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STRONG CESÀRO SUMMABILITY OF TRIPLE FOURIER INTEGRALS

Abstract. The theory of summability is a very extensive field, which has various applications. We prove the following theorem. Assume $f \in L^{\infty}(R^3)$ with bounded support. If f is continuous at some point $(x_1, x_2, x_3) \in R^3$, then the triple Fourier integral of f is strongly q-Cesàro summable at (x_1, x_2, x_3) to the function value $f(x_1, x_2, x_3)$ for every $0 < q < \infty$. Furthermore, if f is continuous on some open subset G of R^3, then the strong q-Cesàro summability of the triple Fourier integral of f is locally uniform on G.

Key words: triple Fourier transform and integral, inversion formula, partial (or Dirichlet) integral, $(C, 1)$ summability and strong q - Cesàro summability.

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1. Introduction

1.1 Strong Cesàro summability of single Fourier integrals

Recall that the Fourier transform of a function $f(x)$, integral in Lebesgue's sense on R, in symbol $f(x) \in L^1(R)$, is defined by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_R f(x) e^{-tix} dx, \quad t \in R.$$

By the dominated convergence theorem, $\hat{f}(t)$ exists for every $t \in R$, f is continuous on R and by the Riemann-Lebesgue lemma, $\hat{f}(t) \to 0$ as $|t| \to \infty$.

One of the main concerns is how to reconstruct the function f in terms of its Fourier transform $\hat{f}$. For example, it is known that if $\hat{f}(x) \in L^1(R)$, then the inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_R \hat{f}(t) e^{tix} dt, \tag{1}$$

holds for almost everywhere $x \in R$. (See, e.g. [[11], p. 11]).

Recall that the right-hand side of (1) is called the Fourier integral of f. However, $\hat{f} \notin L^1(R)$, in general, and thus (1) makes no sense as a Lebesgue integral. This motivates the information of the partial (also called Dirichlet) integral of $\hat{f}$ is defined by

$$s_\mu(f,x) = \frac{1}{\sqrt{2\pi}} \int_{-\mu}^{\mu} \hat{f}(t) e^{tix} dt, \quad \mu > 0.$$

By Fubini's theorem, we find that

$$s_\mu(f,x) = \frac{1}{\pi} \int_R f(x-t) D_\mu(t) dt, \quad x \in R, \tag{2}$$

where

$$D_\mu(t) = \frac{\sin \mu t}{t}, \quad 0 \neq t \in R. \tag{3}$$

This representation justifies the use of the term "Dirichlet Integral". One might expect that (1) could be saved by considering its right- hand side as an improper integral, that is, the limit of $s_\mu(f,x)$ as $\mu \to \infty$. Unfortunately, this is not the case in general. According to [1], there exists a function $f \in L^1(R)$ such that $\lim_{\mu\to\infty} \sup |s_\mu(f,x)| = \infty$ for almost every $x \in R$.

On the other hand, strong Cesàro summability of $s_\mu(f,x)$ with respect to μ may take place. The following theorem was proved in [5] by the author.

Theorem 1. *Let $f \in L^1(R)$ be locally bounded on R, and let $0 < q < \infty$.*

(a) If f is continuous at some point $x \in R$, then

$$\lim_{m\to\infty} \frac{1}{m} \int_0^m |s_\mu(f,x) - f(x)|^q d\mu = 0. \tag{4}$$

(b) If f is continuous on some open subset G of R, then (4) holds locally uniformly on G. Note that if (4) holds for some $0 < q < \infty$, then it holds for every $0 < q_1 < q$. Indeed, by Hölder's inequality, we have

$$\left\{ \frac{1}{m} \int_0^m |s_\mu(f,x) - f(x)|^{q_1} d\mu \right\}^{1/q_1} \leq \left\{ \frac{1}{m} \int_0^m |s_\mu(f,x) - f(x)|^{q} d\mu \right\}^{1/q}, \quad m > 0. \tag{5}$$

Thus, in case $q \geq 1$, the ordinary Cesàro summability of $s_\mu(f,x)$, that is

$$\lim_{m\to\infty} \frac{1}{m} \int_0^m s_\mu(f,x) d\mu = f(x)$$

immediately follows from (4). Concerning Cesàro summability of integrals, we refer to [[3], pp. 10-13], where it is called summability $(C,1)$.

1.2. Strong Cesàro summability of double Fourier integrals [2]

Recall that the double Fourier transform of a function $f(x_1, x_2) \in L^1(R^2)$ is defined by

$$\hat{f}(t_1, t_2) = \frac{1}{2\pi} \iint_{R^2} f(x_1, x_2) e^{-i(t_1 x_1 + t_2 x_2)} dx_1 dx_2, \;\; t_1, t_2 \in R^2. \tag{6}$$

By the dominated convergence theorem, $\hat{f}(t_1, t_2)$ exists for every $(t_1, t_2) \in R^2$, $\hat{f}$ is continuous on R^2 and by the Riemann-Lebesgue lemma, $\hat{f}(t_1, t_2) \to 0$ as $|t_1|, |t_2| \to \infty$. If $\hat{f} \in L^1(R^2)$, then the inversion formula

$$f(x_1, x_2) = \frac{1}{2\pi} \iint_{R^2} \hat{f}(t_1, t_2) e^{i(t_1 x_1 + t_2 x_2)} dt_1 dt_2 \tag{7}$$

holds for almost every $(t_1, t_2) \in R^2$. The reader is referred to [[5], ch. 1] for details. The partial (also called Dirichlet) integral of $\hat{f}$ is defined by

$$s_{\mu_1,\mu_2}(f, x_1, x_2) = \frac{1}{2\pi} \int_{-\mu_1}^{\mu_1} \int_{-\mu_2}^{\mu_2} \hat{f}(t_1, t_2) e^{i(t_1 x_1 + t_2 x_2)} dt_1 dt_2, \;\; \mu_1, \mu_2 > 0.$$

Use of (6) and Fubini's theorem, we get

$$s_{\mu_1,\mu_2}(f, x_1, x_2) = \frac{1}{\pi^2} \iint_{R^2} f(x_1 - t_1, x_2 - t_2) D_{\mu_1}(t_1) D_{\mu_2}(t_2) dt_1 dt_2, \tag{8}$$

where $D_\mu(t)$ is defined in (3). The inversion formula (7) makes no sense if $\hat{f} \notin L^1(R^2)$ and cannot be saved by replacing the right-hand side by the limit of $s_{\mu_1,\mu_2}(f, x_1, x_2)$ as $\mu_1, \mu_2 \to \infty$, because this limit does not exist in general (see [1]).

On the other hand, Cesàro summability of $s_{\mu_1,\mu_2}(f, x_1, x_2)$ with respect to μ_1, μ_2 may take place. The following theorem was proved in [2] by the author.

Theorem 2. *Let $f \in L^\infty(R^2)$ with bounded support and let $0 < q < \infty$.*
(a) If f is continuous at some point $(x_1, x_2) \in R^2$, then

$$\lim_{m_1,m_2 \to \infty} \frac{1}{m_1 m_2} \int_0^{m_1} \int_0^{m_2} |s_{\mu_1,\mu_2}(f, x_1, x_2) - f(x_1, x_2)|^q d\mu_1 d\mu_2 = 0. \tag{9}$$

(b) If f is continuous on some open subset G of R^2, then (9) holds locally uniformly on G. By the term locally uniformly on G we mean that every point (x_1, x_2) in G has a neighborhood in G, on which the limit relation (9) holds uniformly.

In 1974, Khan [4] studied on degree of approximation to a functions belonging to the class $Lip(\alpha, p)$. Recently, Mishra et al. ([6]-[9]) have obtained the degree of approximation of a function belonging to various classes using different summability matrices with monotone and non-monotone rows.

2. Main results

2.1. Strong Cesàro summability of Triple Fourier integrals

Recall that the triple Fourier transform of a function $f(x_1, x_2, x_3) \in L^1(R^3)$ is defined by

$$\hat{f}(t_1, t_2, t_3) = \frac{1}{(2\pi)^{3/2}} \iiint_{R^3} f(x_1, x_2, x_3) e^{-i(t_1x_1+t_2x_2+t_3x_3)} dx_1 dx_2 dx_3, \tag{10}$$

$t_1, t_2, t_3 \in R^3$. By the dominated convergence theorem, $\hat{f}(t_1, t_2, t_3)$ exists for every $(t_1, t_2, t_3) \in R^3$, $\hat{f}$ is continuous on R^3 and by the Riemann-Lebesgue lemma, $\hat{f}(t_1, t_2, t_3) \to 0$ as $|t_1|, |t_2|, |t_3| \to \infty$. If $\hat{f} \in L^1(R^3)$, then the inversion formula

$$f(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} \iiint_{R^3} \hat{f}(t_1, t_2, t_3) e^{i(t_1x_1+t_2x_2+t_3x_3)} dt_1 dt_2 dt_3 \tag{11}$$

holds for almost everywhere $(t_1, t_2, t_3) \in R^3$. The partial (also called Dirichlet) integral of $\hat{f}$ is defined by

$$s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} \int_{-\mu_1}^{\mu_1} \int_{-\mu_2}^{\mu_2} \int_{-\mu_3}^{\mu_3} \hat{f}(t_1, t_2, t_3) e^{i(t_1x_1+t_2x_2+t_3x_3)} dt_1 dt_2 dt_3,$$

$\mu_1, \mu_2, \mu_3 > 0$. Use of (10) and Fubini's theorem we get

$$s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3) = \frac{1}{\pi^3} \iiint_{R^3} f(x_1 - t_1, x_2 - t_2, x_3 - t_3) D_{\mu_1}(t_1) D_{\mu_2} D_{\mu_3}(t_3) dt_1 dt_2 dt_3, \tag{12}$$

where $D_\mu(t)$ is defined in (3). The inversion formula (11) makes no sense if $\hat{f} \notin L^1(R^3)$ and cannot be saved by replacing the right-hand side by the limit of $s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3)$ as $\mu_1, \mu_2, \mu_3 \to \infty$, because this limit does not exist in general.

On the other hand, Cesàro summability of $s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3)$ with respect to μ_1, μ_2, μ_3 may take place. The following theorem is three dimensional analogue of Theorem 2.

Theorem 3. *Let $f \in L^\infty(R^3)$ with bounded support and let $0 < q < \infty$.*
(a) If f is continuous at some point $(x_1, x_2, x_3) \in R^3$, then

$$(13)\quad \lim_{m_1,m_2,m_3\to\infty} \frac{1}{m_1m_2m_3} \int_0^{m_1}\int_0^{m_2}\int_0^{m_3} |s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3) - f(x_1, x_2, x_3)|^q d\mu_1 d\mu_2 d\mu_3 = 0.$$

(b) If f is continuous on some open subset G of R^3, then (13) holds locally uniformly on G. By the term locally uniformly on G we mean that every point (x_1, x_2, x_3) in G has a neighborhood in G, on which the limit relation (13) holds uniformly.

3. Proof of Theorem 3

Part (a)**.** By the three dimensional analogue of inequality (5), without loss of generality we may assume that $3 \le q < \infty$. By the assumption that $f \in L^\infty(R^3)$ is continuous at $(x_1, x_2, x_3) \in R^3$, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$(14)\quad |f(x_1 - t_1, x_2 - t_2, x_3 - t_3) - f(x_1, x_2, x_3)| < \epsilon \text{ if } |t_j| < \delta,\ j = 1, 2, 3$$

and for some constant $B > 0$,we have

$$(15)\qquad |f(y_1, y_2, y_3)| \le B \text{ for almost every } (y_1, y_2, y_3) \in R^3.$$

Since f is bounded support, there exits some constant $M > 0$ such that

$$(16)\qquad f(x_1 - t_1, x_2 - t_2, x_3 - t_3) = 0 \text{ for all } (t_1, t_2, t_3) \in \frac{R^3}{Q_M},$$

where

$$(17)\qquad Q_M = [-M, M]|[-M, M]|[-M, M].$$

Recall (see, e.g.,[[11], vol. 1, pp. 56-58]) that

$$\int \frac{\sin t}{t} dt = \lim_{n\to\infty} \int_{-m}^{m} \frac{\sin t}{t} dt = \pi,$$

and

$$\left| \int_{-m}^{m} \frac{\sin t}{t} dt \right| < 2\pi \text{ for all } m > 0,$$

Thus, we may choose M so large (17) that both (16) and the following inequality hold:

$$\left| \int_{-m}^{m} \frac{\sin t}{t} dt - \pi \right| < \epsilon \text{ whenever } m \ge M.$$

Accordingly, for $\mu > 0$ we have

$$\left| \int_{-m}^{m} \frac{\sin \mu t}{t} dt - \pi \right| < \begin{cases} \epsilon, & \text{if} \quad \mu m \geq M, \\ 3\pi, & \text{if} \ \ \mu m < M, \ \mu > 0. \end{cases} \tag{18}$$

By (12) and (16), the following representation clearly holds:

$$\begin{aligned} &\pi^3[s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3) - f(x_1, x_2, x_3)] \\ &= \iiint_{Q_M} [f(x_1 - t_1, x_2 - t_2, x_3 - t_3) - f(x_1, x_2, x_3)] \\ &\qquad\qquad \times D\mu_1(t_1) D\mu_2(t_2) D\mu_3(t_3) dt_1 dt_2 dt_3 \\ &\quad - f(x_1, x_2, x_3) \left[\pi^3 - \iiint_{Q_M} D\mu_1(t_1) D\mu_2(t_2) D\mu_3(t_3) dt_1 dt_2 dt_3 \right], \end{aligned}$$

where $D_\mu(t)$ is defined in (3). Let $m_j > 1$, $j = 1, 2, 3$.

By Minkowski's inequality, we have

$$\begin{aligned} &\sum_{m_1,m_2,m_3} (f, x_1, x_2, x_3, q) \\ &= \pi^3 \Big\{ \frac{1}{m_1 m_2 m_3} \int_0^{m_1} \int_0^{m_2} \int_0^{m_3} |s_{\mu_1,\mu_2,\mu_3}(f, x_1, x_2, x_3) \\ &\qquad\qquad - f(x_1, x_2, x_3)|^q d\mu_1 d\mu_2 d\mu_3 \Big\}^{1/q} \\ &\leq \Big\{ \frac{1}{m_1 m_2 m_3} \int_0^{m_1} \int_0^{m_2} \int_0^{m_3} \Big| \iiint_{Q_M} [f(x_1 - t_1, x_2 - t_2, x_3 - t_3) \\ &\quad - f(x_1, x_2, x_3)] D\mu_1(t_1) D\mu_2(t_2) D\mu_3(t_3) dt_1 dt_2 dt_3 \Big|^q d\mu_1 d\mu_2 d\mu_3 \Big\}^{1/q} \\ &\quad + |f(x_1, x_2, x_3)| \times \Big\{ \frac{1}{m_1 m_2 m_3} \int_0^{m_1} \int_0^{m_2} \int_0^{m_3} \Big| \iiint_{Q_M} D\mu_1(t_1) \\ &\qquad\qquad \times D\mu_2(t_2) D\mu_3(t_3) - \pi^3 \Big|^q d\mu_1 d\mu_2 d\mu_3 \Big\}^{1/q}. \end{aligned} \tag{19}$$

We claim that the order of magnitude of the second term on the right hand side of (19) is $O(\epsilon)$. Indeed, by (15), (18) and Minkowski's inequality, the second term in question does not exceed the following quantity:

$$\begin{aligned} B\Big\{ \frac{1}{m_1 m_2 m_3} \Big[&\int_0^1\int_0^1\int_0^1 + \int_1^{m_1}\int_0^1\int_0^1 + \int_0^1\int_1^{m_2}\int_0^1 + \int_0^1\int_0^1\int_1^{m_3} \\ &+ \int_1^{m_1}\int_1^{m_2}\int_1^{m_3} + \int_1^{m_1}\int_1^{m_2}\int_0^1 + \int_0^1\int_1^{m_2}\int_1^{m_3} + \int_1^{m_1}\int_0^1\int_1^{m_3} \Big] \end{aligned}$$

$$\times\bigg|\bigg[\int_{-M}^{M}\frac{\sin\mu_1 t_1}{t_1}dt_1-\pi\bigg]\bigg[\int_{-M}^{M}\frac{\sin\mu_2 t_2}{t_2}dt_2-\pi\bigg]$$
$$\times\bigg[\int_{-M}^{M}\frac{\sin\mu_3 t_3}{t_3}dt_3-\pi\bigg]+\pi\bigg[\int_{-M}^{M}\frac{\sin\mu_1 t_1}{t_1}dt_1-\pi\bigg]$$
$$+\pi\bigg[\int_{-M}^{M}\frac{\sin\mu_2 t_2}{t_2}dt_2-\pi\bigg]+\pi\bigg[\int_{-M}^{M}\frac{\sin\mu_3 t_3}{t_3}dt_3-\pi\bigg]\bigg|^q d\mu_1 d\mu_2 d\mu_3\bigg\}^{1/q}$$
$$\leq B\bigg\{[27\pi^3+3\pi^2+3\pi^2+3\pi^2]+(m_1-1)^{1/q}[9\pi^2\epsilon+\pi\epsilon+3\pi^2+3\pi^2]$$
$$+(m_2-1)^{1/q}[9\pi^2\epsilon+\pi\epsilon+3\pi^2+3\pi^2]$$
$$+(m_3-1)^{1/q}[9\pi^2\epsilon+\pi\epsilon+3\pi^2+3\pi^2]$$
$$+(m_1-1)^{1/q}(m_2-1)^{1/q}(m_3-1)^{1/q}[\epsilon^3+\pi\epsilon+\pi\epsilon+\pi\epsilon]$$
$$+(m_1-1)^{1/q}(m_2-1)^{1/q}[3\pi\epsilon^2+\pi\epsilon+3\pi^2+\pi\epsilon]$$
$$+(m_2-1)^{1/q}(m_3-1)^{1/q}[3\pi\epsilon^2+\pi\epsilon+3\pi^2+\pi\epsilon]$$
$$+(m_1-1)^{1/q}(m_3-1)^{1/q}[3\pi\epsilon^2+\pi\epsilon+3\pi^2+\pi\epsilon]\bigg\}=O(\epsilon),$$

provided that m_1, m_2 and m_3 are large enough. Now equation (19), we have

$$(20)\quad \sum_{m_1,m_2,m_3}(f,x_1,x_2,x_3,q)$$
$$\leq\bigg\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\bigg|\iiint_{Q_M}[f(x_1-t_1,x_2-t_2,x_3-t_3)$$
$$-f(x_1,x_2,x_3)]D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3\bigg|^q d\mu_1d\mu_2d\mu_3\bigg\}^{1/q}$$
$$+O(\epsilon),$$

as $m_1, m_2, m_3 \to \infty$. We shall assume that

$$(21)\qquad m_j>\frac{1}{\delta},\quad j=1,2,3,$$

where δ occurs in (14), and consider the decomposition

$$\begin{aligned}Q_M &= E_{m_1}\times E_{m_2}\times E_{m_3}\cup CE_{m_1}\times E_{m_2}\times E_{m_3}\\ &\cup E_{m_1}\times CE_{m_2}\times E_{m_3}\cup E_{m_1}\times E_{m_2}\times CE_{m_3}\\ &\cup CE_{m_1}\times CE_{m_2}\times E_{m_3}\cup CE_{m_1}\times E_{m_2}\times CE_{m_3}\\ &\cup E_{m_1}\times CE_{m_2}\times CE_{m_3}\cup CE_{m_1}\times CE_{m_2}\times CE_{m_3},\end{aligned}$$

where

$$(22)\qquad \begin{cases} E_{m_j}=\{t_j\in R:|t_j|\leq 1/m_j\},\\ CE_{m_j}=\{t_j\in R:1/m_j<|t_j|\leq M\},\quad j=1,2,3.\end{cases}$$

Accordingly, we decompose the inner triple product integral $\int\int\int_{Q_M}$ in (20) into eight parts and denote them in turn by I_j, $j = 1,2,3,4,5,6,7,8$; for example,

$$I_1 = \int_{E_{m_1}}\int_{E_{m_2}}\int_{E_{m_3}} [f(x_1 - t_1, x_2 - t_2, x_3 - t_3) - f(x_1, x_2, x_3)] \times D\mu_1(t_1) D\mu_2(t_2) D\mu_3(t_3) dt_1 dt_2 dt_3.$$

By the trivial estimate

$$|D_\mu(t)| \le \mu, \quad \mu \ge 0 \text{ and } t \in R, \tag{23}$$

Clearly, we have

$$|I_1| \le \mu_1\mu_2\mu_3 \times \int_{E_{m_1}}\int_{E_{m_2}}\int_{E_{m_3}} [f(x_1 - t_1, x_2 - t_2, x_3 - t_3) - f(x_1, x_2, x_3)] dt_1 dt_2 dt_3.$$

By Fubini's theorem, we obtain

$$\begin{aligned} &\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3} |I_1|^q d\mu_1 d\mu_2 d\mu_3 \\ &\le \frac{1}{m_1m_2m_3}\left\{\int_{E_{m_1}}\int_{E_{m_2}}\int_{E_{m_3}} |f(x_1 - t_1, x_2 - t_2, x_3 - t_3) \right. \\ &\quad \left. - f(x_1, x_2, x_3)| dt_1 dt_2 dt_3\right\}^q \int_0^{m_1} \mu_1^q d\mu_1 \int_0^{m_2} \mu_2^q d\mu_2 \int_0^{m_3} \mu_3^q d\mu_3 \\ &= \frac{(m_1m_2m_3)^q}{(q+1)^3}\left\{\int_{E_{m_1}}\int_{E_{m_2}}\int_{E_{m_3}} [f(x_1 - t_1, x_2 - t_2, x_3 - t_3) \right. \\ &\quad \left. - f(x_1, x_2, x_3)] dt_1 dt_2 dt_3\right\}^q. \end{aligned}$$

On combining (14), (21) and (22), hence we conclude that

$$\left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3} |I_1|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q} \le \frac{8\epsilon}{(q+1)^{3/q}} = O(\epsilon). \tag{24}$$

Next

$$I_2 = \int_{CE_{m_1}}\int_{E_{m_2}}\int_{E_{m_3}} [f(x_1 - t_1, x_2 - t_2, x_3 - t_3) - f(x_1, x_2, x_3)] \times D\mu_1(t_1) D\mu_2(t_2) D\mu_3(t_3) dt_1 dt_2 dt_3.$$

Applying Fubini's theorem and Jensen's inequality, we yield

$$
\begin{aligned}
(25)\qquad A_2^q &= \frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_2|^q d\mu_1 d\mu_2 d\mu_3 \\
&\le \frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\Big\{\int_{E_{m_2}}|D_{\mu_2}(t_2)|\int_{E_{m_3}}|D_{\mu_3}(t_3)| \\
&\quad\times\Big|\int_{CE_{m_1}}[f(x_1-t_1,x_2-t_2,x_3-t_3) \\
&\quad - f(x_1,x_2,x_3)]D_{\mu_1}(t_1)dt_1\Big|dt_2dt_3\Big\}^q d\mu_1 d\mu_2 d\mu_3 \\
&\le \frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\Big\{\Big[\int_{E_{m_2}}|D_{\mu_2}(t_2)|dt_2\Big]^{q-1} \\
&\quad\times\Big[\int_{E_{m_3}}|D_{\mu_3}(t_3)dt_3|\Big]^{q-1}\Big[\int_{E_{m_2}}|D_{\mu_2}(t_2)| \\
&\quad\times\int_{E_{m_3}}|D_{\mu_3}(t_3)|\Big]\Big|\int_{CE_{m_1}}[f(x_1-t_1,x_2-t_2,x_3-t_3) \\
&\quad - f(x_1,x_2,x_3)]D_{\mu_1}(t_1)dt_1\Big|^q dt_2dt_3\Big\}d\mu_1 d\mu_2 d\mu_3.
\end{aligned}
$$

By (3), (22) and (23), while applying Fubini's theorem again, we obtain

$$
\begin{aligned}
(26)\qquad A_2^q &\le \frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\Big\{\Big[2\int_0^{1/m_2}m_2dt_2\Big]^{q-1} \\
&\quad\times\Big[2\int_0^{1/m_3}m_3dt_3\Big]^{q-1}\Big[\int_{E_{m_2}}m_2\int_{E_{m_3}}m_3\Big] \\
&\quad\times\Big|\int_{CE_{m_1}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)] \\
&\quad\times D_{\mu_1}(t_1)dt_1\Big|^q dt_2dt_3\Big\}d\mu_1 d\mu_2 d\mu_3 \\
&= \frac{2^{2q-2}}{m_1m_2m_3}\int_0^{m_2}\int_0^{m_3}\Big\{\int_{E_{m_2}}m_2\int_{E_{m_3}}m_3 \\
&\quad\times\Big[\int_0^{m_1}\Big|\int_{CE_{m_1}}[f(x_1-t_1,x_2-t_2,x_3-t_3) \\
&\quad - f(x_1,x_2,x_3)]\frac{\sin\mu_1 dt_1}{t_1}dt_1\Big|^q d\mu_1\Big]dt_2dt_3\Big\}d\mu_2 d\mu_3
\end{aligned}
$$

$$= \frac{2^{2q-2}}{m_1}\int_{E_{m_2}} m_2 \int_{E_{m_3}} m_3 \Big[\int_0^{m_1} \Big| \int_{CE_{m_1}} [f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)] \times \frac{\sin \mu_1 t_1}{t_1} dt_1\Big|^q d\mu_1\Big] dt_2 dt_3.$$

The inner triple integral on the right hand side of (26) involving the q^{th} power can be estimated by the Hausdorff-inequality (see, e.g., [[10], p. 178] or [11, Vol. 2 p. 254]) as follows:

$$(27)\quad \int_0^{m_1} \Big| \int_{CE_{m_1}} [f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)] \times \frac{\sin \mu_1 t_1}{t_1} dt_1\Big|^q d\mu_1$$

$$= \int_0^{m_1} \Big| \int_R \frac{f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)}{t_1} \times \chi_1(t_1) \sin \mu_1 t_1 dt_1 \Big|^q d\mu_1$$

$$\le C_q^q \Big\{ \int_{CE_{m_1}} \Big| \frac{f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)}{t_1} \Big|^p dt_1 \Big\}^{1/p},$$

where $\chi_1(t_1)$ is the characteristic function of the set $[-M, -1/m_1) \cup [M, 1/m_1)$, C_q is constant depending only on q, and $p = q/(q-1)$ is the exponent conjugate to q. Since $3 \le q < \infty$, we have $1 < p \le 3/2$. Combining (25) to (27) and using Minkowski's inequality, we get

$$(28)\quad A_2^q \le \frac{2^{2q-2}}{m_1} C_q^q \int_{E_{m_2}} m_2 \int_{E_{m_3}} m_3 \times \Big\{ \int_{CE_{m_1}} \Big| \frac{f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)}{t_1} \Big|^p dt_1 \Big\}^{1/p} dt_2 dt_3$$

$$\le \frac{2^{2q-2}}{m_1} C_q^q \int_{E_{m_2}} m_2 \int_{E_{m_3}} m_3 \times \Big\{ \Big[\int_{1/m_1}^{M} \Big| \frac{f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)}{t_1} \Big|^p dt_1 \Big]^{1/p} + \Big[\int_{-M}^{-1/m_1} \Big| \frac{f(x_1-t_1,x_2-t_2,x_3-t_3) - f(x_1,x_2,x_3)}{t_1} \Big|^p dt_1 \Big]^{1/p} \Big\} dt_2 dt_3.$$

By (14), (18) and (21), we estimate the inner integral $\int_{1/m_1}^{M}$ as follows:

$$\text{(29)}\quad \left[\left\{\int_{1/m_1}^{\delta}+\int_{\delta}^{M}\right\}\left|\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_1}\right|^p dt_1\right]^{1/p}$$
$$\leq \left[\int_{1/m_1}^{\delta}\frac{\epsilon_1^p}{t_1^p}dt_1\right]^{1/p}+\left[\int_{\delta}^{M}\frac{(2B)^p}{t_1^p}dt_1\right]^{1/p}$$
$$\leq \frac{1}{(p-1)^1/p}\left[\epsilon {m_1}^{1/q}+\frac{2B}{\delta^{1q}}\right]$$

The estimate is valid for the other inner integral $\int_{-M}^{-1/m_1}$. Putting together (25), (28), (29) and its counterpart yields

$$\text{(30)}\qquad A_2 = \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_2|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q}$$
$$\leq \frac{8C_q}{(p-1)^{1/p}}\left[\epsilon+\frac{2B}{(m_1\delta)^{1q}}\right]=O(\epsilon),$$

provided that m_1 is large enough. An analogous estimate is valid for

$$I_3 = \int_{E_{m_1}}\int_{CE_{m_2}}\int_{E_{m_3}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$
$$\times D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3.$$

that is, we have

$$\text{(31)}\qquad A_3 = \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_3|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q}$$
$$\leq \frac{8C_q}{(p-1)^{1/p}}\left[\epsilon+\frac{2B}{(m_2\delta)^{1q}}\right]=O(\epsilon),$$

provided that m_2 is large enough.

$$I_4 = \int_{E_{m_1}}\int_{E_{m_2}}\int_{C_{m_3}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$
$$\times D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3.$$

that is, we have

$$\text{(32)}\qquad A_4 = \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_4|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q}$$
$$\leq \frac{8C_q}{(p-1)^{1/p}}\left[\epsilon+\frac{2B}{(m_3\delta)^{1q}}\right]=O(\epsilon),$$

provided that m_3 is large enough

$$I_5 = \int_{CE_{m_1}}\int_{CE_{m_2}}\int_{C_{m_3}} [f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)] \times D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3.$$

Now, we apply the Hausdroff-Young inequality (see, e.g. [[10], p. 178]) to obtain

$$\begin{aligned}
(33)\quad A_5 &= \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_5|^q d\mu_1d\mu_2d\mu_3\right\}^{1/q} \\
&= \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\right. \\
&\quad \left|\iiint_{R^3}\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_1t_2t_3}\right. \\
&\qquad \times \chi_1(t_1)(\sin\mu_1t_1)\chi_2(t_2)(\sin\mu_2t_2) \\
&\qquad \left.\left.\times \chi_3(t_3)(\sin\mu_3t_3)dt_1dt_2dt_3\right|^q d\mu_1d\mu_2d\mu_3\right\}^{1/q} \\
&\le \tilde{C}_q\left\{\int_{CE_{m_1}}\int_{CE_{m_2}}\int_{CE_{m_3}}\right. \\
&\quad \left.\left|\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_1t_2t_3}\right|^p dt_1dt_2dt_3\right\}^{1/p}
\end{aligned}$$

where $\chi_j(t_j)$ is the characteristic function of the set $[-M,-1/m_j)\cup(1/m_j,M]$, $j=1,2,3$, $\tilde{C}_q$ is constant depending only on q, and $p=q/(q-1)$ is the exponent conjugate to q.

We decompose the domain of integration at the right most side of (33) as follows:

$$\begin{aligned}
CE_{m_1}\times CE_{m_2}\times CE_{m_3} &= (1/m_1,M]\times(1/m_2,M]\times(1/m_3,M] \\
&\cup[-M,-1/m_1)\times[-M,-1/m_2)\times[-M,-1/m_3) \\
&\cup[-M,-1/m_1)\times(1/m_2,M]\times(1/m_3,M] \\
&\cup(1/m_1,M]\times(1/m_2,M]\times[-M,-1/m_3) \\
&\cup(1/m_1,M]\times[-M,-1/m_2)\times(1/m_3,M] \\
&\cup[-M,-1/m_1)\times[-M,-1/m_2)\times(1/m_3,M] \\
&\cup(1/m_1,M]\times[-M,-1/m_2)\times[-M,-1/m_3) \\
&\cup[-M,-1/m_1)\times(1/m_2,M]\times[-M,-1/m_3).
\end{aligned}$$

For example, we consider the corresponding integral over $(1/m_1,M]\times(1/m_2,M]\times(1/m_3,M]$.

By Minkowski's inequality, while using (14), (15) and (21), we find that

$$\begin{aligned}
&\Bigg\{\Bigg[\int_{1/m_1}^{\delta}\int_{1/m_2}^{\delta}\int_{1/m_3}^{\delta}+\int_{1/m_1}^{\delta}\int_{\delta}^{M}\int_{1/m_3}^{\delta}+\int_{\delta}^{M}\int_{1/m_2}^{\delta}\int_{1/m_3}^{\delta}\\
&\quad+\int_{1/m_1}^{\delta}\int_{1/m_2}^{\delta}\int_{\delta}^{M}+\int_{1/m_1}^{\delta}\int_{\delta}^{M}\int_{\delta}^{M}+\int_{\delta}^{M}\int_{1/m_1}^{\delta}\int_{\delta}^{M}\\
&\quad+\int_{\delta}^{M}\int_{1/m_2}^{\delta}\int_{\delta}^{M}+\int_{\delta}^{M}\int_{\delta}^{M}\int_{\delta}^{M}\Bigg]\\
&\quad\times\left|\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_1t_2t_3}\right|^p dt_1dt_2dt_3\Bigg\}^{1/p}\\
&\le\left\{\int_{1/m_1}^{\delta}\int_{1/m_2}^{\delta}\int_{1/m_3}^{\delta}\frac{(\epsilon)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{1/m_1}^{\delta}\int_{\delta}^{M}\int_{1/m_3}^{\delta}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{\delta}^{M}\int_{1/m_2}^{\delta}\int_{1/m_3}^{\delta}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{1/m_1}^{\delta}\int_{1/m_2}^{\delta}\int_{\delta}^{M}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{1/m_1}^{\delta}\int_{\delta}^{M}\int_{\delta}^{M}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{\delta}^{M}\int_{1/m_2}^{\delta}\int_{\delta}^{M}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{\delta}^{M}\int_{\delta}^{M}\int_{1/m_3}^{\delta}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\quad+\left\{\int_{\delta}^{M}\int_{\delta}^{M}\int_{\delta}^{M}\frac{(2B)^p}{t_1^pt_2^pt_3^p}dt_1dt_2dt_3\right\}^{1/p}\\
&\le\frac{1}{(p-1)^{3/p}}\Bigg[\epsilon m_1^{1/q}m_2^{1/q)}m_3^{1/q}+\frac{2B}{\delta^{1/q}}m_1^{1/q}m_3^{1/q}+\frac{2B}{\delta^{1/q}}m_1^{1/q}m_2^{1/q}\\
&\quad+\frac{2B}{\delta^{1/q}}m_2^{1/q}m_3^{1/q}+\frac{2B}{\delta^{2/q}}m_1^{1/q}+\frac{2B}{\delta^{2/q}}m_2^{1/q}+\frac{2B}{\delta^{2/q}}m_3^{1/q}+\frac{2B}{\delta^{3/q}}\Bigg].
\end{aligned}$$

The same estimate is valid for the other seven integrals, too. Combining (33) with (34) and its seven counter parts, while using Minkowski's inequality yields

$$A_5=\left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_5|^q d\mu_1d\mu_2d\mu_3\right\}^{1/q}$$

$$\begin{aligned}
&\leq \frac{8\tilde{C}_q}{(p-1)^{3/p}}\Bigg[\epsilon + \frac{2B}{(\delta m_1)^{1/q}} + \frac{2B}{(\delta m_2)^{1/q}} \\
&\quad + \frac{2B}{(\delta m_3)^{1/q}} + \frac{2B}{(\delta^2 m_1 m_3)^{1/q}} + \frac{2B}{(\delta^2 m_1 m_2)^{1/q}} \\
&\quad + \frac{2B}{(\delta^2 m_2 m_3)^{1/q}} + \frac{2B}{(\delta^3 m_1 m_2 m_3)^{1/q}}\Bigg] = O(\epsilon),
\end{aligned}$$

provided that m_1, m_2 and m_3 are large.

$$\begin{aligned}
I_6 \;=\; &\int_{E_{m_1}}\int_{CE_{m_2}}\int_{CE_{m_3}} [f(x_1-t_1, x_2-t_2, x_3-t_3) - f(x_1,x_2,x_3)] \\
&\qquad\qquad \times D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3.
\end{aligned}$$

Applying Fubini's theorem, we get

$$\begin{aligned}
(34)\quad A_6^q \;=\; &\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3} |I_6|^q d\mu_1 d\mu_2 d\mu_3 \\
\leq\; &\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\Bigg\{\int_{E_{m_1}} |D_{\mu_1}(t_1)| \\
&\times\Bigg|\int_{CE_{m_2}}\int_{CE_{m_3}} [f(x_1-t_1, x_2-t_2, x_3-t_3 - f(x_1,x_2,x_3)] \\
&\times D_{\mu_2}(t_2)D_{\mu_3}(t_3)dt_2dt_3\Bigg|dt_1\Bigg\}^q d\mu_1d\mu_2d\mu_3 \\
\leq\; &\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\Bigg\{\Bigg[\int_{E_{m_1}} |D_{\mu_1}(t_1)|dt_1\Bigg]^{q-1} \\
&\int_{E_{m_1}} |D_{\mu_1}(t_1)|\Bigg|\int_{CE_{m_2}}\int_{CE_{m_3}} [f(x_1-t_1, x_2-t_2, x_3-t_3) \\
&- f(x_1,x_2,x_3)]D_{\mu_2}(t_2)D_{\mu_3}(t_3)dt_2dt_3\Bigg|^q\Bigg\}d\mu_1d\mu_2d\mu_3.
\end{aligned}$$

By (3), (22) and (23), while applying Fubini's theorem again, we obtain

$$\begin{aligned}
(35)\quad A_6^q \;\leq\; &\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}\Bigg\{\Bigg[2\int_0^{1/m_1} m_1 dt_1\Bigg]^{q-1}\int_{E_{m_1}} m_1 \\
&\times\Bigg|\int_{CE_{m_2}}\int_{CE_{m_3}} [f(x_1-t_1, x_2-t_2, x_3-t_3) - f(x_1,x_2,x_3)] \\
&\times D_{\mu_2}(t_2)D_{\mu_3}(t_3)dt_2dt_3\Bigg|^q dt_1\Bigg\}d\mu_1d\mu_2d\mu_3
\end{aligned}$$

$$= \frac{2^{q-1}}{m_1 m_2 m_3}\int_0^{m_1}\Big\{\int_{E_{m_1}} m_1\Big[\int_0^{m_2}\int_0^{m_3}$$
$$\Big|\int_{CE_{m_2}}\int_{CE_{m_3}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$
$$\times\frac{\sin\mu_2 dt_2}{t_2}\frac{\sin\mu_3 dt_3}{t_3}dt_2dt_3\Big|^q d\mu_2 d\mu_3\Big]dt_1\Big\}d\mu_1$$
$$= \frac{2^{q-1}}{m_2 m_3}\int_{E_{m_1}} m_1\Big[\int_0^{m_2}\int_0^{m_3}\Big|\int_{CE_{m_2}}\int_{CE_{m_3}}$$
$$[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$
$$\times\frac{\sin\mu_2 dt_2}{t_2}\frac{\sin\mu_3 dt_3}{t_3}dt_2dt_3\Big|^q d\mu_2 d\mu_3\Big]dt_1.$$

The inner triple integral on the right hand side of (35) involving the q^{th} power can be estimated by the Hausdorff-inequality as follows:

$$\text{(36)}\int_0^{m_2}\int_0^{m_3}\Big|\int_{CE_{m_2}}\int_{CE_{m_3}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$
$$\times\frac{\sin\mu_2 dt_2}{t_2}\frac{\sin\mu_3 dt_3}{t_3}dt_2dt_3\Big|^q d\mu_2 d\mu_3$$
$$= \int_0^{m_2}\int_0^{m_3}\Big|\int\int_{R^2}\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_2t_3}$$
$$\times\chi_2(t_2)\sin\mu_2 t_2\chi_3(t_3)\sin\mu_3 t_3 dt_2dt_3\Big|^q d\mu_2 d\mu_3$$
$$\le C_q^q\Big\{\int_{CE_{m_2}}\int_{CE_{m_3}}\Big|\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_2t_3}\Big|^p dt_2dt_3\Big\}^{1/p},$$

where $\chi_j(t_j)$ is the characteristic function of the set $[-M,-1/m_j)\cup(1/m_j,M]$, $j=2,3$, C_q is constant depending only on q, and $p=q/(q-1)$ is the exponent conjugate to q.

We decompose the domain of integration at the right most side of (36) as follows:

$$\begin{aligned} CE_{m_2}\times CE_{m_3} &= (1/m_2,M]\times(1/m_3,M]\\ &\quad\cup[-M,-1/m_2)\times(1/m_3,M]\\ &\quad\cup(1/m_2,M]\times[-M,-1/m_3)\\ &\quad\cup[-M,-1/m_2)\times[-M,-1/m_3).\end{aligned}$$

For example, we consider the corresponding integral over $(1/m_2,M]\times(1/m_3,M]$. By Minkowski's inequality, while using (14), (15) and (21), we find that

$$\text{(37)}\qquad \Big\{\Big[\int_{1/m_2}^{\delta}\int_{1/m_3}^{\delta}+\int_{\delta}^{M}\int_{1/m_3}^{\delta}+\int_{1/m_2}^{\delta}\int_{\delta}^{M}+\int_{\delta}^{M}\int_{\delta}^{M}\Big]$$

$$\times\left|\frac{f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)}{t_2t_3}\right|^p dt_2dt_3\Bigg\}^{1/p}$$

$$\leq \left\{\int_{1/m_2}^{\delta}\int_{1/m_3}^{\delta}\frac{\epsilon_p}{t_2^pt_3^p}dt_2dt_3\right\}^{1p}+\left\{\int_{\delta}^{M}\int_{1/m_3}^{\delta}\frac{(2B)^p}{t_2^pt_3^p}dt_2dt_3\right\}^{1p}$$

$$+\left\{\int_{1/m_2}^{\delta}\int_{\delta}^{M}\frac{(2B)^p}{t_2^pt_3^p}dt_2dt_3\right\}^{1p}+\left\{\int_{\delta}^{M}\int_{\delta}^{M}\frac{(2B)^p}{t_2^pt_3^p}dt_2dt_3\right\}^{1p}$$

$$\leq \frac{1}{(p-1)^{2/p}}\left[\epsilon m_2^{1/q}m_3^{1/q}+\frac{2Bm_2^{1/q}}{\delta^{1/q}}+\frac{2Bm_3^{1/q}}{\delta^{1/q}}+\frac{2B}{\delta^{2/q}}\right].$$

The same estimate is valid for the other three integrals, too.

Combining (36) with (37) and its three counter parts, while using Minkowski's inequality yields

$$A_6 = \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_6|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q} \tag{38}$$

$$\leq \frac{8C_q}{(p-1)^{2/p}}\left[\epsilon+\frac{2B}{(\delta m_2)^{1/q}}+\frac{2B}{(\delta m_3)^{1/q}}+\frac{2B}{(\delta^2 m_2m_3)^{1/q}}\right]$$

$$= O(\epsilon),$$

provided that m_2 and m_3 are large. An analogous estimate is valid for

$$I_7 = \int_{CE_{m_1}}\int_{E_{m_2}}\int_{CE_{m_3}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$

$$\times D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3.$$

that is, we have

$$A_7 = \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_7|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q} \tag{39}$$

$$\leq \frac{8C_q}{(p-1)^{2/p}}\left[\epsilon+\frac{2B}{(\delta m_1)^{1/q}}+\frac{2B}{(\delta m_3)^{1/q}}+\frac{2B}{(\delta^2 m_1m_3)^{1/q}}\right]=O(\epsilon),$$

provided that m_1, and m_3 are large. An analogous estimate is valid for

$$I_8 = \int_{CE_{m_1}}\int_{CE_{m_2}}\int_{E_{m_3}}[f(x_1-t_1,x_2-t_2,x_3-t_3)-f(x_1,x_2,x_3)]$$

$$\times D\mu_1(t_1)D\mu_2(t_2)D\mu_3(t_3)dt_1dt_2dt_3.$$

that is, we have

$$A_8 = \left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|I_8|^q d\mu_1 d\mu_2 d\mu_3\right\} \tag{40}$$

$$\leq \frac{8C_q}{(p-1)^{2/p}}\left[\epsilon+\frac{2B}{(\delta m_1)^{1/q}}+\frac{2B}{(\delta m_2)^{1/q}} + \frac{2B}{(\delta^2 m_1 m_2)^{1/q}}\right] = O(\epsilon),$$

provided that m_1, and m_2 are large enough collecting together (19), (20), (22), (24), (30), (31), (32), (34), (38), (39) and (40), gives finally that

$$\left\{\frac{1}{m_1m_2m_3}\int_0^{m_1}\int_0^{m_2}\int_0^{m_3}|s_{\mu_1,\mu_2,\mu_3}(f,x_1,x_2,x_3) - f(x_1,x_2,x_3)|^q d\mu_1 d\mu_2 d\mu_3\right\}^{1/q} = O(\epsilon), \tag{41}$$

as $m_1, m_2, m_3 \to \infty$. Being $\epsilon > 0$ arbitrary, this proves (13).

Part (b)**.** Let (x_1, x_2, x_3) be an arbitrary point in G. Let $\eta > 0$ be such that

$$N = N(x_1,x_2,x_3) = \left\{(y_1,y_2,y_3)\in R^3 : [(y_1-x_1)^2+(y_2-x_2)^2 + (y_3-x_3)^2]^{1/3} \leq \eta\right\} \subset G.$$

We claim that (13) holds uniformly on N. In fact, by the uniform continuity of f on N, inequality (14) holds for all $(y_1, y_2, y_3) \in N$ in place of (x_1, x_2, x_3), possibly with a smaller $\delta > 0$. Since N is a compact set, M can be chosen so large that (16) also holds for all $(y_1, y_2, y_3) \in N$ in place of (x_1, x_2, x_3). It follows that the constant in the term $O(\epsilon)$ does not depend on $(y_1, y_2, y_3) \in N$ in the estimate (20), (24), (30), (31), (32), (34),(38), (39), (40) and consequently in (41). This proves part (b) and completes the proof of Theorem 3.

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