0 \leq j \leq k$

$$\lim_{i\to\infty} x_{2(k+1)i+2j} = L_{2j} \in (\bar{x}_2, \infty), \quad \text{and} \quad \lim_{i\to\infty} x_{2(k+1)i+2j+1} = L_{2j+1} \in [0, \bar{x}_2)$$

We will show that for each $0 \leq j \leq k$, $L_{2j+1} = 0$. For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, k\}$ with $L_{2j+1} \in (0, \bar{x}_2)$. Then

$$\begin{aligned} L_{2j+1} &= \lim_{i\to\infty} x_{2(k+1)(i+1)+2j+1} \\ &= \lim_{i\to\infty} \frac{a x_{2(k+1)i+2j+1}}{1 + x^p_{2(k+1)(i+1)+2j-2l} x^q_{2(k+1)i+2j+2}} \\ &= \frac{a L_{2j+1}}{1 + L^p_{2j-2l} L^q_{2j+2}}. \end{aligned}$$

So as

$$\lim_{i\to\infty} x_{2(k+1)i+2j+1} = L_{2j+1} \in (0, \bar{x}_2),$$

we have

$$a = 1 + L^p_{2j-2l} L^q_{2j+2} > a,$$

which is a contradiction. Thus it is true that for each $0 \leq j \leq k$, $L_{2j+1} = 0$, and so $\lim_{n\to\infty} x_{2n+1} = 0$.

Now we show that for each $0 \leq j \leq k$, $L_{2j} = \infty$. For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, k\}$ with $L_{2j} \in (\bar{x}_2, \infty)$. Then

$$\begin{aligned} L_{2j} &= \lim_{i\to\infty} x_{2(k+1)(i+1)+2j} \\ &= \lim_{i\to\infty} \frac{a x_{2(k+1)i+2j}}{1 + x^p_{2(k+1)(i+1)+2j-2l-1} x^q_{2(k+1)i+2j+1}} = a L_{2j}, \end{aligned}$$

so $\gamma = 1$, which is contradiction. Hence $\lim_{n\to\infty} x_{2n} = \infty$, and the proof is complete. ■

7. Numerical simulation

In this section, we give some numerical simulations supporting our theoretical analysis via the software package Matlab 7.13.

Example 1. Consider the following difference equation

$$y_{n+1} = \frac{4y_{n-3}}{20 + y_{n-2}^5 y_{n-4}^3}, \tag{10}$$

where $l = r = 1$, $k = 2$, $p = 5$, $q = 3$, $a = \frac{1}{5} < 1$. Figure 1 shows that the zero equilibrium point of equation (10) is globally asymptotically stable with initial data $y_{-4} = 15$, $y_{-3} = 3$, $y_{-2} = 10$, $y_{-1} = 2$, $y_0 = 5$.

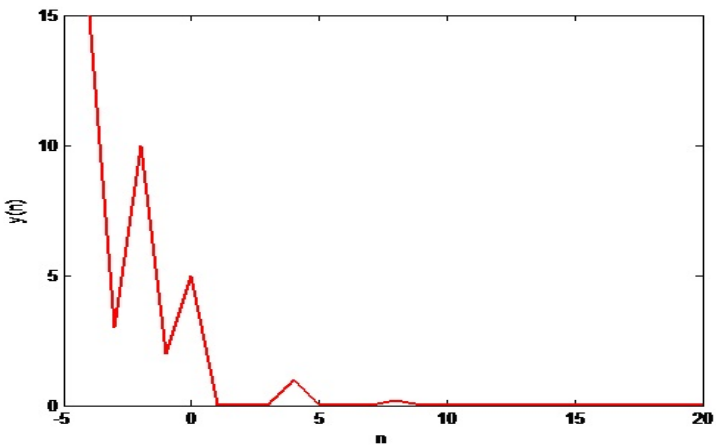

Figure 1.

Example 2. Consider the difference equation

$$y_{n+1} = \frac{5y_{n-3}}{10 + y_n^{0.5} y_{n-2}}, \tag{11}$$

where $l = 0$, $r = k = 1$, $p = 0.5$, $q = 1$, $a = \frac{1}{2} < 1$. Figure 2 shows that the zero equilibrium point of equation (11) is globally asymptotically stable, with initial data $y_{-3} = 5$, $y_{-2} = 2$, $y_{-1} = 2.8$, $y_0 = 3.5$.

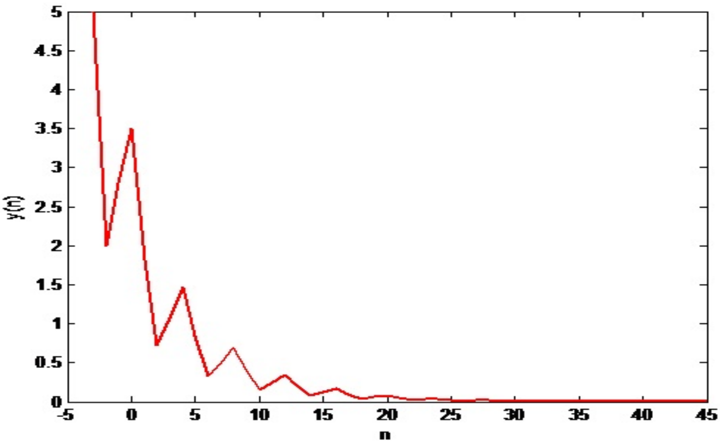

Figure 2.

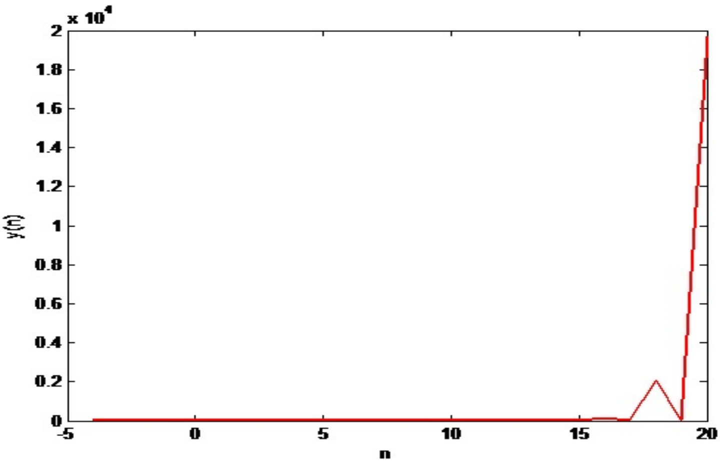

Figure 3.

Example 3. Consider the difference equation

$$y_{n+1} = \frac{126 y_{n-3}}{1 + y_{n-2} y_{n-4}^2}, \tag{12}$$

where $l = 1$, $r = 1$, $k = 2$, $p = 1$, $q = 2$, $a = 126 > 1$. Figure 3 shows that the solution of equation (12) oscillates about $(a-1)^{\frac{1}{3}} = 5$ with semicycles of length one. Where the initial data satisfies condition (6) of Theorem 6 $y_{-4} = 5.01$, $y_{-3} = 4.98$, $y_{-2} = 5.1$, $y_{-1} = 4.97$, $y_0 = 5.05$. (see Table 1)

n	y_n	n	y_n
1	4.863789779607879	11	2.500160694572724
2	5.171498511628885	12	12.229466489231854
3	4.731527270391876	13	1.409074511183059
4	5.252600948999005	14	29.820705455533090
5	4.611748889495370	15	0.399724489299515
6	5.769965613085051	16	1.571101831719999e+02
7	4.213898520842077	17	0.039799187249847
8	6.348781701777266	18	2.094840249200388e+03
9	3.627392718413308	19	3.604856918910693e-04
10	8.022493807080705	20	1.967079451259480e+04

Example 4. Consider the difference equation

$$y_{n+1} = \frac{y_{n-5}}{1 + y_n^2 y_{n-4}^3}, \tag{13}$$

where $l = 0$, $r = k = 2$, $p = 2$, $q = 3$, $a = 1$. Figure 4 shows that every solution of equation (13) converges to a periodic solution of equation (13) with period 6, with initial data $y_{-5} = 3$, $y_{-4} = 1$, $y_{-3} = 5$, $y_{-2} = 2$, $y_{-1} = 9$, $y_0 = 8$.

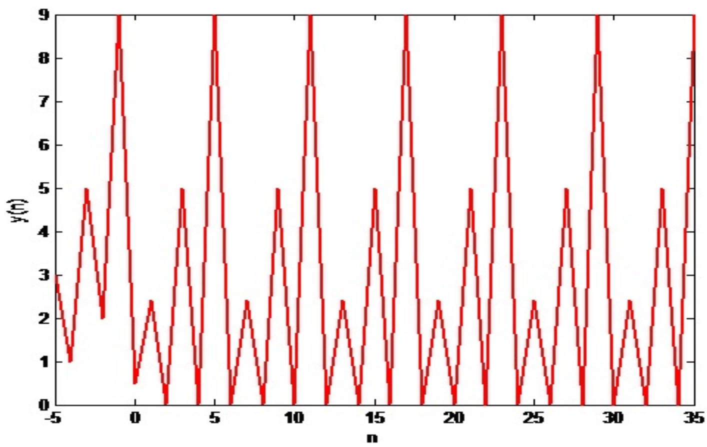

Figure 4.

Example 5. Consider the difference equation

$$y_{n+1} = \frac{y_{n-1}}{1 + y_{n-2}^{0.5} y_{n-6}^{0.4}}, \tag{14}$$

where $l = 1$, $r = 0$, $k = 3$, $p = 0.5$, $q = 0.4$, $a = 1$. Figure 5 shows that every solution of equation (14) is bounded, with initial data $y_{-6} = 3$, $y_{-5} = 5$, $y_{-4} = 10$, $y_{-3} = 12$, $y_{-2} = 15$, $y_{-1} = 20$, $y_0 = 25$.

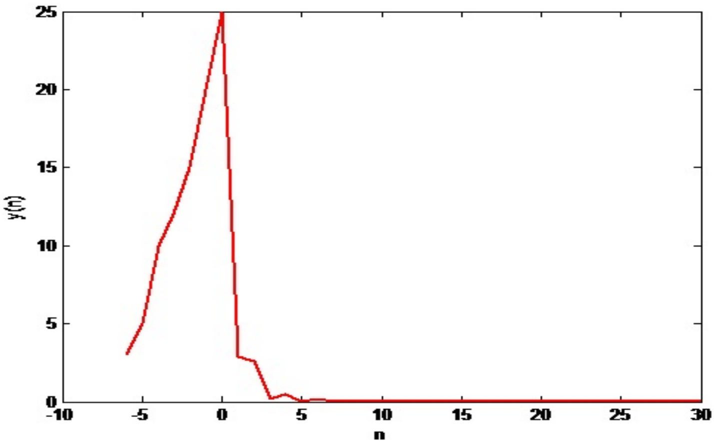

Figure 5.

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