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REMARKS ON SEQUENCE-COVERING CLOSED MAPS

Abstract. In this paper, we prove that each sequence-covering closed map on spaces with point-countable weak bases is 1-sequence-covering (or weak-open).

Key words: g-metrizable space, weak base, boundary-compact map, sequence-covering map, 1-sequence-covering map, weak-open map, closed map.

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1. Introduction and preliminaries

A study of images of topological spaces under certain sequence-covering maps is an important question in general topology ([1], [2], [6]-[9], [11], [13], [20], for example). In 2000, P. Yan, S. Lin and S.L. Jiang proved that each sequence-covering closed map on metric spaces is 1-sequence-covering ([20]). Recently, F.C. Lin and S. Lin proved that each sequence-covering closed map on g-metrizable spaces is 1-sequence-covering ([7]).

In this paper, we prove that each sequence-covering closed map on spaces with point-countable weak bases is 1-sequence-covering (or weak-open).

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers and ω denotes $\mathbb{N} \cup \{0\}$. Let $\mathcal{P}$ be a collection of subsets of X, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$.

Definition 1. *Let X be a space, and $P \subset X$.*

(*a*) *A sequence $\{x_n\}$ in X is called* eventually *in P, if $\{x_n\}$ converges to x, and there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq m\} \subset P$.*

(*b*) *P is called a* sequential neighborhood *of x in X [5], if whenever $\{x_n\}$ is a sequence converging to x in X, then $\{x_n\}$ is eventually in P.*

Definition 2. *Let $\mathcal{P}$ be a collection of subsets of X.*

(*a*) *$\mathcal{P}$ is* point-countable, *if each point $x \in X$ belongs to only countably many members of $\mathcal{P}$.*

(b) $\mathcal{P}$ *is* locally finite *(resp.,* locally countable*), if for each* $x \in X$*, there exists a neighborhood* V *of* x *such that* V *meets only countably (resp., finite) many members of* $\mathcal{P}$*.*

(c) $\mathcal{P}$ *is* σ-locally finite *(resp.,* σ-locally countable*), if* $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$*, where each* $\mathcal{P}_n$ *is locally finite* (*resp., locally countable*).

(d) $\mathcal{P}$ *is* compact-countable*, if each compact subset of* X *meets only countably many members of* $\mathcal{P}$*.*

(e) $\mathcal{P}$ *is a* network at x in X*, if* $x \in P$ *for every* $P \in \mathcal{P}$*, and whenever* $x \in U$ *with* U *open in* X*, then* $x \in P \subset U$ *for some* $P \in \mathcal{P}$*.*

Definition 3 ([3])**.** *Let* $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ *be a cover of a space* X*. Assume that* $\mathcal{P}$ *satisfies the following (1) and (2) for every* $x \in X$*.*

(a) $\mathcal{P}_x$ *is a network at* x*.*

(b) *If* P_1*,* $P_2 \in \mathcal{P}_x$*, then there exists* $P \in \mathcal{P}_x$ *such that* $P \subset P_1 \cap P_2$*.*

$\mathcal{P}$ *is called a* weak base *of* X*, if for* $G \subset X$*,* G *is open in* X *if and only if for every* $x \in G$*, there exists* $P \in \mathcal{P}_x$ *such that* $P \subset G$*;* $\mathcal{P}_x$ *is said to be a* weak neighborhood base at x *in* X*.*

Definition 4. *Let* X *be a space.*

(a) X *is* gf-countable *[3], if* X *has a weak base* $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ *such that* $\mathcal{P}_x$ *is countable for every* $x \in X$*.*

(b) X *is* g-metrizable *[14], if* X *is regular and has a* σ*-locally finite weak base.*

Definition 5. *Let* $f : X \longrightarrow Y$ *be a map.*

(a) f *is a* weak-open *map [18], if there exists a weak base* $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ *for* Y*, and for* $y \in Y$*, there exists* $x_y \in f^{-1}(y)$ *such that for each open neighborhood* U *of* x_y*,* $P_y \subset f(U)$ *for some* $P_y \in \mathcal{P}_y$*.*

(b) f *is an* 1-sequence-covering *map [8], if for each* $y \in Y$*, there is* $x_y \in f^{-1}(y)$ *such that whenever* $\{y_n\}$ *is a sequence converging to* y *in* Y*, there is a sequence* $\{x_n\}$ *converging to* x_y *in* X *with* $x_n \in f^{-1}(y_n)$ *for every* $n \in \mathbb{N}$*.*

(c) f *is a* sequence-covering *map [14], if every convergent sequence of* Y *is the image of some convergent sequence of* X*.*

(d) f *is a* boundary-compact *map [4], if each* $\partial f^{-1}(y)$ *is compact in* X*.*

2. Main results

Lemma 1. *Let* $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ *be a point-countable weak base for* X*, and* K *be a compact subset of* X*. If* $x \in K$*, then* $x \in \boldsymbol{int}_K(P \cap K)$ *for all* $P \in \mathcal{P}_x$*.*

Proof. By Lemma 7(2) in [10], $\mathcal{P}$ is a point-countable k-network for X. Since K is compact, it follows that K is metrizable. Let $P \in \mathcal{P}_x$ and

$\{V_n : n \in \mathbb{N}\}$ be a local base at the point x in K. Then $x \in V_n \subset P \cap K$ for some $n \in \mathbb{N}$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in V_n - (P \cap K)$. It implies that the sequence $\{x_n\}$ converges to x in X. Since P is a weak neighborhood of x in X, $\{x_n\}$ is eventually in P. This contradicts to $x_n \notin P$ for all $n \in \mathbb{N}$.

Therefore, $V_n \subset P \cap K$ for some $n \in \mathbb{N}$, and $x \in \mathtt{int}_K(P \cap K)$. ■

Lemma 2. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a point-countable weak base for X. If K is a compact subset of X, then $\bigcup\{\mathcal{P}_x : x \in K\}$ is countable.*

Proof. Similar the proof of Lemma 1, it implies that K is a metrizable compact subset for X. Thus, there exists a countable subset $D \subset K$ such that $K = \mathtt{cl}_K(D)$. Now, let $P \in \bigcup\{\mathcal{P}_x : x \in K\}$. Then $P \in \mathcal{P}_x$ for some $x \in K$. By Lemma 1, $x \in \mathtt{Int}_K(P \cap K)$. Therefore, $D \cap \mathtt{Int}_K(P \cap K) \neq \emptyset$, it implies that $P \cap D \neq \emptyset$. This follows that

$$\bigcup\{\mathcal{P}_x : x \in K\} \subset \{P \in \mathcal{P} : P \cap D \neq \emptyset\}.$$

Finally, since $\mathcal{P}$ is point-countable and D is countable, it implies that $\bigcup\{\mathcal{P}_x : x \in K\}$ is countable. ■

Theorem 1. *Let $f : X \longrightarrow Y$ be a sequence-covering closed map. If X has a point-countable weak base, then f is 1-sequence-covering.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ be a point-countable weak base for X. By Lemma 3.1 in [12], Y is gf-countable. Furthermore, since Y is gf-countable and f is a closed map, it follows from Corollary 8 in [10] and Corollary 10 in [19] that Y contains no closed copy of S_ω. By Lemma 3.2 in [12], f is a boundary-compact map.

Let $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ be a weak base for Y with each $\mathcal{P}_y$ is countable. Firstly, we prove that each non-isolated point $y \in Y$, there exists $x_y \in \partial f^{-1}(y)$ such that for each $B \in \mathcal{B}_{x_y}$, there exists $P \in \mathcal{P}_y$ satisfying $P \subset f(B)$. Otherwise, there exists a non-isolated point $y \in Y$ so that for each $x \in \partial f^{-1}(y)$, there exists $B_x \in \mathcal{B}_x$ such that $P \not\subset f(B_x)$ for all $P \in \mathcal{P}_y$. Since $\mathcal{P}_y$ is a weak neighborhood base at y, we can choose a decreasing countable network $\{P_{y,n} : n \in \mathbb{N}\} \subset \mathcal{P}_y$ at y. Furthermore, since $\mathcal{B}$ is a point-countable weak base, it follows from Lemma 2 that $\bigcup\{\mathcal{B}_x : x \in \partial f^{-1}(y)\}$ is countable. Thus, $\{B_x : x \in \partial f^{-1}(y)\}$ is countable. Assume that

$$\{B_x : x \in \partial f^{-1}(y)\} = \{B_m : m \in \mathbb{N}\}.$$

Hence, for each $m, n \in \mathbb{N}$, there exists $x_{n,m} \in P_{y,n} - f(B_m)$. For $n \geq m$, we denote $y_k = x_{n,m}$ with $k = m + n(n-1)/2$. Since $\{P_{y,n} : n \in \mathbb{N}\}$ is a decreasing network at y, $\{y_k\}$ is a sequence converging to y in Y. On the other hand, because f is a sequence-covering map, $\{y_k\}$ is an image of some sequence $\{x_n\}$ converging to $x \in \partial f^{-1}(y)$ in X. Furthermore, since $B_x \in \{B_m : m \in \mathbb{N}\}$, there exists $m_0 \in \mathbb{N}$ such that $B_x = B_{m_0}$. Because B_{m_0} is a weak neighborhood of x, $\{x\} \bigcup \{x_k : k \geq k_0\} \subset B_{m_0}$ for some $k_0 \in \mathbb{N}$. Thus, $\{y\} \bigcup \{y_k : k \geq k_0\} \subset f(B_{m_0})$. But if we take $k \geq k_0$, then there exists $n \geq m_0$ such that $y_k = x_{n,m_0}$, and it implies that $x_{n,m_0} \in f(B_{m_0})$. This contradicts to $x_{n,m_0} \in P_{y,n} - f(B_{m_0})$.

We now prove that f is an 1-sequence-covering map. Suppose $y \in Y$, by the above proof there is $x_y \in \partial f^{-1}(y)$ such that whenever $B \in \mathcal{B}_{x_y}$, there exists $P \in \mathcal{P}_y$ satisfying $P \subset f(B)$. Let $\{y_n\}$ be an any sequence in Y, which converges to y. Since $\mathcal{B}_{x_y}$ is a weak neighborhood base at x_y, we can choose a decreasing countable network $\{B_{y,n} : n \in \mathbb{N}\} \subset \mathcal{B}_{x_y}$ at x_y. We choose a sequence $\{z_n\}$ in X as follows.

Since $B_{y,n} \in \mathcal{B}_{x_y}$, by the above argument, there exists $P_{y,k_n} \in \mathcal{P}_y$ satisfying $P_{y,k_n} \subset f(B_{y,n})$ for all $n \in \mathbb{N}$. On the other hand, since each element of $\mathcal{P}_y$ is a sequential neighborhood of y, it follows that for each $n \in \mathbb{N}$, $f(B_{y,n})$ is a sequential neighborhood of y in Y. Hence, for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $y_i \in f(B_{y,n})$ for every $i \geq i_n$. Assume that $1 < i_n < i_{n+1}$ for each $n \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, we take

$$z_j = \begin{cases} z_j \in f^{-1}(y_j) & \text{if} \quad j < i_1 \\ z_{j,n} \in f^{-1}(y_j) \cap B_{y,n} & \text{if} \quad i_n \leq j < i_{n+1}. \end{cases}$$

If we put $S = \{z_j : j \geq 1\}$, then S converges to x_y in X, and $f(S) = \{y_n\}$. Therefore, f is 1-sequence-covering. ■

By Theorem 1, the following corollary holds.

Corollary 1. *Let $f : X \longrightarrow Y$ be a sequence-covering closed map. If one of the following properties holds, then f is 1-sequence-covering.*

(*a*) *X has a compact-countable weak base;*
(*b*) *X has a locally countable weak base;*
(*c*) *X has a σ-locally countable weak base;*
(*d*) *X is g-metrizable.*

Remark 1. By Corollary 2, we obtain Theorem 4.5 in [7].

By Theorem 1 and Corollary 3.6 in [18], we have the following corollary.

Corollary 2. *Let $f : X \longrightarrow Y$ be a sequence-covering closed map. If one of the following properties holds, then f is weak-open.*

(*a*) *X has a point-countable weak base;*
(*b*) *X has a compact-countable weak base;*
(*c*) *X has a locally countable weak base;*
(*d*) *X has a σ-locally countable weak base;*
(*e*) *X is g-metrizable.*

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