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APPROXIMATING COMMON RANDOM FIXED POINT FOR TWO FINITE FAMILIES OF ASYMPTOTICALLY NONEXPANSIVE RANDOM MAPPINGS

Abstract. The aim of this paper is to study weak and strong convergence of an implicit random iterative process with errors to a common random fixed point of two finite families of asymptotically nonexpansive random mappings in a uniformly convex separable Banach space.

Key words: asymptotically nonexpansive random mappings, implicit iterative process, weak and strong convergence, common random fixed points, condition (B), Opial's condition.

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1. Introduction

Random approximations and random fixed point theorems are stochastic generalizations of classical approximations and fixed point theorems. The study of random fixed point theorems was initiated by Prague school of probabilities in the 1950's by Spacek [25] and Hans [14], [15]. The interest in these problems was enhanced after the publication of the survey article of Bharucha-Reid [8] in 1976. Random fixed point theory and applications have been further developed rapidly in recent years (see e.g. [4], [2], [3], [6], [7], [17], [19], [29] and references therein). The class of asymptotically nonexpansive self-mappings introduced by Goebel and Kirk [13] in 1972. In 2001, Xu and Ori [30] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{n(modN)}x_n, \ \ n \geq 1, \ \ x_0 \in K, \tag{1}$$

for a finite family of nonexpansive mappings $\{T_1, T_2, ..., T_N\} : K \to K$, where K is a nonempty closed convex subset of a Hilbert space E and $\{\alpha_n\}_{n\geq 1}$ is a real sequence in $(0,1)$. They proved the weakly convergence of the sequence $\{x_n\}$ defined by (1) to a common fixed point $p \in F = \cap_{i=1}^{N} F(T_i)$.

In 2003, Sun [27] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n, \ \ n \geq 1, \ \ x_0 \in K, \tag{2}$$

for a finite family of asymptotically quasi-nonexpansive self-mappings on a bounded closed convex subset K of a Hilbert space E with $\{\alpha_n\}_{n\geq 1}$ a sequence in $(0,1)$, where $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, ..., N\}$, and proved the strong convergence of the sequence $\{x_n\}$ defined by (2) to a common fixed point $p \in F = \cap_{i=1}^{N} F(T_i)$.

In 2010, Filomena Cianciaruso et al. [12] considered the following implicit iterative process for a finite family of asymptotically nonexpansive mappings

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, \\ y_n &= (1-\beta_n-\delta_n) x_n + \beta_n T_{i(n)}^{k(n)} x_n + \delta_n v_n, \ n \geq 1, \end{aligned} \tag{3}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, ..., N\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, are sequences of real numbers in $(0,1)$ with $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$ and $\{u_n\}$, $\{v_n\}$, are two bounded sequences and x_0 is a given point. They proved convergence of the implicit iterative process defined by (3) to a common fixed point of asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Very recently, Hao et al. [16] studied the convergence of an implicit iterative process with errors for two finite families $\{T_i\}_{i=1}^{N}$, $\{S_i\}_{i=1}^{N} : K \to K$ of asymptotically nonexpansive mappings defined as follows:

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, \\ y_n &= (1-\beta_n-\delta_n) x_n + \beta_n S_{i(n)}^{k(n)} x_n + \delta_n v_n, \ \ n \geq 1, \end{aligned} \tag{4}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, ..., N\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, are sequences of real numbers in $[0,1]$ with $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$ and $\{u_n\}$, $\{v_n\}$, are two bounded sequences.

The development of random fixed point iterations was initiated by Choudhury in [10] where random Ishikawa iteration scheme was defined and its strong convergence to a random fixed point in Hilbert spaces was discussed. After that several authors have worked on random fixed point iterations some of which are noted in ([5], [9], [21], [20], [22], [23]) and many others. Banerjee et al. [1] construct a composite implicit random iterative process with errors for a finite family $\{T_i : i \in I = \{1, 2, \ldots, N\}\}$ of N continuous asymptotically nonexpansive random operators from $\Omega \times C$ to C, where C be nonempty closed convex subset of a separable Banach space E. They discuss the necessary and sufficient conditions for the convergence of this composite implicit random iterative process defined in the compact form as follows:

$$
\begin{aligned}
(5) \qquad \xi_n(t) &= \alpha_n \xi_{n-1}(t) + \beta_n T_{i(n)}^{k(n)}(t, \eta_n(t)) + \gamma_n f_n(t),\\
\eta_n(t) &= a_n \xi_n(t) + b_n T_{i(n)}^{k(n)}(t, \xi_n(t)) + c_n g_n(t), \quad n \geq 1, \ \forall t \in \Omega,
\end{aligned}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences of real numbers in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$ and $\{f_n(t)\}$, $\{g_n(t)\}$ are bounded sequences of measurable functions from Ω to C.

Inspired and motivated by theses facts, we investigate convergence of the following implicit random iterative process:

Definition 1. *Let $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ be two finite families of 2N asymptotically nonexpansive random mappings form $\Omega \times C$ to C. where C is a nonempty closed convex subset of a separable Banach space E. Let $\xi_0 : \Omega \to C$ be a measurable function. Then define the sequence $\{\xi_n(w)\}$ as*

$$
\begin{aligned}
(6) \qquad \xi_n(w) &= (1 - \alpha_n - \gamma_n)\xi_{n-1}(w) + \alpha_n T_{i(n)}^{k(n)}(w, \eta_n(w)) + \gamma_n f_n(w),\\
\eta_n(w) &= (1 - \beta_n - \delta_n)\xi_n(w) + \beta_n S_{i(n)}^{k(n)}(w, \xi_n(w)) + \delta_n g_n(w),
\end{aligned}
$$

where $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are sequences of real numbers in $[0,1]$ with $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $w \in \Omega$ and for all $n \geq 1$ and $\{f_n(w)\}$, $\{g_n(w)\}$ are bounded sequences of measurable functions from Ω to C.

We extend the random iterative process (5) to the case of two finite families of asymptotically nonexpansive random mappings $\{T_i, S_i : i = 1, 2, \ldots, N\}$ and also study the random version of the implicit iterative process (4). We obtain the weak and strong convergence of an implicit random iterative process (6) in a uniformly convex Banach space.

2. Preliminaries

Let (Ω, Σ) be a measurable space, C a nonempty subset of E. A mapping $\xi : \Omega \to C$ is called measurable if $\xi^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of a Banach space E. A mapping $T : \Omega \times C \to C$ is said to be random mapping if for each fixed $x \in C$, the mapping $T(., x) : \Omega \to C$ is measurable. A measurable mapping $\xi : \Omega \to C$ is called a random fixed point of the random mapping $T : \Omega \times C \to C$ if $T(w, \xi(w)) = \xi(w)$ for each $w \in \Omega$.

We denote the set of all random fixed points of random mapping T by $RF(T)$.

Definition 2 ([18]). *A Banach space E is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ weakly as $n \to \infty$ and $x \neq y$ implying that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$.

Definition 3. *A map $T : C \to E$ is called demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ weakly and $Tx_n \to y$ strongly imply that $x \in C$ and $Tx = y$.*

Definition 4 ([1]). *A finite family $\{T_i : i \in I = \{1, 2, 3, \dots, N\}\}$ of N continuous random operators from $\Omega \times C$ to E with $F = \bigcap\limits_{i=1}^{N} RF(T_i) \neq \emptyset$, is said to satisfy condition B on C if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) \geq 0$ for all $r \in (0, \infty)$ such that for all $w \in \Omega$, $f(d(\xi(w), F)) \leq \max\limits_{1\leq i\leq N}\{\|\xi(w) - T_i(w, \xi(w))\|\}$ for all $\xi(w)$, where $\xi : \Omega \to C$ is a measurable function and $d(\xi(w), F) = \inf\{\|\xi(w) - q(w)\| : q(w) \in F = \bigcap\limits_{i=1}^{N} RF(T_i)\}$.*

Definition 5 ([5]). *Let C be a nonempty closed convex subset of a separable Banach space E and $T : \Omega \times C \to E$ be a random mapping. Then T is said to be*

(a) *Nonexpansive random operator if for arbitrary $x, y \in C$,*

$$\|T(w, x) - T(w, y)\| \leq \|x - y\|, \ \forall w \in \Omega.$$

(b) *Asymptotically nonexpansive random mapping if there exists a measurable mapping sequence $r_n(w) : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} r_n(w) = 1$ for each $w \in \Omega$ such that for arbitrary $x, y \in C$ and for each $w \in \Omega$*

$$\|T^n(w, x) - T^n(w, y)\| \leq r_n(w)\|x - y\|, \ \ n = 1, 2, \dots$$

(c) *Uniformly L-Lipschitzian random mapping if there exists a constant $L > 0$ such that for arbitrary $x, y \in C$ and $w \in \Omega$*

$$\|T^n(w, x) - T^n(w, y)\| \leq L\|x - y\|, \ \ n = 1, 2, ...$$

(d) *Semicompact random mapping if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to C with $\lim_{n\to\infty} \|\xi_n(w) - T(w, \xi_n(w))\| = 0$ for all $w \in \Omega$ there exists a subsequence $\{\xi_{n_k}(w)\}$ of $\{\xi_n(w)\}$ such that $\{\xi_{n_k}(w)\} \to \{\xi(w)\}$ as $k \to \infty$ for each $w \in \Omega$, where $\{\xi(w)\}$ is a measurable mapping from Ω to C.*

Remark 1. Every asymptotically nonexpansive random mapping is uniformly L-Lipschitzian, where $L = \sup_{w\in\Omega, n\geq 1} r_n(w)$.

The following lemmas are useful for proving our main results.

Lemma 1 ([28]). *Let $\{a_n\}$, $\{b_n\}$ and $\{m_n\}$ be nonnegative real sequences satisfying*

$$a_{n+1} \leq (1+m_n)a_n + b_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} m_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(a) $\lim_{n\to\infty} a_n$ *exists.*
(b) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2 ([24]). *Let E be a uniformly convex Banach space, and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3 (Demiclosedness Principle, [11]). *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $T : C \to E$ be asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero. i.e., if $x_n \to x$ weakly and $\|x_n - Tx_n\| \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 4 ([26]). *Let E be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be sequence in E. Let $u, v \in E$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequence of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3. Main results

Before proving our main results, we shall prove the following crucial lemmas:

Lemma 5. *Let E be a separable Banach space and C be a nonempty closed convex subset of E. Let $\{T_i, S_i : i \in I = \{1, 2, \ldots, N\}\}$ be 2N asymptotically nonexpansive random mappings with sequences of measurable mappings $\{r_{i_n}\} : \Omega \to [1, \infty)$ such that $\sum_{n=1}^{\infty}(r_{i_n}(w) - 1) < \infty$, $r_{i_n}(w) \to 1$ as $n \to \infty$, for all $w \in \Omega$ and $i \in I = \{1, 2, \ldots, N\}$. Suppose that $F = \bigcap_{i=1}^{N}(RF(T_i) \cap RF(S_i)) \neq \emptyset$. Let $\{\xi_n(w)\}$ be the sequence defined as in (6)* *with the additional assumption* $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\delta_n<\infty$ *and* $\limsup_{n\to\infty}\alpha_n<1$. *Then*

(a) $\lim_{n\to\infty}\|\xi_n(w)-\xi(w)\|$ *exists for all* $\xi(w)\in F=\bigcap_{i=1}^{N}(RF(T_i)\cap RF(S_i))$.

(b) $\lim_{n\to\infty}d(\xi_n(w),F)$ *exists where* $d(\xi_n(w),F)=\inf_{\xi(w)\in F}\|\xi_n(w)-\xi(w)\|$.

Proof. Let $\xi(w)\in F$. Since $\{f_n\}$ and $\{g_n\}$ are bounded sequence of measurable function from Ω to C, we can put for each $w\in\Omega$

$$M(w)=\sup_{n\geq 1}\|f_n(w)-\xi(w)\|\vee\sup_{n\geq 1}\|g_n(w)-\xi(w)\|. \tag{7}$$

Then $M(w)<\infty$ for each $w\in\Omega$ and $n\geq 1$. For $n\geq 1$, let $r_n(w)=\max\{r_{i_n}(w):i\in I=\{1,2,\ldots,N\}\}$, then we can write

$$\begin{aligned}\|T_{i(n)}^{k(n)}(w,x)-T_{i(n)}^{k(n)}(w,y)\|&\leq r_n(w)\|x-y\|\\ \|S_{i(n)}^{k(n)}(w,x)-S_{i(n)}^{k(n)}(w,y)\|&\leq r_n(w)\|x-y\|,\quad w\in\Omega.\end{aligned}\tag{8}$$

Using (6), (7) and (8), we have for $\xi(w)\in F$ and $w\in\Omega$ that

$$\begin{aligned}&\|\xi_n(w)-\xi(w)\|\\ &=\|(1-\alpha_n-\gamma_n)\xi_{n-1}(w)+\alpha_nT_{i(n)}^{k(n)}(w,\eta_n(w))+\gamma_nf_n(w)\|\\ &=\|(1-\alpha_n-\gamma_n)(\xi_{n-1}(w)-\xi(w))\\ &\quad+\alpha_n(T_{i(n)}^{k(n)}(w,\eta_n(w))-\xi(w))+\gamma_n(f_n(w)-\xi(w))\|\\ &\leq(1-\alpha_n-\gamma_n)\|\xi_{n-1}(w)-\xi(w)\|\\ &\quad+\alpha_n\|T_{i(n)}^{k(n)}(w,\eta_n(w))-\xi(w)\|+\gamma_n\|f_n(w)-\xi(w)\|\\ &\leq(1-\alpha_n-\gamma_n)\|\xi_{n-1}(w)-\xi(w)\|\\ &\quad+\alpha_nr_n(w)\|\eta_n(w)-\xi(w)\|+\gamma_nM(w)\\ &\leq(1-\alpha_n)\|\xi_{n-1}(w)-\xi(w)\|\\ &\quad+\alpha_nr_n(w)\|\eta_n(w)-\xi(w)\|+\gamma_nM(w)\end{aligned}\tag{9}$$

On the other hand,

$$\begin{aligned}&\|\eta_n(w)-\xi(w)\|\\ &=\|(1-\beta_n-\delta_n)\xi_n(w)+\beta_nS_{i(n)}^{k(n)}(w,\xi_n(w))+\delta_ng_n(w)-\xi(w)\|\\ &\leq(1-\beta_n-\delta_n)\|\xi_n(w)-\xi(w)\|+\beta_n\|S_{i(n)}^{k(n)}(w,\xi_n(w))-\xi(w)\|\\ &\quad+\delta_n\|g_n(w)-\xi(w)\|\\ &\leq(1-\beta_n)\|\xi_n(w)-\xi(w)\|\\ &\quad+\beta_n\|S_{i(n)}^{k(n)}(w,\xi_n(w))-\xi(w)\|+\delta_n\|g_n(w)-\xi(w)\|\\ &\leq(1-\beta_n)\|\xi_n(w)-\xi(w)\|+\beta_nr_n(w)\|\xi_n(w)-\xi(w)\|+\delta_nM(w)\end{aligned}\tag{10}$$

$$
\begin{aligned}
&= (1-\beta_n+\beta_n r_n(w))\|\xi_n(w)-\xi(w)\|+\delta_n M(w) \\
&\le r_n(w)\|\xi_n(w)-\xi(w)\|+\delta_n M(w),
\end{aligned}
$$

where the last inequality follows from $r_n(w) \ge 1$. Putting (10) into (9), we get

$$
\begin{aligned}
(11) \qquad \|\xi_n(w)-\xi(w)\| \;\le\; & (1-\alpha_n)\|\xi_{n-1}(w)-\xi(w)\| \\
& + \alpha_n r_n(w)\,[r_n(w)\|\xi_n(w)-\xi(w)\| \\
& + \ \delta_n M(w)] + \gamma_n M(w) \\
= \; & (1-\alpha_n)\|\xi_{n-1}(w)-\xi(w)\| \\
& + \alpha_n r_n^2(w)\|\xi_n(w)-\xi(w)\| \\
& + (\alpha_n r_n(w)\delta_n+\gamma_n)M(w)
\end{aligned}
$$

Rearranging both sides, we obtain

$$
\begin{aligned}
(12) \qquad \|\xi_n(w)-\xi(w)\| \;\le\; & \frac{1-\alpha_n}{1-\alpha_n r_n^2(w)}\|\xi_{n-1}(w)-\xi(w)\| \\
& + \frac{\alpha_n r_n(w)\delta_n+\gamma_n}{1-\alpha_n r_n^2(w)M(w)} \\
= \; & 1+\frac{\alpha_n r_n^2(w)-\alpha_n}{1-\alpha_n r_n^2(w)}\|\xi_{n-1}(w)-\xi(w)\| \\
& + \frac{\alpha_n r_n(w)\delta_n+\gamma_n}{1-\alpha_n r_n^2(w)}M(w) \\
= \; & (1+A_n(w))\|\xi_{n-1}(w)-\xi(w)\|+B_n(w).
\end{aligned}
$$

Since $\limsup\limits_{n\to\infty}\alpha_n < 1$, then there exists $\lambda < 1$ such that $\alpha_n \le \lambda$ for big n, therefore

$$
\begin{aligned}
A_n(w) &= \frac{\alpha_n r_n^2(w)-\alpha_n}{1-\alpha_n r_n^2(w)} = \frac{\alpha_n(r_n^2(w)-1)}{1-\alpha_n r_n^2(w)} \\
&\le \frac{\lambda(r_n^2(w)-1)}{1-\lambda r_n^2(w)} = \frac{\lambda(r_n(w)+1)(r_n(w)-1)}{1-\lambda r_n^2(w)},
\end{aligned}
$$

and since $\lim\limits_{n\to\infty} r_n(w) = 1$, we obtain $\lim\limits_{n\to\infty}\frac{\lambda(r_n(w)+1)}{1-\lambda r_n^2(w)} \le \frac{2\lambda}{1-\lambda}$, then there exists a real constant k such that $\frac{\lambda(r_n(w)+1)}{1-\lambda r_n^2(w)} \le k$, $\forall n \ge 1$. it follows that $\sum\limits_{n=1}^{\infty} A_n(w) = \sum\limits_{n=1}^{\infty}\frac{\alpha_n(r_n^2(w)-1)}{1-\alpha_n r_n^2(w)} < \infty$. Similarly, we can prove that $\sum\limits_{n=1}^{\infty} B_n(w) = \sum\limits_{n=1}^{\infty}\frac{\alpha_n r_n(w)\delta_n+\gamma_n}{1-\alpha_n r_n^2(w)}M(w) < \infty$. It follows by lemma 1 and inequality (12) that $\lim\limits_{n\to\infty}\|\xi_n(w)-\xi(w)\|$ exists for all $\xi(w)\in F$. To prove (2). Putting $inf_{\xi\in F}$ on both sides of (12), we get $d(\xi_n(w), F) \le (1 + A_n(w))d(\xi_{n-1}(w), F) + B_n(w)$, then also by lemma 1, we obtain that $\lim\limits_{n\to\infty} d(\xi_n(w), F)$ exists and for all $w \in \Omega$ ■

Lemma 6. *Let E be a uniformly convex separable Banach space and C be a nonempty closed convex subset of E. Let $\{T_i, S_i : i \in I = \{1, 2, \dots, N\}\}$ be 2N asymptotically nonexpansive random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(r_{i_n}(w) - 1) < \infty$, $r_{i_n}(w) \to 1$ as $n \to \infty$, for all $w \in \Omega$ and $i \in I = \{1, 2, \dots, N\}$. Suppose that $F = \bigcap\limits_{i=1}^{N}(RF(T_i) \cap RF(S_i)) \neq \emptyset$. Let $\{\xi_n(w)\}$ be the sequence defined as in (6) with the additional assumption $\sum\limits_{n=1}^{\infty}\gamma_n < \infty$, $\sum\limits_{n=1}^{\infty}\delta_n < \infty$ and $\limsup\limits_{n\to\infty}\alpha_n < 1$. Then*

(*a*) $\lim\limits_{n\to\infty} \|\xi_n(w) - T_l(w, \xi_n(w)\| = 0$,
(*b*) $\lim\limits_{n\to\infty} \|\xi_n(w) - S_l(w, \xi_n(w)\| = 0$,
(*c*) $\lim\limits_{n\to\infty} \|T_l(w, \xi_n(w)) - S_l(w, \xi_n(w))\| = 0$,

for all $w \in \Omega$ and for all $l = 1, 2, \dots, N$.

Proof. Let $\xi(w) \in F$. Since $\{f_n\}$ and $\{g_n\}$ are bounded sequence of measurable function from Ω to C, we can put for each $w \in \Omega$

$$M(w) = \sup_{n\ge1}\|f_n(w) - \xi(w)\| \vee \sup_{n\ge1}\|g_n(w) - \xi(w)\|.$$

Then $M(w) < \infty$ for each $w \in \Omega$ and $n \ge 1$. By Lemma 5, we see that $\lim_{n\to\infty}\|\xi_n(w) - \xi(w)\|$ exists for each $w \in \Omega$. Assume that $\lim_{n\to\infty}\|\xi_n(w) - \xi(w)\| = c$. Similarly, by using (10), we have

$$\|\eta_n(w) - \xi(w)\| \le r_n(w)\|\xi_n(w) - \xi(w)\| + \delta_n M(w).$$

Taking $\limsup_{n\to\infty}$ on both sides of the inequality, (where $\lim\limits_{n\to\infty}\delta_n = 0$) we have

$$\limsup_{n\to\infty}\|\eta_n(w) - \xi(w)\| \le c. \tag{13}$$

In addition $\|T_{i(n)}^{k(n)}(w, \eta_n(w)) - \xi(w)\| \le r_n\|\eta_n(w) - \xi(w)\|$, taking $\limsup_{n\to\infty}$ on both sides of the inequality, we have

$$\limsup_{n\to\infty}\|T_{i(n)}^{k(n)}(w, \eta_n(w)) - \xi(w)\| \le c. \tag{14}$$

Since $\lim_{n\to\infty}\gamma_n = 0$, it follows from (14) that

$$\|T_{i(n)}^{k(n)}(w, \eta_n(w)) - \xi(w) + \gamma_n(f_n(w) - \xi_{n-1}(w))\| \tag{15}$$

$$\leq \|T_{i(n)}^{k(n)}(w,\eta_n(w)) - \xi(w)\| + \gamma_n\|f_n(w) - \xi_{n-1}(w)\|$$
$$\Rightarrow \limsup_{n\to\infty} \|T_{i(n)}^{k(n)}(w,\eta_n(w)) - \xi(w) + \gamma_n(f_n(w) - \xi_{n-1}(w))\| \leq c.$$

Also,

$$\begin{aligned}\limsup_{n\to\infty} &\|\xi_{n-1}(w) - \xi(w) + \gamma_n(f_n(w) - \xi_{n-1}(w))\| \\ &\leq \limsup_{n\to\infty} \|\xi_{n-1}(w) - \xi(w)\| = c.\end{aligned} \tag{16}$$

Now, by using (6) we have

$$\begin{aligned} c &= \lim_{n\to\infty} \|\xi_n(w) - \xi(w)\| = \lim_{n\to\infty} \|(1-\alpha_n-\gamma_n)\xi_{n-1}(w) \\ &\quad + \alpha_n T_{i(n)}^{k(n)}(w,\eta_n(w)) + \gamma_n f_n(w) - \xi(w)\| \\ &= \lim_{n\to\infty} \|\alpha_n T_{i(n)}^{k(n)}(w,\eta_n(w)) + (1-\alpha_n)\xi_{n-1}(w) \\ &\quad - \gamma_n\xi_{n-1}(w) + \gamma_n f_n(w) - (1-\alpha_n)\xi(w) - \alpha_n\xi(w)\| \\ &= \lim_{n\to\infty} \|\alpha_n T_{i(n)}^{k(n)}(w,\eta_n(w)) - \alpha_n\xi(w) + \alpha_n\gamma_n f_n(w) \\ &\quad - \alpha_n\gamma_n\xi_{n-1}(w) + (1-\alpha_n)\xi_{n-1}(w) - (1-\alpha_n)\xi(w) \\ &\quad - \gamma_n\xi_{n-1}(w) + \gamma_n f_n(w) - \alpha_n\gamma_n f_n(w) + \alpha_n\gamma_n\xi_{n-1}(w)\| \\ &= \lim_{n\to\infty} \|\alpha_n(T_{i(n)}^{k(n)}(w,\eta_n(w)) - \xi(w) + \gamma_n(f_n(w) - \xi_{n-1}(w))) \\ &\quad + (1-\alpha_n)(\xi_{n-1}(w) - \xi(w) + \gamma_n(f_n(w) - \xi_{n-1}(w)))\| \end{aligned} \tag{17}$$

From (15), (16), (17) and Lemma 2, we obtain

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)}(w,\eta_n(w)) - \xi_{n-1}(w)\| = 0. \tag{18}$$

On the other hand,

$$\begin{aligned} &\|\xi_n(w) - T_{i(n)}^{k(n)}(w,\eta_n(w))\| \\ &\quad \leq \|\xi_n(w) - \xi_{n-1}(w)\| + \|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w,\eta_n(w))\| \\ &\quad = \|(1-\alpha_n-\gamma_n)\xi_{n-1}(w) + \alpha_n T_{i(n)}^{k(n)}(w,\eta_n(w)) \\ &\qquad + \gamma_n f_n(w) - \xi_{n-1}(w)\| + \|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w,\eta_n(w))\| \\ &\quad = \|\xi_{n-1}(w) - \alpha_n\xi_{n-1}(w) - \gamma_n\xi_{n-1}(w) + \alpha_n T_{i(n)}^{k(n)}(w,\eta_n(w)) \\ &\qquad + \gamma_n f_n(w) - \xi_{n-1}(w)\| + \|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w,\eta_n(w))\| \\ &\quad \leq \alpha_n\|T_{i(n)}^{k(n)}(w,\eta_n(w)) - \xi_{n-1}(w)\| + \gamma_n\|f_n(w) - \xi_{n-1}(w)\| \\ &\qquad + \|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w,\eta_n(w))\| \\ &\quad = (1+\alpha_n)\|T_{i(n)}^{k(n)}(w,\eta_n(w)) - \xi_{n-1}(w)\| + \gamma_n\|f_n(w) - \xi_{n-1}(w)\| \end{aligned}$$

By (18), we have

$$\lim_{n\to\infty} \|\xi_n(w) - T_{i(n)}^{k(n)}(w, \eta_n(w))\| = 0 \tag{19}$$

Also, we have

$$\begin{aligned} \|\xi_n(w) - \xi(w)\| &\leq \|\xi_n(w) - T_{i(n)}^{k(n)}(w, \eta_n(w))\| + \|T_{i(n)}^{k(n)}(w, \eta_n(w)) - \xi(w)\| \\ &\leq \|\xi_n(w) - T_{i(n)}^{k(n)}(w, \eta_n(w))\| + r_n(w)\|\eta_n(w) - \xi(w)\|, \end{aligned}$$

which implies by (19) that

$$c = \lim_{n\to\infty} \|\xi_n(w) - \xi(w)\| \leq \liminf_{n\to\infty} \|\eta_n(w) - \xi(w)\|.$$

Since $c \leq \liminf_{n\to\infty} \|\eta_n(w) - \xi(w)\| \leq \limsup_{n\to\infty} \|\eta_n(w) - \xi(w)\| \leq c$, Thus,

$$\lim_{n\to\infty} \|\eta_n(w) - \xi(w)\| = c. \tag{20}$$

Now, we have

$$\begin{aligned} &\limsup_{n\to\infty} \|S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi(w)\| \\ &\qquad \leq \limsup_{n\to\infty} r_n(w)\|\xi_n(w) - \xi(w)\| = c \end{aligned} \tag{21}$$

Also,

$$\begin{aligned} &\|S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi(w) + \delta_n(g_n(w) - \xi(w))\| \\ &\qquad \leq \|S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi(w)\| + \delta_n\|g_n(w) - \xi(w)\| \end{aligned}$$

Using (21), we have

$$\lim_{n\to\infty} \|S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi(w) + \delta_n(g_n(w) - \xi(w))\| \leq c. \tag{22}$$

In addition,

$$\begin{aligned} &\limsup_{n\to\infty} \|\xi_n(w) - \xi(w) + \delta_n(g_n(w) - \xi(w))\| \\ &\qquad \leq \limsup_{n\to\infty} \|\xi_n(w) - \xi(w)\| = c. \end{aligned} \tag{23}$$

On the other hand,

$$\begin{aligned} c &= \lim_{n\to\infty} \|\eta_n(w) - \xi(w)\| = \lim_{n\to\infty} \|(1 - \beta_n - \delta_n)\xi_n(w) \\ &\quad + \beta_n S_{i(n)}^{k(n)}(w, \xi_n(w)) + \delta_n g_n(w) - \xi(w)\| \\ &= \lim_{n\to\infty} \|\beta_n S_{i(n)}^{k(n)}(w, \xi_n(w)) + (1 - \beta_n)\xi_n(w) - \delta_n\xi_n(w) \\ &\quad + \delta_n g_n(w) - (1 - \beta_n)\xi(w) - \beta_n\xi(w)\| \end{aligned} \tag{24}$$

$$= \lim_{n\to\infty} \|\beta_n S_{i(n)}^{k(n)}(w, \xi_n(w)) - \beta_n \xi(w) + \beta_n \delta_n g_n(w) - \beta_n \delta_n \xi_n(w) + (1-\beta_n)\xi_n(w) - (1-\beta_n)\xi(w) - \delta_n \xi_n(w) + \delta_n g_n(w) - \beta_n \delta_n g_n(w) + \beta_n \delta_n \xi_n(w)\|$$
$$= \lim_{n\to\infty} \|\beta_n (S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi(w) + \delta_n(g_n(w) - \xi_n(w))) + (1-\beta_n)(\xi_n(w) - \xi(w) + \delta_n(g_n(w) - \xi_n(w)))\|$$

From (22), (23), (24) and Lemma 2, we obtain

$$\lim_{n\to\infty} \|S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi_n(w)\| = 0. \tag{25}$$

Notice that,

$$\begin{aligned}
&\|\eta_n(w) - \xi_n(w)\| \\
&\quad = \|(1-\beta_n-\delta_n)\xi_n(w) + \beta_n S_{i(n)}^{k(n)}(w, \xi_n(w)) + \delta_n g_n(w) - \xi_n(w)\| \\
&\quad \le \beta_n \|S_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi_n(w)\| + \delta_n \|g_n(w) - \xi_n(w)\|,
\end{aligned}$$

Using (25) we obtain

$$\lim_{n\to\infty} \|\eta_n(w) - \xi_n(w)\| = 0, \tag{26}$$

Since,

$$\begin{aligned}
&\|T_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi_n(w)\| \\
&\quad \le \|T_{i(n)}^{k(n)}(w, \xi_n(w)) - T_{i(n)}^{k(n)}(w, \eta_n(w))\| + \|T_{i(n)}^{k(n)}(w, \eta_n(w)) - \xi_n(w)\| \\
&\quad \le r_n \|\xi_n(w) - \eta_n(w)\| + \|T_{i(n)}^{k(n)}(w, \eta_n(w)) - \xi_n(w)\|.
\end{aligned}$$

By using (19), (26), we get

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)}(w, \xi_n(w)) - \xi_n(w)\| = 0. \tag{27}$$

also,

$$\begin{aligned}
&\|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w, \xi_n(w))\| \\
&\quad \le \|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w, \eta_n(w))\| + \|T_{i(n)}^{k(n)}(w, \eta_n(w)) - T_{i(n)}^{k(n)}(w, \xi_n(w))\|.
\end{aligned}$$

Both (18) and (26) imply that

$$\lim_{n\to\infty} \|\xi_{n-1}(w) - T_{i(n)}^{k(n)}(w, \xi_n(w))\| = 0. \tag{28}$$

Now,

$$\begin{aligned}&\|\xi_n(w)-\xi_{n-1}(w)\|\\&\quad\le \alpha_n\|T_{i(n)}^{k(n)}(w,\eta_n(w))-\xi_{n-1}(w)\|+\gamma_n\|f_n(w)-\xi_{n-1}(w)\|.\end{aligned}$$

Using (18), we get $\lim_{n\to\infty}\|\xi_n(w)-\xi_{n-1}(w)\|=0$. Hence

$$\lim_{n\to\infty}\|\xi_n(w)-\xi_{n+l}(w)\|=0, \tag{29}$$

for all $w\in\Omega$ and for all $l\in I$. Since

$$\begin{aligned}&\|T_{i(n)}^{k(n)}(w,\xi_n(w))-S_{i(n)}^{k(n)}(w,\xi_n(w))\|\\&\quad\le \|T_{i(n)}^{k(n)}(w,\xi_n(w))-\xi_n(w)\|+\|\xi_n(w)-S_{i(n)}^{k(n)}(w,\xi_n(w))\|,\end{aligned}$$

By (25) and (27), we get

$$\lim_{n\to\infty}\|T_{i(n)}^{k(n)}(w,\xi_n(w))-S_{i(n)}^{k(n)}(w,\xi_n(w))\|=0. \tag{30}$$

Notice that

$$\begin{aligned}&\|\xi_{n-1}(w)-T_{i(n)}(w,\xi_n(w))\|\\&\quad\le \|\xi_{n-1}(w)-T_{i(n)}^{k(n)}(w,\xi_n(w))\|\\&\qquad+\|T_{i(n)}^{k(n)}(w,\xi_n(w))-T_{i(n)}(w,\xi_n(w))\|\\&\quad\le \|\xi_{n-1}(w)-T_{i(n)}^{k(n)}(w,\xi_n(w))\|\\&\qquad+L\|T_{i(n)}^{k(n)-1}(w,\xi_n(w))-\xi_n(w)\|\\&\quad\le \|\xi_{n-1}(w)-T_{i(n)}^{k(n)}(w,\xi_n(w))\|\\&\qquad+L[\|T_{i(n)}^{k(n)-1}(w,\xi_n(w))-T_{i(n-N)}^{k(n)-1}(w,\xi_{n-N}(w))\|\\&\qquad+\|T_{i(n-N)}^{k(n)-1}(w,\xi_{n-N}(w))-\xi_{(n-N)-1}(w)\|\\&\qquad+\|\xi_{(n-N)-1}(w)-\xi_n(w)\|].\end{aligned} \tag{31}$$

Since for each $n>N$, $n=(n-N)(modN)$ and $n=(K(n)-1)N+i(n)$, we have $k(n-N)=k(n)-1$ and $i(n-N)=i(n)$.

$$\|T_{i(n)}^{k(n)-1}(w,\xi_n(w))-T_{i(n-N)}^{k(n)-1}(w,\xi_{n-N}(w))\|\le L\|\xi_n(w)-\xi_{n-N}(w)\|, \tag{32}$$

and

$$\begin{aligned}&\|T_{i(n-N)}^{k(n)-1}(w,\xi_{n-N}(w))-\xi_{(n-N)-1}(w)\|\\&\qquad=\|T_{i(n-N)}^{k(n-N)}(w,\xi_{n-N}(w))-\xi_{(n-N)-1}(w)\|.\end{aligned} \tag{33}$$

Substituting (33) and (32) into (31), we obtain

$$\begin{aligned}&\|\xi_{n-1}(w)-T_{i(n)}(w,\xi_n(w))\|\\&\quad\le\|\xi_{n-1}(w)-T^{k(n)}_{i(n)}(w,\xi_n(w))\|+L^2\|\xi_n(w)-\xi_{n-N}(w)\|\\&\qquad+L\|T^{k(n-N)}_{i(n-N)}(w,\xi_{n-N}(w))-\xi_{(n-N)-1}(w)\|\\&\qquad+L\|\xi_{(n-N)-1}(w)-\xi_{n(w)}\|\end{aligned}$$

It follows by (28) and (29) that

$$\lim_{n\to\infty}\|\xi_{n-1}(w)-T_{i(n)}(w,\xi_n(w))\|=0. \tag{34}$$

and

$$\begin{aligned}&\|\xi_n(w)-T_{i(n)}(w,\xi_n(w))\|\\&\quad\le\|\xi_n(w)-\xi_{n-1}(w)\|+\|\xi_{n-1}(w)-T_{i(n)}(w,\xi_n(w))\|\to 0 \text{ as } (n\to\infty).\end{aligned} \tag{35}$$

Now for each $l=1,2,\dots,N$, we have

$$\begin{aligned}&\|\xi_n(w)-T_{n+l}(w,\xi_n(w))\|\\&\quad\le\|\xi_n(w)-\xi_{n+l}(w)\|+\|\xi_{n+l}(w)-T_{n+l}(w,\xi_{n+l}(w)))\|\\&\qquad+\|T_{n+l}(w,\xi_{n+l}(w))-T_{n+l}(w,\xi_n(w)))\|\\&\quad\le\|\xi_n(w)-\xi_{n+l}(w)\|+\|\xi_{n+l}(w)-T_{n+l}(w,\xi_{n+l}(w)))\|\\&\qquad+L\|\xi_{n+l}(w)-\xi_n(w)\|\to 0 \text{ as } n\to\infty \text{ for each } w\in\Omega.\end{aligned} \tag{36}$$

Consequently, we have

$$\|\xi_n(w)-T_l(w,\xi_n(w))\|\to 0, \tag{37}$$

for each $w\in\Omega$ and for each $l=1,2,\dots,N$. Similarly, we can prove that

$$\lim_{n\to\infty}\|\xi_n(w)-S_l(w,\xi_n(w))\|=0, \tag{38}$$

for each $w\in\Omega$ and for each $l=1,2,\dots,N$. Finally, since

$$\begin{aligned}&\|T_l(w,\xi_n(w))-S_l(w,\xi_n(w))\|\\&\qquad\le\|T_l(w,\xi_n(w))-\xi_n(w)\|+\|\xi_n(w)-S_l(w,\xi_n(w))\|\end{aligned}$$

Thus by (37) and (38), we obtain

$$\lim_{n\to\infty}\|T_l(w,\xi_n(w))-S_l(w,\xi_n(w))\|=0, \tag{39}$$

for each $w\in\Omega$ and for each $l=1,2,\dots,N$. ■

In the next, we study strong convergence of the sequence $\{\xi_n(w)\}$ defined by (6) to a common random fixed point of $\{T_i,S_i : i=1,2,\dots,N\}$.

Theorem 1. *Let E be a separable Banach space and C be a nonempty closed convex subset of E. Let $\{T_i, S_i : i \in I = \{1, 2, \ldots, N\}\}$ be 2N asymptotically nonexpansive random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(r_{i_n}(w) - 1) < \infty$, $r_{i_n}(w) \to 1$ as $n \to \infty$, for all $w \in \Omega$ and $i \in \{1, 2, \ldots, N\}$. Suppose that $F = \bigcap\limits_{i=1}^{N}(RF(T_i) \cap RF(S_i)) \neq \emptyset$. Let $\{\xi_n(w)\}$ be the sequence defined as in (6) with the additional assumption $\sum\limits_{n=1}^{\infty}\gamma_n < \infty$, $\sum\limits_{n=1}^{\infty}\delta_n < \infty$ and $\limsup\limits_{n\to\infty}\alpha_n < 1$. Then $\{\xi_n(w)\}$ converges to a common random fixed point of $\{T_i, S_i : i = 1, 2, \ldots, N\}$ if and only if*

$$\liminf_{n\to\infty} d(\xi_n(w), F) = 0, \quad w \in \Omega. \tag{40}$$

Proof. The necessity of (40) is obvious. To prove the sufficiency of (40), we have by Lemma 5, that $\lim\limits_{n\to\infty} d(\xi_n(w), F)$ exists for $w \in \Omega$ and we have from the hypothesis of the Theorem that $\liminf\limits_{n\to\infty} d(\xi_n(w), F) = 0$, $w \in \Omega$, then $\lim\limits_{n\to\infty} d(\xi_n(w), F) = 0$. Now, since $1 + x \leq e^x$ for $x > 0$ and from (12), we have that

$$\begin{aligned}
&\|\xi_{n+m}(w) - \xi(w)\| \\
&\leq (1 + A_{n+m}(w))\|\xi_{n+m-1}(w) - \xi(w)\| + B_{n+m}(w) \\
&\leq e^{A_{n+m}(w)}\|\xi_{n+m-1}(w) - \xi(w)\| + B_{n+m}(w) \\
&\leq e^{A_{n+m}(w)+A_{n+m-1}(w)}\|\xi_{n+m-2}(w) - \xi(w)\| \\
&+ e^{A_{n+m}(w)}B_{n+m-1}(w) + B_{n+m}(w) \\
&\qquad\vdots \\
&\leq e^{\sum\limits_{i=n+1}^{n+m} A_i(w)}\|\xi_n(w) - \xi(w)\| \\
&+ \sum_{k=n+1}^{n+m-1} B_k(w) e^{\sum\limits_{i=k+1}^{n+m} A_i(w)} + B_{n+m}(w) \\
&\leq R(w)\|\xi_n(w) - \xi(w)\| + R(w)\sum_{k=n+1}^{\infty} B_k(w),
\end{aligned} \tag{41}$$

for each $w \in \Omega$ and for all natural numbers m, n where $R(w) = e^{\sum\limits_{n=1}^{\infty} A_n(w)} < \infty$. Therefore, for any $\xi(w) \in F$, (41) implies that

$$\begin{aligned}
&\|\xi_{n+m}(w) - \xi_n(w)\| \\
&\qquad \leq \|\xi_{n+m}(w) - \xi(w)\| + \|\xi_n(w) - \xi(w)\|
\end{aligned} \tag{42}$$

$$\leq R(w)\|\xi_n(w) - \xi(w)\| + R(w) \sum_{k=n+1}^{\infty} B_k(w) + \|\xi_n(w) - \xi(w)\|$$

$$= (R(w) + 1)\|\xi_n(w) - \xi(w)\| + R(w) \sum_{k=n+1}^{\infty} B_k(w).$$

Since $\lim\limits_{n\to\infty} d(\xi_n(w), F) = 0$, and $\sum\limits_{n=1}^{\infty} B_n(w) < \infty$, given $\epsilon > 0$, there exists a natural number n_0 such that $d(\xi_n(w), F) < \frac{\epsilon}{2(R(w)+1)}$ and $\sum\limits_{n=1}^{\infty} B_n(w) < \frac{\epsilon}{2R(w)}$ for all $n \geq n_0$. So there exists $\xi^*(w) \in F$ such that $\|\xi_n(w) - \xi^*(w)\| < \frac{\epsilon}{2(R(w)+1)}$ for all $n \geq n_0$. Therefore from (42), we have for all $n \geq n_0$ that

$$\begin{aligned}\|\xi_{n+m}(w) - \xi_n(w)\| &\leq (R(w) + 1)\|\xi_n(w) - \xi^*(w)\| + R(w) \sum_{k=n+1}^{\infty} B_k(w) \\ &< (R(w) + 1)\frac{\epsilon}{2(R(w) + 1)} + R(w)\frac{\epsilon}{2R(w)} = \epsilon,\end{aligned}$$

which implies that $\{\xi_n(w)\}$ is a Cauchy sequence in C for each $w \in \Omega$. Since C is closed subset of E, then there exists $p(w)$ such that $\lim\limits_{n\to\infty} \xi_n(w) = p(w)$, where p being the limit of measurable functions is also measurable. Now we show that $p(w) \in F$. Since for each $w \in \Omega$, $\lim\limits_{n\to\infty} \xi_n(w) = p(w)$, there exists $n_1 \in \mathbb{N}$ such that $\|\xi_n(w) - p(w)\| < \frac{\epsilon}{2(1+r_l(w)}$ for all $n \geq n_1$. Since $\lim\limits_{n\to\infty} d(\xi_n(w), F) = 0$ for each $w \in \Omega$ there exists $n_2 \in N$ such that $d(\xi_n(w), F) < \frac{\epsilon}{2(1+r_l(w))}$ for all $n \geq n_2$. So there exists $q \in F$ such that $\|\xi_n(w) - q(w)\| < \frac{\epsilon}{2(1+r_l(w)}$ for all $n \geq n_2$. Let $n_3 = \max\{n_1, n_2\}$. For all $l \in I = \{1, 2, ..., N\}$ and for all $n \geq n_3$

$$\begin{aligned}\|T_l(w, p(w)) - p(w)\| & \\ &\leq \|T_l(w, p(w)) - q(w)\| + \|q(w) - p(w)\| \\ &\leq \|T_l(w, p(w)) - T_l(w, q(w))\| + \|q(w) - p(w)\| \\ &\leq r_l(w)\|q(w) - p(w)\| + \|q(w) - p(w)\| \\ &= (1 + r_l(w))\|q(w) - p(w)\| \\ &\leq (1 + r_l(w))\|q(w) - \xi_n(w)\| \\ &\quad + (1 + r(w))\|\xi_n(w) - p(w)\| \\ &< (1 + r(w))\frac{\epsilon}{2(1 + r_l(w)} \\ &\quad + (1 + r_l(w))\frac{\epsilon}{2(1 + r_l(w)} = \epsilon,\end{aligned} \tag{43}$$

which implies that $T_l(w, p(w)) = p(w)$ for all $l \in \{1, 2, \ldots, N\}$ and for all $w \in \Omega$. In addition, by (38), we have $S_l(w, \xi_n(w)) \to \xi_n(w)$, then there exists $n_4 \in \mathbb{N}$ such that $\|S_l(w, \xi_n(w)) - \xi_n(w)\| < \frac{\epsilon}{2}$ for all $n \geq n_4$. Let $n_5 = \max\{n_1, n_4\}$, then we have

$$
\begin{aligned}
(44) \qquad & \|S_l(w, p(w)) - p(w)\| \\
& \leq \|S_l(w, p(w)) - S_l(w, \xi_n(w))\| + \|S_l(w, \xi_n(w)) - \xi_n(w)\| \\
& \quad + \|\xi_n(w) - p(w)\| \\
& \leq r_l(w)\|\xi_n(w) - p(w)\| + \|S_l(w, \xi_n(w)) - \xi_n(w)\| \\
& \quad + \|\xi_n(w) - p(w)\| \\
& = (1 + r_l(w))\|\xi_n(w) - p(w)\| + \|S_l(w, \xi_n(w)) - \xi_n(w)\| \\
& < (1 + r_l(w))\frac{\epsilon}{2(1 + r_l(w)} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}
$$

which implies that $S_l(w, p(w)) = p(w)$ for all $l \in \{1, 2, \ldots, N\}$ and for all $w \in \Omega$. Thus $p \in F = \bigcap_{i=1}^{N} (RF(T_i) \cap RF(S_i))$. ■

Theorem 2. *Let E be a uniformly convex separable Banach space and C be a nonempty closed convex subset of E. Let $\{T_i, S_i : i \in I = \{1, 2, \ldots, N\}\}$ be 2N asymptotically nonexpansive random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (r_{i_n}(w) - 1) < \infty$, $r_{i_n}(w) \to 1$ as $n \to \infty$, for all $w \in \Omega$ and $i \in I = \{1, 2, \ldots, N\}$. Suppose that $F = \bigcap_{i=1}^{N} (RF(T_i) \cap RF(S_i)) \neq \emptyset$. Let $\{\xi_n(w)\}$ be the sequence defined as in (6) with the additional assumption $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ and $\limsup_{n\to\infty} \alpha_n < 1$. If one of the families $\{T_i : i \in I\}$ or $\{S_i : i \in I\}$ satisfy the condition B for all $w \in \Omega$. Then $\{\xi_n(w)\}$ converges strongly to a common random fixed point of $\{T_i, S_i : i = 1, 2, \ldots, N\}$.*

Proof. By Lemma 6, we have $\lim_{n\to\infty} \|\xi_n(w) - T_i(w, \xi_n(w))\| = 0, i = 1, 2, \ldots, N$. Suppose $\{T_i : i = 1, 2, \ldots, N\}$ satisfy the condition B, then

$$
\begin{aligned}
f(d(\xi_n(w), F)) \; &\leq \; \max_{1\leq i\leq N} \{\|\xi_n(w) - T_i(w, \xi_n(w))\|\} \\
&\Rightarrow \; \lim_{n\to\infty} f(d(\xi_n(w), F)) = 0.
\end{aligned}
$$

Lemma 5, says that $\lim_{n\to\infty} d(\xi_n(w), F)$ exists and since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty} d(\xi_n(w), F) = 0$ and hence the result follows from Theorem 1.

We can get the same result if $\{S_i : i = 1, 2, \ldots, N\}$ satisfy the condition B. ■

Theorem 3. *Let E be a uniformly convex separable Banach space and C be a nonempty closed convex subset of E. Let $\{T_i, S_i : i \in I = \{1, 2, \dots, N\}\}$ be 2N asymptotically nonexpansive random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(r_{i_n}(w) - 1) < \infty$, $r_{i_n}(w) \to 1$ as $n \to \infty$, for all $w \in \Omega$ and $i \in \{1, 2, \dots, N\}$. Suppose that $F = \bigcap_{i=1}^{N}(RF(T_i) \cap RF(S_i)) \neq \emptyset$. Let $\{\xi_n(w)\}$ be the sequence defined as in (6) with the additional assumption $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\limsup_{n\to\infty} \alpha_n < 1$. If one of $\{T_i : i = 1, 2, \dots, N\}$ is semicompact. Then $\{\xi_n(w)\}$ converge strongly to a common random fixed point of $\{T_i, S_i : i = 1, 2, \dots, N\}$.*

Proof. Suppose that T_1 is semicompact. By Lemma 6, we have $\lim_{n\to\infty} \|\xi_n(w) - T_1(w, \xi_n(w))\| = 0$ and $\lim_{n\to\infty} \|\xi_n(w) - S_1(w, \xi_n(w))\| = 0$, so there exists subsequence $\{\xi_{n_j}(w)\}$ of $\{\xi_n(w)\}$ such that $\{\xi_{n_j}(w)\}$ converge strongly to $\{\xi(w)\}$ for all $w \in \Omega$, where $\{\xi(w)\}$ is a measurable mapping from Ω to C. Again by Lemma 6, we have

$$\|\xi(w) - T_l(w, \xi(w))\| = \lim_{j\to\infty} \|\xi_{n_j}(w) - T_l(w, \xi_{n_j}(w))\| = 0,$$

for all $w \in \Omega$ and for all $l \in I$, and

$$\|\xi(w) - S_l(w, \xi(w))\| = \lim_{j\to\infty} \|\xi_{n_j}(w) - S_l(w, \xi_{n_j}(w))\| = 0,$$

for all $w \in \Omega$ and for all $l \in I$. It follows that $\xi \in F = \bigcap_{i=1}^{N}(RF(T_i) \cap RF(S_i))$. From Lemma 5, we see that $\|\xi_n(w) - \xi(w)\|$ exists and since $\{\xi_n(w)\}$ has a subsequence $\{\xi_{n_j}(w)\}$ such that $\{\xi_{n_j}(w)\}$ converge strongly to $\{\xi(w)\}$ for all $w \in \Omega$, then we have $\lim_{n\to\infty} \|\xi_n(w) - \xi(w)\| = 0$ for all $w \in \Omega$ and hence $\{\xi_n(w)\}$ converges strongly to a common random fixed point of $\{T_i, S_i : i = 1, 2, ..., N\}$. ■

Finally, we prove weak convergence of the iterative scheme (6) for 2N asymptotically nonexpansive random mappings in a uniformly convex separable Banach space satisfying Opial's condition.

Theorem 4. *Let E be a uniformly convex separable Banach space which satisfy Opial's condition and C be a nonempty closed convex subset of E. Let $\{T_i, S_i : i \in I = \{1, 2, \dots, N\}\}$ be 2N asymptotically nonexpansive random mappings with sequences of measurable mappings $\{r_{i_n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(r_{i_n}(w) - 1) < \infty$, $r_{i_n}(w) \to 1$ as $n \to \infty$, for all $w \in \Omega$ and $i \in$*

$\{1, 2, \ldots, N\}$. *Suppose that* $F = \bigcap_{i=1}^{N} (RF(T_i) \cap RF(S_i)) \neq \emptyset$. *Let* $\{\xi_n(w)\}$ *be the sequence defined as in (6) with the additional assumption* $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\limsup_{n\to\infty} \alpha_n < 1$. *Then* $\{\xi_n(w)\}$ *converges weakly to a common random fixed point of* $\{T_i, S_i : i = 1, 2, \ldots, N\}$.

Proof. From Lemma 6, we have that $\lim_{n\to\infty} \|\xi_n(w) - T_l(w, \xi_n(w))\| = 0$ and $\lim_{n\to\infty} \|\xi_n(w) - S_l(w, \xi_n(w))\| = 0$ for $l = 1, 2, \ldots, N$. Since E is uniformly convex and $\{\xi_n(w)\}$ is bounded, we may assume that $\xi_n(w) \to \xi(w)$ weakly as $n \to \infty$, without loss of generality. Hence by Lemma 3, we have $\xi(w) \in F = \bigcap_{i=1}^{N} (RF(T_i) \cap RF(S_i))$. Suppose that subsequences $\xi_{n_k}(w)$ and $\xi_{m_k}(w)$ of $\xi_n(w)$ converge weakly to $u(w)$ and $v(w)$, respectively. by Lemma 3, we have $u(w), v(w) \in F = \bigcap_{i=1}^{N} (RF(T_i) \cap RF(S_i))$, and by lemma 5, $\lim_{n\to\infty} \|\xi_n(w) - u(w)\|$ and $\lim_{n\to\infty} \|\xi_n(w) - v(w)\|$ exist. It follows from lemma 4, that $u(w) = v(w)$. Therefore $\{\xi_n(w)\}$ converges weakly to a common fixed point of $\{T_i, S_i : i = 1, 2, \ldots, N\}$. ■

Remark.

1. Our results improve and extend the corresponding results in [1] to the case of two finite families of asymptotically nonexpansive random mappings.
2. Our results also improve and extend the results in [16] to the case of two finite families of implicit random iterative process.

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