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δ-LOCAL FUNCTION AND ITS PROPERTIES IN IDEAL TOPOLOGICAL SPACES

Abstract. In this paper, we investigated δ-local function and its properties in ideal topological space. Moreover, the relationships other local functions such as local function [1, 3] and semi-local function [2] are investigated.

Key words: ideal topological space, local function, δ-open set.

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1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930. This topic has won its importance by Vaidyanathaswamy [6]. In [1] Jankovic and Hamlett investigated further properties of ideal topological space. In this paper, we investigated δ-local function and its properties in ideal topological space. Moreover, the relationships other local functions [1, 3, 2] are investigated.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y), always mean topological spaces on which no separation axiom is assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively.

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [7] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called δ-open [7] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a δ-open set is called δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(U)) \cap A \neq \phi$ for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta Cl(A)$. The δ-interior of A is the union of all regular open sets of X contained in A and it is denoted by $\delta Int(A)$. A is δ-open if $\delta Int(A) = A$. δ-open sets forms a topology τ^{δ}. Actually τ^{δ} is the same as the collection of all δ-open sets of (X, τ) and is denoted by $\delta O(X)$. A subset A of a space (X, τ) is said to be semi-open [4] if $A \subset Cl(Int(A))$. The complement of a semi-open set is said to be semi-closed. The collection of all semi-open sets in X is denoted by $SO(X)$. The semi-closure of A in (X, τ) is defined by the intersection of all semi-closed sets containing A and is denoted by $sCl(A)$.

An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal topological space is a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$ called a local function [1, 3] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. We simply write A^* instead of $A^*(\mathcal{I},\tau)$. X^* is often a proper subset of X. The hypothesis $X = X^*$ [5] is equivalent to the hypothesis $\tau \cap \mathcal{I} = \phi$. For every ideal topological space, there exists a topology $\tau^*(\mathcal{I})$ or briefly τ^*, finer than τ, generated by $\beta(\mathcal{I},\tau) = \{U - I : U \in \tau$ and $I \in \mathcal{I}\}$, but in general $\beta(\mathcal{I},\tau)$ is not always a topology [1]. Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(\mathcal{I})$. If $\mathcal{I}$ is an ideal on X then $(X,\tau,\mathcal{I})$ is called an ideal topological space. Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A a subset of X. Then $A_*(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in SO(X,x)\}$ is called semi local function of A with respect to $\mathcal{I}$ and τ [2]. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. We say that the topology τ is *compatible* with the ideal $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following hold for every $A \subset X$, if for every $x \in A$ there exists a $U \in \tau$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ [1].

2. δ-local functions

Definition 1. *Let $(X,\tau,\mathcal{I})$ an ideal topological space and A a subset of X. Then $A^{\delta *}(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \delta O(X,x)\}$ is called δ-local function of A with respect to $\mathcal{I}$ and τ, where $\delta O(X,x) = \{U \in \delta O(X) : x \in U\}$. We denote simply $A^{\delta *}$ for $A^{\delta *}(\mathcal{I},\tau)$.*

Remark 1. The notions of the local function, semi local function and δ-local functions are independent notions as in the following example. Therefore, Remark 3.2(1) in [2] is false.

Example 1. Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a,b\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. Take $A = \{a,d\}$. Then $A^{\delta *} = X$, $A^* = \{c,d\}$ and $A_* = \{d\}$.

Example 2. Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a\}, \{a,c\}, \{a,b\}, \{a,b,c\}\}$ and $\mathcal{I} = \{\phi, \{a,c\}\}$. Take $A = \{a,c\}$. Then $A^{\delta *} = \phi$, $A^* = \{a,b\}$ and $A_* = \{a,b,d\}$.

Remark 2. (1) The simplest ideals are $\{\phi\}$ and $\mathcal{P}(X) = \{A : A \subset X\}$. It can be deduce that $A^{\delta *}(\{\phi\}) = \delta Cl(A) \neq Cl(A)$ and $A^{\delta *}(\mathcal{P}(X)) = \phi$ for every $A \subset X$.
(2) If $A \in \mathcal{I}$, then $A^{\delta *} = \phi$.
(3) Neither $A \subset A^{\delta *}$ nor $A^{\delta *} \subset A$ in general.

Theorem 1. *Let* $(X, \tau, \mathcal{I})$ *an ideal topological space and* A, B *subsets of* X. *Then for* δ*-local functions the following properties hold:*

(1) *If* $A \subset B$, *then* $A^{\delta*} \subset B^{\delta*}$,
(2) $A^{\delta*} = \delta Cl(A^{\delta*}) \subset \delta Cl(A)$ *and* $A^{\delta*}$ *is* δ*-closed,*
(3) $(A^{\delta*})^{\delta*} \subset A^{\delta*}$,
(4) $(A \cup B)^{\delta*} = A^{\delta*} \cup B^{\delta*}$,
(5) $A^{\delta*} - B^{\delta*} = (A - B)^{\delta*} - B^{\delta*} \subset (A - B)^{\delta*}$,
(6) *If* $U \in \tau^\delta$, *then* $U \cap A^{\delta*} = U \cap (U \cap A)^{\delta*} \subset (U \cap A)^{\delta*}$,
(7) *If* $U \in \mathcal{I}$, *then* $(A - U)^{\delta*} = A^{\delta*}$,
(8) *If* $A \subseteq A^{\delta*}$, *then* $A^{\delta*} = \delta Cl(A^{\delta*}) = \delta Cl(A)$.

Proof. (1) Suppose that $A \subset B$ and $x \notin B^{\delta*}$. Then there exists $U \in \delta O(X, x)$ such that $U \cap B \in \mathcal{I}$. Since $A \subset B$, $U \cap A \in \mathcal{I}$ and $x \notin A^{\delta*}$, i.e., $A^{\delta*} \subset B^{\delta*}$.

(2) $A^{\delta*} \subset \delta Cl(A^{\delta*})$ holds in general. Let $x \in \delta Cl(A^{\delta*})$. Then $A^{\delta*} \cap U \neq \phi$ for every $U \in \delta O(X, x)$. Therefore, there exists some $y \in A^{\delta*} \cap U$ and $U \in \delta O(X, y)$ since $y \in A^{\delta*}$, $A \cap U \notin \mathcal{I}$ and hence $x \in A^{\delta*}$. Thus $\delta Cl(A^{\delta*}) \subset A^{\delta*}$ and $\delta Cl(A^{\delta*}) = A^{\delta*}$. Now, let $x \in A^{\delta*}$, then $A \cap U \notin \mathcal{I}$ for every $U \in \delta O(X, x)$. This implies that $A \cap U \neq \phi$ for every $U \in \delta O(X, x)$ and so, $x \in \delta Cl(A)$. Consequently, $A^{\delta*} = \delta Cl(A^{\delta*}) \subset \delta Cl(A)$ and $A^{\delta*}$ is δ-closed.

(3) $x \in (A^{\delta*})^{\delta*}$. Then, for every $U \in \delta O(X, x)$, $A^{\delta*} \cap U \notin \mathcal{I}$ and hence $A^{\delta*} \cap U \neq \phi$. Let $y \in A^{\delta*} \cap U$. Then $U \in \delta O(X, y)$ and $y \in A^{\delta*}$. Thus we have $A \cap U \notin \mathcal{I}$ and $x \in A^{\delta*}$, i.e., $(A^{\delta*})^{\delta*} \subset A^{\delta*}$.

(4) $A^{\delta*} \cup B^{\delta*} \subset (A \cup B)^{\delta*}$ holds by (1). Now let $x \in (A \cup B)^{\delta*}$. Then, for every $U \in \delta O(X, x)$, $(U \cap A) \cup (U \cap B) = U \cap (A \cup B) \notin \mathcal{I}$. Therefore, $U \cap A \notin \mathcal{I}$ or $U \cap B \notin \mathcal{I}$. This implies that $x \in A^{\delta*}$ or $x \in B^{\delta*}$, that is, $x \in A^{\delta*} \cup B^{\delta*}$. So we obtain the equality.

(5) Since $A = (A - B) \cup (B \cap A)$, by (4), $A^{\delta*} = (A - B)^{\delta*} \cup (B \cap A)^{\delta*}$ and hence

$$\begin{aligned} A^{\delta*} - B^{\delta*} &= A^{\delta*} \cap (X - B^{\delta*}) \\ &= ((A - B)^{\delta*} \cup (B \cap A)^{\delta*}) \cap (X - B^{\delta*}) \\ &= ((A - B)^{\delta*} \cap (X - B^{\delta*})) \cup ((B \cap A)^{\delta*} \cap (X - B^{\delta*})) \\ &= ((A - B)^{\delta*} - B^{\delta*}) \cup \phi \subset (A - B)^{\delta*}. \end{aligned}$$

(6) Assume that $U \in \delta O(X)$ and $x \in U \cap A^{\delta*}$. Then $x \in U$ and $x \in A^{\delta*}$. For $V \in \delta O(X, x)$, $V \cap U \in \delta O(X, x)$, since δ-open sets forms a topology. Thus $V \cap (U \cap A) = (V \cap U) \cap A \notin \mathcal{I}$. Hence $x \in (U \cap A)^{\delta*}$. Therefore, $U \cap A^{\delta*} \subset (U \cap A)^{\delta*}$. Also, $U \cap A^{\delta*} \subset U \cap (U \cap A)^{\delta*}$ and by (1), $A^{\delta*} \supset (U \cap A)^{\delta*}$ and $U \cap A^{\delta*} \supset U \cap (U \cap A)^{\delta*}$. So, we get the result.

(7) Since $A \cap U \subset U \in I$, $A \cap U \in I$ and by Remark 2 $(A \cap U)^{\delta*} = \phi$. Since $A = (A - U) \cup (A \cap U)$, by (4) $A^{\delta*} = (A - U)^{\delta*} \cup (A \cap U)^{\delta*} = (A - U)^{\delta*}$. So, we get the result.

(8) For any subset A of X, by (2) we have $A^{\delta *} = \delta Cl(A^{\delta *}) \subset \delta Cl(A)$. Since $A \subseteq A^{\delta *}$, $\delta Cl(A) \subseteq \delta Cl(A^{\delta *})$ and hence $A^{\delta *} = \delta Cl(A^{\delta *}) = \delta Cl(A)$. ■

Theorem 2. *Let (X, τ) be a topological space with ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X and $A \subset X$. Then the following properties hold:*
(1) *If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $A^{\delta *}(\mathcal{I}_2) \subset A^{\delta *}(\mathcal{I}_1)$,*
(2) *$A^{\delta *}(\mathcal{I}_1 \cap \mathcal{I}_2) = A^{\delta *}(\mathcal{I}_1) \cup A^{\delta *}(\mathcal{I}_2)$.*

Proof. (1) Let $\mathcal{I}_1 \subset \mathcal{I}_2$ and $x \in A^{\delta *}(\mathcal{I}_2)$. Then $A \cap U \notin \mathcal{I}_2$ for every $U \in \delta O(X, x)$ and hence $A \cap U \notin \mathcal{I}_1$, i.e., $x \in A^{\delta *}(\mathcal{I}_1)$. Therefore, we have the result.

(2) By (1), we have $A^{\delta *}(\mathcal{I}_1) \subset A^{\delta *}(\mathcal{I}_1 \cap \mathcal{I}_2)$ and $A^{\delta *}(\mathcal{I}_2) \subset A^{\delta *}(\mathcal{I}_1 \cap \mathcal{I}_2)$. Therefore, $A^{\delta *}(\mathcal{I}_1) \cup A^{\delta *}(\mathcal{I}_2) \subset A^{\delta *}(\mathcal{I}_1 \cap \mathcal{I}_2)$. Now, let $x \in A^{\delta *}(\mathcal{I}_1 \cap \mathcal{I}_2)$. Then for every $U \in \delta O(X, x)$, $U \cap A \notin \mathcal{I}_1 \cap \mathcal{I}_2$ and hence $U \cap A \notin \mathcal{I}_1$ or $U \cap A \notin \mathcal{I}_2$. This shows that $x \in A^{\delta *}(\mathcal{I}_1)$ or $x \in A^{\delta *}(\mathcal{I}_2)$. Thus $x \in A^{\delta *}(\mathcal{I}_1) \cup A^{\delta *}(\mathcal{I}_2)$. So, we get the result. ■

3. The open sets of $\tau^{\delta *}$

In this section, we have defined $\tau^{\delta *}$ in terms of the closure operator $\delta Cl^*(A) = A \cup A^{\delta *}$.

Theorem 3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, $\delta Cl^*(A) = A \cup A^{\delta *}$. and A, B subsets of X. Then*
(1) *$\delta Cl^*(\phi) = \phi$.*
(2) *$A \subseteq \delta Cl^*(A)$.*
(3) *$\delta Cl^*(A \cup B) = \delta Cl^*(A) \cup \delta Cl^*(B)$.*
(4) *$\delta Cl^*(A) = \delta Cl^*(\delta Cl^*(A))$.*

Proof. By Theorem 1, we obtain
(1) $\delta Cl^*(\phi) = (\phi)^{\delta *} \cup \phi = \phi$.
(2) $A \subseteq A \cup A^{\delta *} = \delta Cl^*(A)$.
(3) $\delta Cl^*(A \cup B) = (A \cup B)^{\delta *} \cup (A \cup B) = (A^{\delta *} \cup B^{\delta *}) \cup (A \cup B) = Cl^{\delta *}(A) \cup Cl^{\delta *}(B)$.
(4) $\delta Cl^*(\delta Cl^*(A)) = \delta Cl^*(A^{\delta *} \cup A) = (A^{\delta *} \cup A)^{\delta *} \cup (A^{\delta *} \cup A) = ((A^{\delta *})^{\delta *} \cup A^{\delta *}) \cup (A^{\delta *} \cup A) = A^{\delta *} \cup A = \delta Cl^*(A)$.

■

By Theorem 3, we obtain that $\delta Cl^*(A) = A \cup A^{\delta *}$ is a Kuratowski closure operator. We will denote by $\tau^{\delta *}$ the topology generated by δCl^*, that is, $\tau^{\delta *} = \{U \subseteq X : \delta Cl^*(X - U) = X - U\}$.

Lemma 1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B subsets of X. Then $A^{\delta *} - B^{\delta *} = (A - B)^{\delta *} - B^{\delta *}$.*

Proof. We have, by Theorem 1 $A^{\delta *} = [(A-B)\cup(A\cap B)]^{\delta *} = (A-B)^{\delta *}\cup(A\cap B)^{\delta *} \subseteq (A-B)^{\delta *}\cup B^{\delta *}$. Thus $A^{\delta *}-B^{\delta *} \subseteq (A-B)^{\delta *}-B^{\delta *}$. Also by Theorem 1, $(A-B)^{\delta *} \subseteq A^{\delta *}$ and hence $(A-B)^{\delta *}-B^{\delta *} \subseteq A^{\delta *}-B^{\delta *}$. Hence $A^{\delta *}-B^{\delta *} = (A-B)^{\delta *}-B^{\delta *}$. ■

Corollary 1. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A, B subsets of X with $B\in\mathcal{I}$. Then $(A\cup B)^{\delta *} = A^{\delta *} = (A-B)^{\delta *}$.*

Proof. Since $B\in\mathcal{I}$, by Remark 2 $B^{\delta *}=\phi$. By Lemma 1, $A^{\delta *} = (A-B)^{\delta *}$ and by Theorem 1 $(A\cup B)^{\delta *} = A^{\delta *}\cup B^{\delta *} = A^{\delta *}$ ■

Lemma 2. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A, B subsets of X. Then*

(1) *If $A\subseteq B$, then $\delta Cl^*(A)\subseteq \delta Cl^*(B)$.*
(2) *$\delta Cl^*(A\cap B)\subseteq \delta Cl^*(A)\cap\delta Cl^*(B)$.*
(3) *If $U\in\tau^{\delta}$, then $U\cap\delta Cl^*(A)\subseteq\delta Cl^*(U\cap A)$.*

Proof. (1) Since $A\subseteq B$, by Theorem 1 we have $\delta Cl^*(A) = A\cup A^{\delta *} \subseteq B\cup B^{\delta *} = \delta Cl^*(B)$.

(2) This is obvious by (1).

(3) Since $U\in\tau^{\delta}$, by Theorem 1 we have $U\cap\delta Cl^*(A) = U\cap(A\cup A^{\delta *}) = (U\cap A)\cup(U\cap A^{\delta *}) \subseteq (U\cap A)\cup(U\cap A)^{\delta *} = \delta Cl^*(U\cap A)$. ■

The proof of the following Corollary follows from Theorem 1.

Corollary 2. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A\subseteq X$. If $A\subseteq A^{\delta *}$, then*

1. *$\delta Cl(A) = \delta Cl^*(A)$.*
2. *$\delta Int(X-A) = \delta Int^*(X-A)$.*

A basis for the open sets of $\tau^{\delta *}$ described as follow:

Let (X,τ) be a space, $\mathcal{I}$ an ideal on X and A is $\tau^{\delta *}$-closed if and only if $A^{\delta *}\subset A$. Thus we have $U\in\tau^{\delta *}$ if and only if $X-U$ is $\tau^{\delta *}$-closed if and only if $U\subset X-(X-U)^{\delta *}$. Thus if $x\in U$, $x\notin(X-U)^{\delta *}$, i.e., there exists a $V\in\delta O(X,x)$ such that $V\cap(X-U)\in\mathcal{I}$. Hence, let $I_o = V\cap(X-U)$ and we have $x\in V-I_o\subset U$, where $V\in\delta O(X,x)$ and $I_o\in\mathcal{I}$. Let us denote $\beta(\mathcal{I},\tau) = \{V-I_o\ :\ V\in\delta O(X),\ I_o\in\mathcal{I}\}$, simplicity $\beta(\mathcal{I},\tau)$ for β.

Theorem 4. *Let (X,τ) be a space, $\mathcal{I}$ an ideal on X. Then β is a basis for $\tau^{\delta *}$.*

Proof. Since $\phi\in\mathcal{I}$, then $\delta O(X)\subset\beta$ from which it follows that $X=\cup\beta$ (recall that δ-open sets forms a topology). Also for every $\beta_1,\beta_2\in\beta$, we have $\beta_1 = V_1-I_1$ and $\beta_2 = V_2-I_2$, where $V_1,V_2\in\delta O(X)$ and $I_1,I_2\in\mathcal{I}$. Then

$\beta_1 \cap \beta_2 = (V_1 - I_1) \cap (V_2 - I_2) = (V_1 \cap (X - I_1)) \cap (V_2 \cap (X - I_2)) = (V_1 \cap V_2) - (I_1 \cup I_2) \in \beta$, where $V_1 \cap V_2 \in \delta O(X)$, $I_1 \cup I_2 \in \mathcal{I}$. ■

Remark 3. The topology $\tau^{\delta *}$ finer than τ^{δ}. See the following example.

Example 3. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}, \{c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\phi, \{b\}\}$. Here, $\{a, c\} \in \tau^{\delta *}$, but $\{a, c\} \notin \delta O(X)$.

Remark 4. If $(X, \tau, \mathcal{I})$ is an ideal topological space, β is a basis for $\tau^{\delta *}$. If β is itself a topology, then we have $\beta = \tau^{\delta *}$ and all the open sets of $\tau^{\delta *}$ are of the simple form $V - I_0$ where $V \in \tau^{\delta *}$ and $I_0 \in \mathcal{I}$. The Example 3.6 in [1] also shows that β is not a topology in general. In the following section, we can see the condition relating τ and $\mathcal{I}$ that will guarantee β is a topology and hence all sets in $\tau^{\delta *}$ will be of simple form.

4. δ-compatible topology with an ideal

Definition 2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that the topology τ is δ-compatible with the ideal $\mathcal{I}$, denoted $\tau \sim^{\delta} \mathcal{I}$, if the following hold for every $A \subset X$, if for every $x \in A$ there exists a $U \in \delta O(X, x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.*

Remark 5. A δ-compatible space is a compatible, but the converse is not true in general.

Theorem 5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following are equivalent:*

(1) $\tau \sim^{\delta} \mathcal{I}$,

(2) *If a subset A of X has a cover of δ-open sets each of whose intersection with A is in $\mathcal{I}$, then A is in $\mathcal{I}$,*

(3) *For every $A \subset X$, if $A \cap A^{\delta *} = \phi$, $A \in \mathcal{I}$,*

(4) *For every $A \subset X$, if $A - A^{\delta *} \in \mathcal{I}$,*

(5) *For every $A \subset X$, if A contains no nonempty subset B with $B \subset B^{\delta *}$, then $A \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2) The proof is obvious.

(2) $\Rightarrow$ (3) Let $A \subset X$ and $x \in A$. Then $x \notin A^{\delta *}$ and there exists $U_x \in \delta O(X, x)$ such that $U_x \cap A \in \mathcal{I}$. Thus, $A \subset \cup\{U_x : x \in A\}$ and $U_x \in \delta O(X, x)$ by (2) $A \in \mathcal{I}$.

(3) $\Rightarrow$ (4) For any $A \subset X$, $A - A^{\delta *} \subset A$ and $(A - A^{\delta *}) \cap (A - A^{\delta *})^{\delta *} \subset (A - A^{\delta *}) \cap A^{\delta *} = \phi$. By (3), $A - A^{\delta *} \in \mathcal{I}$.

(4) $\Rightarrow$ (5) By (4), for every $A \subset X$, $A - A^{\delta *} \in \mathcal{I}$. Let $A - A^{\delta *} = J \in \mathcal{I}$, then $A = J \cup (A \cap A^{\delta *})$ and by Theorem 1 $A^{\delta *} = J^{\delta *} \cup (A \cap A^{\delta *})^{\delta *} = (A \cap A^{\delta *})^{\delta *}$ because Remark 2. Therefore, we have $A \cap A^{\delta *} = A \cap (A \cap A^{\delta *})^{\delta *} \subset$

$(A \cap A^{\delta*})^{\delta*}$ and $A \cap A^{\delta*} \subset A$. By the assumption $A \cap A^{\delta*} = \phi$ and hence $A = A - A^{\delta*} \in \mathcal{I}$.

(5) $\Rightarrow$ (1) Let $A \subset X$ and assume that for every $x \in A$, there exists $U \in \delta O(X, x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap A^{\delta*} = \phi$. Since $(A - A^{\delta*}) \cap (A - A^{\delta*})^{\delta*} \subset (A - A^{\delta*}) \cap A^{\delta*} = \phi$. So, $A - A^{\delta*}$ contains no nonempty subset B with $B \subset B^{\delta*}$. By (5), $A - A^{\delta*} \in \mathcal{I}$ and hence $A = A \cap (X - A^{\delta*}) = A - A^{\delta*} \in \mathcal{I}$. ■

Theorem 6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*
(1) $\tau \sim^{\delta} \mathcal{I}$;
(2) *For every $\tau^{\delta*}$-closed subset A, $A - A^{\delta*} \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2) It is clear by Theorem 5.

(2) $\Rightarrow$ (1) Let $A \subseteq X$ and assume that for every $x \in A$, there exists an δ-open set U containing x such that $U \cap A \in \mathcal{I}$. Then $A \cap A^{\delta*} = \phi$. Since $Cl^{\delta*}(A) = A \cup A^{\delta*}$ is $\tau^{\delta*}$-closed, we have $(A \cup A^{\delta*}) - (A \cup A^{\delta*})^{\delta*} \in \mathcal{I}$. Moreover, $(A \cup A^{\delta*}) - (A \cup A^{\delta*})^{\delta*} = (A \cup A^{\delta*}) - (A^{\delta*} \cup (A^{\delta*})^{\delta*}) = (A \cup A^{\delta*}) - A^{\delta*} = A$. Therefore $A \in \mathcal{I}$. ■

Theorem 7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is δ-compatible with $\mathcal{I}$, then the following equivalent properties hold:*
(1) *For every $A \subseteq X$, $A \cap A^{\delta*} = \phi$ implies that $A^{\delta*} = \phi$.*
(2) *For every $A \subseteq X$, $(A - A^{\delta*})^{\delta*} = \phi$.*
(3) *For every $A \subseteq X$, $(A \cap A^{\delta*})^{\delta*} = A^{\delta*}$.*

Proof. First, we show that (1) holds if τ is δ-compatible with $\mathcal{I}$. Let A be any subset of X and $A \cap A^{\delta*} = \phi$. By Theorem 5, $A \in \mathcal{I}$ and by Remark 2 $A^{\delta*} = \phi$.

(1) $\Rightarrow$ (2) Assume that for every $A \subseteq X$, $A \cap A^{\delta*} = \phi$ implies that $A^{\delta*} = \phi$. Let $B = A - A^{\delta*}$, then

$$\begin{aligned} B \cap B^{\delta*} &= (A - A^{\delta*}) \cap (A - A^{\delta*})^{\delta*} \\ &= (A \cap (X - A^{\delta*})) \cap (A \cap (X - A^{\delta*}))^{\delta*} \\ &\subseteq [A \cap (X - A^{\delta*})] \cap [A^{\delta*} \cap (X - A^{\delta*})^{\delta*}] = \phi. \end{aligned}$$

By (1) we have $B^{\delta*} = \phi$. Hence $(A - A^{\delta*})^{\delta*} = \phi$.

(2) $\Rightarrow$ (3) Assume for every $A \subseteq X$, $(A - A^{\delta*})^{\delta*} = \phi$.

$$\begin{aligned} A &= (A - A^{\delta*}) \cup (A \cap A^{\delta*}) \\ A^{\delta*} &= [(A - A^{\delta*}) \cup (A \cap A^{\delta*})]^{\delta*} \\ &= (A - A^{\delta*})^{\delta*} \cup (A \cap A^{\delta*})^{\delta*} \\ &= (A \cap A^{\delta*})^{\delta*}. \end{aligned}$$

(3) $\Rightarrow$ (1) Assume for every $A \subseteq X$, $A \cap A^{\delta *} = \phi$ and $(A \cap A^{\delta *})^{\delta *} = A^{\delta *}$. This implies that $\phi = \phi^{\delta *} = A^{\delta *}$. ■

Corollary 3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is δ-compatible with $\mathcal{I}$, then $()^{\delta *}$ is an idempotent operator i.e. $A^{\delta *} = (A^{\delta *})^{\delta *}$ for any subset A of X.*

Proof. By Theorems 7 and 1 we obtain $A^{\delta *} = (A \cap A^{\delta *})^{\delta *} \subseteq A^{\delta *} \cap (A^{\delta *})^{\delta *} = (A^{\delta *})^{\delta *}$ and by Theorem 1 we have $A^{\delta *} = (A^{\delta *})^{\delta *}$ for any subset A of X. ■

Theorem 8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and τ δ-compatible with $\mathcal{I}$. A set is closed with respect to $\tau^{\delta *}$-topology if and only if it is the union of a set which is δ-closed with respect to τ and a set in $\mathcal{I}$.*

Proof. Let A be $\tau^{\delta *}$-closed, then $A^{\delta *} \subseteq A$ implies that $A = (A - A^{\delta *}) \cup A^{\delta *}$. Now $A - A^{\delta *} \in \mathcal{I}$ by Theorem 6 and $A^{\delta *}$ is δ-closed with respect to τ by Theorem 1.

Conversely, if $A = B \cup I$, where B is δ-closed with respect to τ and $I \in \mathcal{I}$, then by Theorem 1 and Remark 2 we have $A^{\delta *} = B^{\delta *} \cup I^{\delta *} = B^{\delta *} \subseteq \delta Cl(B) = B \subseteq A$. Thus $A^{\delta *} \subseteq A$ and A is $\tau^{\delta *}$-closed. ■

Corollary 4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is δ-compatible with $\mathcal{I}$, then $\beta(\tau, \mathcal{I}) = \tau^{\delta *}$.*

Proof. Let $U \in \tau^{\delta *}$. Then by Theorem 8 $X - U = F \cup B$, where F is δ-closed and $B \in \mathcal{I}$. Then $U = X - (F \cup B) = (X - F) \cap (X - B) = (X - F) - B = V - B$ where $V = X - F \in \delta O(X)$. Thus every $\tau^{\delta *}$-open set is of the form $V - B$, where $V \in \delta O(X)$ and $B \in \mathcal{I}$. It follows from Theorem 4 that $\beta(\tau, \mathcal{I}) = \tau^{\delta *}$. ■

Theorem 9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*
(1) *$\tau^{\delta} \cap \mathcal{I} = \phi$;*
(2) *If $I \in \mathcal{I}$, then $\delta Int(I) = \phi$;*
(3) *For every $G \in \tau^{\delta}$, $G \subseteq G^{\delta *}$;*
(4) *$X = X^{\delta *}$.*

Proof. (1) $\Rightarrow$ (2) Let $\tau^{\delta} \cap \mathcal{I} = \phi$ and $I \in \mathcal{I}$. Suppose that $x \in \delta Int(I)$. Then there exists $U \in \tau^{\delta}$ such that $x \in U \subseteq I$. Since $I \in \mathcal{I}$ and hence $\phi \neq \{x\} \subseteq U \in \tau^{\delta} \cap \mathcal{I}$. This is contrary that $\tau^{\delta} \cap \mathcal{I} = \phi$. Therefore, $\delta Int(I) = \phi$.

(2) $\Rightarrow$ (3) Let $x \in G$. Assume $x \notin G^{\delta *}$ then there exists $U_x \in \tau^{\delta}(x)$ such that $G \cap U_x \in \mathcal{I}$. By (2), $x \in G \cap U_x = \delta Int(G \cap U_x) = \phi$. Hence $x \in G^{\delta *}$ and $G \subseteq G^{\delta *}$.

(3) $\Rightarrow$ (4) Since X is δ-open, then $X = X^{\delta *}$.

(4) $\Rightarrow$ (1) $X = X^{\delta *} = \{x \in X : U \cap X = U \notin \mathcal{I}$ for each δ-open set U containing $x\}$. Hence $\tau^{\delta} \cap \mathcal{I} = \phi$. ■

Theorem 10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, τ be δ-compatible with $\mathcal{I}$ and $\tau^{\delta} \cap \mathcal{I} = \phi$. Let G be an $\tau^{\delta *}$-open set such that $G = U - A$, where $U \in \tau^{\delta}$ and $A \in \mathcal{I}$. Then $\delta Cl(G^{\delta *}) = \delta Cl(G) = G^{\delta *} = U^{\delta *} = \delta Cl(U) = \delta Cl(U^{\delta *})$.*

Proof. (1) Let $G = U - A$, where $U \in \tau^{\delta}$ and $A \in \mathcal{I}$. Since $\tau^{\delta} \cap \mathcal{I} = \phi$, by Theorem 9 we have $U \subseteq U^{\delta *}$. Hence by Theorem 1 $U^{\delta *} = \delta Cl(U^{\delta *}) = \delta Cl(U)$.

(2) Now, by using $G \in \tau^{\delta *}$, we show that $G \subseteq G^{\delta *}$. In fact, $\delta Cl^{*}(X - G) = X - G$ which implies that $(X - G)^{\delta *} \subseteq X - G$ and by Lemma 1, $X^{\delta *} - G^{\delta *} \subseteq X - G$. Since $\tau^{\delta} \cap \mathcal{I} = \phi$, by Theorem 9, $X - G^{\delta *} \subseteq X - G$ and hence we have $G \subseteq G^{\delta *}$. Hence by Theorem 1, $G^{\delta *} = \delta Cl(G) = \delta Cl(G^{\delta *})$.

(3) Again, $G \subseteq U$ implies that $G^{\delta *} \subseteq U^{\delta *}$. By Lemma 1, $G^{\delta *} = (U - A)^{\delta *} \supseteq U^{\delta *} - A^{\delta *} = U^{\delta *}$ since $A \in \mathcal{I}$. Thus $U^{\delta *} = G^{\delta *}$.

By (1), (2) and (3), we obtain the result. ■

Theorem 11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and τ be δ-compatible with $\mathcal{I}$. Then for every $G \in \tau^{\delta}$ and any subset A of X, $(G \cap A)^{\delta *} = (G \cap A^{\delta *})^{\delta *} = \delta Cl(G \cap A^{\delta *})$.*

Proof. (1) Let $G \in \tau^{\delta}$. Then by Theorem 1, $G \cap A^{\delta *} = G \cap (G \cap A)^{\delta *} \subseteq (G \cap A)^{\delta *}$ and hence $(G \cap A^{\delta *})^{\delta *} \subseteq ((G \cap A)^{\delta *})^{\delta *} \subseteq (G \cap A)^{\delta *}$ by Theorem 1.

(2) Now by using Theorem 1 and Theorem 7 we obtain $(G \cap (A - A^{\delta *}))^{\delta *} \subseteq G^{\delta *} \cap (A - A^{\delta *})^{\delta *} = G^{\delta *} \cap \phi = \phi$. Moreover, $(G \cap A)^{\delta *} - (G \cap A^{\delta *})^{\delta *} \subseteq ((G \cap A) - (G \cap A^{\delta *}))^{\delta *} = (G \cap (A - A^{\delta *}))^{\delta *} = \phi$, which implies that $(G \cap A)^{\delta *} \subseteq (G \cap A^{\delta *})^{\delta *}$.

By (1) and (2) we obtain $(G \cap A)^{\delta *} = (G \cap A^{\delta *})^{\delta *}$. By Theorem 1, $(G \cap A)^{\delta *} = (G \cap A^{\delta *})^{\delta *} \subseteq \delta Cl(G \cap A^{\delta *})$. Also, in view of Theorem 1 we have $G \cap A^{\delta *} \subseteq (G \cap A)^{\delta *}$ and hence $\delta Cl(G \cap A^{\delta *}) \subseteq \delta Cl((G \cap A)^{\delta *}) = (G \cap A)^{\delta *}$. Consequently, we obtain $(G \cap A^{\delta *})^{\delta *} = (G \cap A)^{\delta *} = \delta Cl(G \cap A^{\delta *})$. ■

5. δ_{*}-$\mathcal{I}$-open sets

In this section, we introduce δ_{*}-$\mathcal{I}$-open sets and the δ_{*}-$\mathcal{I}$-closure of a set in an ideal topological space and investigated their basic properties similarly with δ-open sets and δ-closure due to Veličko [7] and δ-$\mathcal{I}$-open sets and δ-$\mathcal{I}$-closure due to Yuksel et al. [8].

Definition 3. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be a δR-$\mathcal{I}$-open (resp. regular open and R-$\mathcal{I}$-open) set if $Int(\delta Cl^*(A)) = A$ (resp. $Int(Cl(A)) = A$, $Int(Cl^*(A)) = A$). The complement of a δR-$\mathcal{I}$-open set is δR-$\mathcal{I}$-closed.*

Remark 6. (1) Every regular open set is δR-$\mathcal{I}$-open.
(2) The notions of a δR-$\mathcal{I}$-open set and an R-$\mathcal{I}$-open set are independent notions. See example below.

Proof. (1) Let A be a regular open set. Then we have $Int(\delta Cl(A)) = A$ [8]. Since $\tau^{\delta} \subset \tau^{\delta *}$, $A = Int(A) \subset Int(\delta Cl^*(A)) \subset Int(\delta Cl(A)) = A$. Therefore A is δR-$\mathcal{I}$-open. ■

Example 4. $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. Take $A = \{a, b\}$. Then A is R-$\mathcal{I}$-open, but it is not δR-$\mathcal{I}$-open. If we take the ideal $\mathcal{I} = \{\phi, \{a, c\}\}$ in the same topology, then the set $A = \{a, c\}$ is δR-$\mathcal{I}$-open, but it is not R-$\mathcal{I}$-open.

Definition 4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, A a subset of X and $x \in X$.*
(1) *x is called a δ_*-$\mathcal{I}$-cluster point of A if $A \cap Int(\delta Cl^*(U)) \neq \phi$ for each open neighborhood of x,*
(2) *The family of all δ_*-$\mathcal{I}$-cluster points of A is called δ_*-$\mathcal{I}$-closure of A and is denoted by $\delta_* Cl(A)$,*
(3) *A subset A is said to be δ_*-$\mathcal{I}$-closed if $\delta_* Cl(A) = A$. The complement of a δ_*-$\mathcal{I}$-closed set of X is said to be δ_*-$\mathcal{I}$-open.*

Lemma 3. *Let A and B be subsets of an ideal topological spaces $(X, \tau, \mathcal{I})$. Then the following properties hold:*
(1) *$Int(\delta Cl^*(A))$ is δR-$\mathcal{I}$-open,*
(2) *If A and B are δR-$\mathcal{I}$-open, then $A \cap B$ is δR-$\mathcal{I}$-open,*
(3) *If A is δR-$\mathcal{I}$-open, then it is δ_*-$\mathcal{I}$-open,*
(4) *Every δ_*-$\mathcal{I}$-open set is the union of a family of δR-$\mathcal{I}$-open sets.*

Proof. (1) Let A be a subset of X and $U = Int(\delta Cl^*(A))$. Then, we get $Int(\delta Cl^*(U)) = Int(\delta Cl^*(Int(\delta Cl^*(A)))) \subset Int(\delta Cl^*(\delta Cl^*(A))) = Int(\delta Cl^*(A)) = U$ and every time $U = Int(U) \subset Int(\delta Cl^*(U))$ holds. So we obtain the result.

(2) Let A and B be δR-$\mathcal{I}$-open. Then $A \cap B = Int(\delta Cl^*(A)) \cap Int(\delta Cl^*(B)) = Int(\delta Cl^*(A) \cap \delta Cl^*(B)) \supset Int(\delta Cl^*(A \cap B)) \supset Int(A \cap B) = A \cap B$. (since every δR-$\mathcal{I}$-open set is open)

(3) Let A be a δR-$\mathcal{I}$-open set. For each $x \in A$, $(X - A) \cap A = \phi$. Thus $x \notin \delta_* Cl(X - A)$ for each $x \in A$. So, $x \notin (X - A)$ implies that $x \notin \delta_* Cl(X - A)$. Therefore, $\delta_* Cl(X - A) \subset (X - A)$. This shows that $(X - A)$ is δ_*-$\mathcal{I}$-closed and so A is δ_*-$\mathcal{I}$-open.

(4) Let A be a δ_*-$\mathcal{I}$-open set. Then $(X-A)$ is δ_*-$\mathcal{I}$-closed and $(X-A) = \delta_*Cl(X-A)$. For each $x \in A$, $x \notin \delta_*Cl(X-A)$ and there exists an open neighborhood U_x such that $Int(\delta Cl^*(U_x)) \cap (X-A) = \phi$. Therefore, we have $x \in U_x \subset Int(\delta Cl^*(U_x)) \subset A$. This shows that $A = \cup\{Int(\delta Cl^*(U_x)) \mid x \in A\}$. By (1), $Int(\delta Cl^*(U_x))$ is δR-$\mathcal{I}$-open for each $x \in A$. ■

Lemma 4. *Let A and B be subsets of an ideal topological space (X, τ, I). Then the following statements hold:*

(1) $A \subset \delta_*Cl(A)$,
(2) *If* $A \subset B$, *then* $\delta_*Cl(A) \subset \delta_*Cl(B)$,
(3) $\delta_*Cl(A) = \cap\{F \subset X \mid$ *and* F *is* δ_*-$\mathcal{I}$-*closed* $\}$,
(4) *If* A *is a* δ_*-$\mathcal{I}$-*closed set of* X *for each* $i \in \nabla$, $\cap\{A_i \mid i \in \nabla\}$ *is* δ_*-$\mathcal{I}$-*closed,*
(5) $\delta_*Cl(A)$ *is* δ_*-$\mathcal{I}$-*closed.*

Proof. Straightforward. ■

Theorem 12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\delta\tau^* = \{A \subset X \mid A$ is a δ_*-$\mathcal{I}$-open set of $(X, \tau, \mathcal{I})\}$. Then $\delta\tau^*$ is a topology such that $\tau^\delta \subset \delta\tau^* \subset \tau$.*

Proof. It follows from Lemma 3 and Lemma 4. ■

Example 5. $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. Take $A = \{a, c\}$. Then A is δ_*-$\mathcal{I}$-open, but it is not δR-$\mathcal{I}$-open.

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