DE GRUYTER OPEN

FASCICULI MATHEMATICI

Nr 54 2015

DOI:10.1515/fascmath-2015-0002

R. A. Borzooei, A. Borumand Saeid, A. Rezaei,
A. Radfar and R. Ameri

DISTRIBUTIVE PSEUDO BE-ALGEBRAS

Abstract. In this paper, we introduce the notion of distributive pseudo BE–algebra and show that the related relation defined on this structure is transitive and prove that every pseudo upper set is a pseudo filter. Also, the pseudo filter generated by a set is define and show that the set of all pseudo filters is distributive complete lattice but it is not complemented. the notion of prime and irreducible subset and prove that every irreducible subset is prime.

Key words: (distributive)pseudo BE-algebra, (normal)pseudo filter, pseudo upper set.

AMS Mathematics Subject Classification: 06F35, 03G25, 03B52.

1. Introduction

As a generalizations of BCK/BCI-algebras, BE-algebras were introduced by H. S. Kim and Y. H. Kim [5]. By using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras.

G. Georgescu and A. Iorgulescu [2], and independently J. Rachunek [6], introduced pseudo-MV algebras which are a non-commutative generalization of MV-algebras. After pseudo-MV algebras, the pseudo-BL algebras [3], and the pseudo-BCK algebras as an extended notion of BCK-algebras by G. Georgescu and A. Iorgulescu [4], were introduced and studied. A. Walendziak give a system of axioms defining pseudo-BCK algebras [7].

R. Borzooei et al. introduced the notion of pseudo BE-algebras and discuss on the relationship between the subalgebras with pseudo filters. They show that every pseudo filter is a subalgebra and give an example such that the converse it is not true. Also, they prove that every pseudo filter is an union of pseudo upper sets [1].

The purpose of this work is to define the notion of distributive pseudo BE-algebras and show that in this algebraic structure, a subset F of distributive pseudo BE-algebra X is a pseudo filter if and only if it is a normal pseudo filter. We characterize the conditions for a pseudo BE-algebra to be a BE-algebra.

2. Preliminaries

In this section we recall definitions and results that will be used in this paper.

Definition 1 ([5]). *An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called a* BE*–algebra if the following axioms hold:*
$(BE1)$ $x * x = 1$,
$(BE2)$ $x * 1 = 1$,
$(BE3)$ $1 * x = x$,
$(BE4)$ $x * (y * z) = y * (x * z)$, *for all* $x, y, z \in X$
We introduce a relation " $\leq$ *" on* X *by* $x \leq y$ *if and only if* $x * y = 1$.

Definition 2 ([1]). *An algebra* $(X; *, \diamond, 1)$ *of type* $(2, 2, 0)$ *is called a pseudo* BE*-algebra if it satisfies the following axioms:*
$(pBE1)$ $x * x = 1$ *and* $x \diamond x = 1$,
$(pBE2)$ $x * 1 = 1$ *and* $x \diamond 1 = 1$,
$(pBE3)$ $1 * x = x$ *and* $1 \diamond x = x$,
$(pBE4)$ $x * (y \diamond z) = y \diamond (x * z)$,
$(pBE5)$ $x * y = 1 \Leftrightarrow x \diamond y = 1$, *for all* $x, y, z \in X$.

In a pseudo BE-algebra, one can define a binary relation " $\leq$ " by

$$x \leq y \Leftrightarrow x * y = 1 \Leftrightarrow x \diamond y = 1, \; for \; all x, y \in X.$$

Remark 1. If X is a pseudo BE-algebra satisfying $x * y = x \diamond y$, for all $x, y \in X$, then X is a BE-algebra.

Proposition 1 ([1]). *In a pseudo* BE*-algebra* X*, the following holds:*
(1) $x * (y \diamond x) = 1$, $x \diamond (y * x) = 1$,
(2) $x \diamond (y \diamond x) = 1$, $x * (y * x) = 1$,
(3) $x \diamond ((x \diamond y) * y) = 1$, $x * ((x * y) \diamond y) = 1$,
(4) $x * ((x \diamond y) * y) = 1$, $x \diamond ((x * y) \diamond y) = 1$,
(5) *if* $x \leq y * z$, *then* $y \leq x \diamond z$,
(6) *if* $x \leq y \diamond z$, *then* $y \leq x * z$,
(7) $1 \leq x$, *implies* $x = 1$,
(8) *if* $x \leq y$, *then* $x \leq z * y$ *and* $x \leq z \diamond y$,
(9) *if* $x * y = z$, *then* $y * z = y \diamond z = 1$ (*If* $x \diamond y = z$, *then* $y * z = y \diamond z = 1$),
(10) *if* $x * y = x$ *and* $x \neq 1$, *then* $x \diamond y \neq y$,
(11) *if* $x * y = y$ *and* $x \neq 1$, *then* $x \diamond y \neq x$,
(12) *if* $x * y = x$ *and* $x \diamond y = z$, *then* $x * z = x \diamond z = 1$, *and*

$$x * (y * z) = (x * y) * (x * z) = x \diamond (y * z) = (x \diamond y) * (x \diamond z) = 1,$$

(13) *if* $x * y = y$ *and* $x \diamond y = z$*, then* $x * z = z$ *and*

$$x * (y * z) = (x * y) * (x * z) = x * (y \diamond z) = (x * y) \diamond (x * z) = 1,$$

(14) *if* $x * y = z$ *and* $x \diamond y = t$*, then* $x \diamond z = x * t$*,*

for all $x, y, z, t \in X$*.*

Definition 3 ([1]). *A non-empty subset* F *of* X *is called a pseudo filter of* X *if it satisfies in the following axioms:*
($pF1$) $1 \in F$,
($pF2$) $x \in F$ *and* $x * y \in F$ *imply* $y \in F$.

Proposition 2 ([1]). *Let* $F \subseteq X$ *and* $1 \in F$*.* F *is a pseudo filter if and only if* $x \in F$ *and* $x \diamond y \in F$ *imply* $y \in F$*, for all* $x, y \in X$*.*

For any $x, y \in X$, put $A(x, y) := \{z \in X : x*(y\diamond z) = 1\}$. We call $A(x, y)$ a pseudo upper set of x and y.

Proposition 3 ([1]). *Let* $F \subseteq X$ *be a pseudo filter and* $x \in X$*. Then* $x \diamond a \in F$*, for all* $a \in F$*.*

3. Distributive pseudo BE-algebras

Definition 4. *A pseudo* BE*-algebra* X *is said to be distributive if it satisfies only one of the following conditions:*
(i) $x * (y \diamond z) = (x * y) \diamond (x * z)$,
(ii) $x \diamond (y * z) = (x \diamond y) * (x \diamond z)$*, for all* $x, y, z \in X$*.*

We note that if $(X; *, \diamond, 1)$ is a pseudo BE-algebra, then $(X; \diamond, *, 1)$ is a pseudo BE-algebra, too. From now, in this paper, X is a distributive pseudo BE-algebra, which satisfy the condition (i) of Definition 4, unless otherwise is state.

Example 1. (i) Let $X = \{1, a, b, c, d, e\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	c	d	1
b	1	a	1	1	1	e
c	1	a	1	1	1	e
d	1	a	1	1	1	e
e	1	a	d	d	d	1

$\diamond$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	d	1
b	1	a	1	1	1	e
c	1	a	1	1	1	e
d	1	a	1	1	1	e
e	1	a	c	c	d	1

Then $(X; *, \diamond, 1)$ is a distributive pseudo BE-algebra.

(ii) Let $X = \{1, a, b, c, d\}$. Define operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	b	c
c	1	a	1	1	b
d	1	a	1	1	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	b	c
c	1	a	1	1	b
d	1	a	1	1	1

Then X is a pseudo BE-algebra. Also, we have

$$c * (c \diamond d) = c * b = 1 \neq b = 1 \diamond b = (c * c) \diamond (c * d).$$

Therefore, it is not distributive.

Proposition 4. *The relation "$\leq$" is defined on X is transitive.*

Proof. Let $x \leq y$ and $y \leq z$, for some $x, y, z \in X$. Then

$$\begin{aligned}
x * z &= 1 \diamond (x * z) && (by\ (pBE3)) \\
&= (x * y) \diamond (x * z) && (by\ (pBE5)) \\
&= x * (y \diamond z) && (by\ Definition\ 4) \\
&= x * 1 && (by\ (pBE2)) \\
&= 1.
\end{aligned}$$

Now, by $(pBE5)$ we have $x \diamond z = 1$. Hence $x \leq z$. ■

Lemma 1. *For any $x, y \in X$,*
(i) $x * (x \diamond y) = x * y$,
(ii) *if* $x * y = y$, *then* $x \diamond y \neq x$,
(iii) $x * y \neq x$ *and* $x \diamond y \neq x$, *for all* $x, y \in X$ *and* $1 \neq x$.

Proof. (i) Let $x, y \in X$. Then

$$x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond (x * y) = x * y.$$

(ii) Let $x * y = y$ and $x \neq 1$. If $x \diamond y = x$, then

$$x * (x \diamond y) = x * x = 1, \ x \diamond (x * y) = x \diamond y = x.$$

By $(pBE4)$, $x = 1$, which is a contradiction.

(iii) If X is distributive, $x * y = x$ and $x \neq 1$, then

$$1 = x \diamond (x * y) = x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond x = x.$$

If $x \diamond y = x$ and $x \neq 1$, then

$$1 = x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond x = x.$$

Which is a contradiction. ■

Theorem 1. *Let x, y, z, t be distinct elements of X. Then one of the following holds:*
(i) $x * y = x \diamond y \neq x$,
(ii) $x * y = z$ and $x \diamond y = y$,
(iii) $x * y = z$ and $x \diamond y = t$.

Proof. By Lemma 1(*iii*), $x * y \neq x$ and $x \diamond y \neq x$. Thus $x * y = y$ or $x * y = z$. By Lemma 1(*i*), if $x * y = y$, then $x \diamond y \neq x$. If $x * y = y$ and $x \diamond y = z$, then

$$x * z = x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond y = y.$$

Also, by ($pBE4$) and distributivity we have

$$\begin{aligned} x \diamond y = x \diamond (x * y) &= x * (x \diamond y) \\ &= (x * x) \diamond (x * y) \\ &= 1 \diamond y \\ &= y. \end{aligned}$$

Which is a contradiction. ■

Theorem 2. *Let $(X; *, \diamond, 1)$ be a distributive BE-algebra. Then $(X, \diamond, *, 1)$ is distributive if and only if X is a BE-algebra.*

Proof. Let $(X, *, \diamond)$ and $(X, \diamond, *)$ are distributive. It is enough to show that $x * y = x \diamond y$, for all $x, y \in X$. Without lose of generality one of the following cases occur.

Case 1. $x * y = x$ and $x \diamond y = y$. By using Proposition 1 (10), we have a contradiction.

Case 2. $x * y = y$ and $x \diamond y = x$, in which, it is a contradiction by using Proposition 1 (11).

Case 3. $x * y = x$ and $x \diamond y = z$. Then $x \diamond (x * y) = x \diamond x = 1$ and

$$x \diamond (x * y) = (x \diamond x) * (x \diamond y) = 1 * (x \diamond y) = 1 * z = z.$$

Case 4. $x * y = y$ and $x \diamond y = z$. Then $x * (x \diamond y) = x * z$ and

$$x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond (x * y) = 1 * y = y.$$

Thus $x * z = y$ and so

$$x * (x \diamond y) = x * z = y \ \ and \ \ x * (x \diamond y) = x \diamond (x * y) = x \diamond y = z.$$

Hence $y = z$ and so $x * y = x \diamond y$.

Case 5. $x * y = z$ and $x \diamond y = t$. Then $x \diamond (x * y) = x \diamond z$ and

$$x \diamond (x * y) = (x \diamond x) * (x \diamond y) = 1 * (x \diamond y) = 1 * t = t.$$

Thus $x \diamond z = t$. Also,

$$x * (x \diamond y) = x * t \ \ and \ \ x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond z = z.$$

Hence $x * t = z$ and so

$$x \diamond (x * t) = x \diamond z = t \ \ and \ \ x * (x \diamond t) = x * t = z.$$

Thus $z = t$ and so $x * y = x \diamond y$. Therefore, X is a BE-algebra.

Conversely, the proof is clear. ■

Note. By Theorem 2, we conclude that if X is a pseudo BE-algebra satisfies in the following axioms:

(i) $x * (y \diamond z) = (x * y) \diamond (x * z)$,

(ii) $x \diamond (y * z) = (x \diamond y) * (x \diamond z)$,

then $(X; *, 1) = (X; \diamond, 1)$ and it is a BE-algebra.

For $x \in X$ we denote by $A(x) = \{y \in X : x * y = 1\}$, which is called terminal segment of an element x.

Proposition 5. *For any* $x, y, z \in X$, *we have*

(i) $1, x \in A(x)$ *(i. e.,* $A(x)$ *is a nonempty subset of* X*)*,

(ii) $A(x, y) = \{z \in X \mid y \diamond z \in A(x)\} = \{z \in X \mid y * z \in A(x)\}$,

(iii) $A(x) = A(x, 1)$ *and* $A(1) = \{1\}$,

(iv) *let* $\emptyset \neq F \subseteq X$. *If* F *is a pseudo filter, then* $A(x) \subseteq F$,

(v) $z \in A(x)$ *implies* $y * z \in A(x)$ *and* $y \diamond z \in A(x)$.

Proof. By definition of $A(x)$, the proof (i), (ii), (iii) and (iv) are clear. (v). Let $z \in A(x)$. Then $x * z = x \diamond z = 1$. By using $pBE4$,

$$x * (y \diamond z) = y \diamond (x * z) = y \diamond 1 = 1.$$

Thus $y \diamond z \in A(x)$. By a similar way, $y * z \in A(x)$. ■

In the next example we show that the converse of Proposition 5(iv), is not correct in general.

Example 2. Let $X = \{1, a, b, c, d\}$. Define operations "$*$" and "$\diamond$" on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	1	1	1	1
c	1	d	d	1	d
d	1	1	c	c	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	1	1	1	1
c	1	a	d	1	d
d	1	1	c	c	1

Then X is a distributive pseudo BE-algebra. We have

$$A(a) = \{1, a, d\} \subseteq \{1, a, c, d\}, \qquad A(c) = \{1, c\} \subseteq \{1, a, c, d\},$$
$$A(d) = \{1, a, d\} \subseteq \{1, a, c, d\}.$$

Since $d * b = d \diamond b = c \in \{1, a, c, d\}$ and $d \in \{1, a, c, d\}$, we see that $\{1, a, c, d\}$ is not a pseudo filter.

Lemma 2. *$x * y = t$ and $x \diamond y = z$ implies $x \diamond t = x * z = t$, for all $x, y, z, t \in X$.*

Proof. By using $(pBE4)$, $x \diamond t = x \diamond (x * y) = x * (x \diamond y) = x * z$. Also,

$$x * z = x * (x \diamond y) = (x * x) \diamond (x * y) = 1 \diamond (x * y) = 1 \diamond t = t.$$

■

Theorem 3. *$y * z \in A(x)$ if and only if $y \diamond z \in A(x)$, for all $x, y, z \in X$.*

Proof. Let $x, y, z, t, r, k \in X$ and $t = y \diamond z \in A(x)$. Then we have

$$y * t = y * (y \diamond z) = (y * y) \diamond (y * z) = y * z.$$

By Proposition 5 (v), $y * t \in A(x)$. Thus $y * z \in A(x)$.

Conversely, let $y * z = k \in A(x)$ and $x \diamond z = t$. It means that $k * x = k \diamond x = 1$. By using $(pBE4)$, $1 = x \diamond k = x \diamond (y * z) = y \diamond (x * z) = y \diamond t$. Now by $(pBE5)$ we have $y * t = 1$. Since $y * z = t \neq 1$, we see that $x \diamond z = t \neq z$. By Theorem 1, $x * z = x \diamond z = t$ or $x \diamond z = t$ and $x * z = t$. If $x * z = x \diamond z = t$, since $y \diamond t = 1$, we have

$$x * (y \diamond z) = y \diamond (x * z) = y \diamond t = 1.$$

Therefore, $y \diamond z \in A(x)$.

If $x * z = t$ and $x \diamond z = r$, then by Lemma 2, $x * t = r$. Since $y \diamond t = 1$, we have

$$x * (y \diamond z) = y \diamond (x * z) = y \diamond r = y \diamond (x * t) = x * (y \diamond t) = x * 1 = 1.$$

Therefore, $y \diamond z \in A(x)$. ■

Corollary 1. *$x*(y*z)=1$ if and only if $z\in A(x,y)$, for all $x,y,z\in X$.*

Proof. Let $x*(y*z)=1$, for some $x,y,z\in X$. Then $y*z\in A(x)$. By Theorem 3, $y\diamond z\in A(x)$ and so $x*(y\diamond z)=1$. Thus $z\in A(x,y)$.

Conversely, let $z\in A(x,y)$. Then $x*(y\diamond z)=1$ and so $y\diamond z\in A(x)$. By using Theorem 3, $y*z\in A(x)$. Therefore, $x*(y*z)=1$. ■

Theorem 4. *(i) $x*y=z$ and $x\diamond y=y$ implies $z\leq y$,*
(ii) $x*y=z$ and $x\diamond y=t$ implies $z\leq t$, $t\leq z$, $y\leq z$ and $y\leq t$,
(iii) $x*y\leq x\diamond y$ and $x\diamond y\leq x*y$, for all $x,y,z,t\in X$.

Proof. (i) By Proposition 5(i), $x*y=z\in A(z)$. Now, by Theorem 3, $x\diamond y=y\in A(z)$, which is $z\leq y$.

(ii) By Theorem 1(9), $y\leq z$ and $y\leq t$. By Proposition 5(i), $x*y=z\in A(z)$. Using Theorem 3, we have $x\diamond y=t\in A(z)$. It means that $z\leq t$. By a similar way we have $x\diamond y=t\in A(t)$. By Theorem 3, $x*y=z\in A(t)$, which is $t\leq z$.

(iii) Let $x,y\in X$. By Theorem 1, $x*y=x\diamond y$ or $x*y=z$ and $x\diamond y=y$ or $x*y=z$ and $x\diamond y=t$, for some $z,t\in X$. If $x*y=x\diamond y$, then it is obvious that $x*y\leq x\diamond y$ and $x\diamond y\leq x*y$. If $x*y=z$ and $x\diamond y=y$, then by (i), $x*y\leq x\diamond y$ and $x\diamond y\leq x*y$. Now, if $x*y=z$ and $x\diamond y=t$, then by (ii), $x*y\leq x\diamond y$ and $x\diamond y\leq x*y$. ■

Theorem 5. *$A(x*y)=A(x\diamond y)$, for all $x,y\in X$.*

Proof. Using Theorem 1, we have the following cases.

Case 1. if $x*y=x\diamond y$, then the proof is obvious.

Case 2. let $x*y=z$ and $x\diamond y=y$ and $a\in A(x*y)$. Therefore, $z*a=1$. By Proposition 1(9), $y*z=1$. We have

$$1=y\diamond 1=y*(z\diamond a)=(y*z)\diamond(y*a)=1\diamond(y*a)=y*a.$$

Hence $a\in A(y)=A(x\diamond y)$. Thus $A(x*y)\subseteq A(x\diamond y)$.

Conversely, let $a\in A(x\diamond y)$. Then $y*a=y\diamond a=1$. By Theorem 4(i), $z\leq y$ and so $z*y=1$. Thus

$$1=z*1=z*(y\diamond a)=(z*y)\diamond(z*a)=1\diamond(z*a)=z*a.$$

Hence $z*a=1$ and so $a\in A(z)=A(x*y)$. Therefore, $A(x\diamond y)=A(x*y)$. Let $x*y=z$, $x\diamond y=t$ and $a\in A(x*y)$. Then $z*a=1$. By Theorem 4(ii), $t\leq z$ and so $t*z=1$. Thus

$$1=t\diamond 1=t*(z\diamond a)=(t*z)\diamond(t*a)=1\diamond(t*a)=t*a.$$

Hence $a\in A(t)=A(x\diamond y)$ and so $A(x*y)\subseteq A(x\diamond y)$. By a similar way, $a\in A(x\diamond y)$ implies $a\in A(z)=A(x*y)$. Therefore, $A(x\diamond y)=A(x*y)$, for all $x,y\in X$. ■

Theorem 6. *Let $x \leq y$. Then*

(i) $z * x \leq z * y$, *and* $z * x \leq z \diamond y$,

(ii) $z \diamond x \leq z * y$, *and* $z \diamond x \leq z \diamond y$, *for all* $x, y, z \in X$.

Proof. Let $x \leq y$ and $x, y, z \in X$. Then $x * y = 1$. So,

$$(z * x) * (z * y) = z * (x * y) = z * 1 = 1.$$

Beside, by using $(pBE5)$ we have $(z * x) \diamond (z * y) = 1$. Hence $z * x \leq z * y$. Thus $z * y \in A(z * x)$. By Theorems 3 and 5, $z \diamond y \in A(z * x) = A(z \diamond x)$. It means that $z \diamond x \leq z \diamond y$. ■

Theorem 7. *(i)* $y * z \leq (x * y) * (x * z)$, *and* $y * z \leq (x * y) \diamond (x \diamond z)$,
(ii) $y \diamond z \leq (x * y) * (x * z)$, *and* $y \diamond z \leq (x * y) \diamond (x * z)$, *for all* $x, y, z \in X$.

Proof. We have

$$\begin{aligned}(y \diamond z) \diamond ((x * y) \diamond (x * z)) &= (y \diamond z) \diamond (x * (y \diamond z)) \\ &= x * ((y \diamond z) \diamond (y \diamond z)) \\ &= x * 1 \\ &= 1.\end{aligned}$$

Thus $y \diamond z \leq (x * y) \diamond (x * z)$. Also, we have $(x * y) \diamond (x * z) \in A(y \diamond z)$. Then by Theorem 3, $(x * y) * (x * z) \in A(y \diamond z)$. It means that, $y \diamond z \leq ((x * y) * (x * z))$ and so (ii) is hold. Also, by Theorem 5, $A(y \diamond z) = A(y * z)$. Thus $(x * y) \diamond (x * z) \in A(y * z)$ and $(x * y) * (x * z) \in A(y * z)$ and so (i) is hold. ■

Theorem 8. $a * b \in A(x, y)$ *if and only if* $a \diamond b \in A(x, y)$, *for all* $a, b, x, y \in X$.

Proof. Let $a * b \in A(x, y)$. By Theorem 1, we have the following cases.

Case 1. if $a \diamond b = a * b$, then the proof is obvious.

Case 2. let $a * b = c$ and $a \diamond b = b$, for some $c \in X$. Since $c = a * b \in A(x, y)$ and using the Corollary 1, we have $x * (y * c) = 1$. Also, by Theorem 4(i), $c \leq b$ and so $c \diamond b = 1$. Thus

$$\begin{aligned}x * (y * b) = 1 \diamond (x * (y * b)) &= (x * (y * c)) \diamond (x * (y * b)) \\ &= x * ((y * c) \diamond (y * b)) \\ &= x * (y * (c \diamond b)) \\ &= x * (y * 1) \\ &= x * 1 \\ &= 1.\end{aligned}$$

By Corollary 1, $a \diamond b = b \in A(x, y)$.

Case 3. if $a*b = c$ and $a\diamond b = t$, for some $t, c \in X$. Since $c = a*b \in A(x, y)$ and by Theorem 1, we see that $x * (y * c) = 1$. Also, by Theorem 4(i), $c < t$ and so $c \diamond t = 1$. Thus

$$\begin{aligned} x * (y * t) = 1 \diamond (x * (y * t)) &= (x * (y * c)) \diamond (x * (y * t)) \\ &= x * ((y * c) \diamond (y * t)) \\ &= x * (y * (c \diamond t)) \\ &= x * (y * 1) \\ &= x * 1 \\ &= 1. \end{aligned}$$

By Theorem 3, $a \diamond b = t \in A(x, y)$. Therefore, $a * b \in A(x, y)$ implies $a \diamond b \in A(x, y)$. By a similar way $a \diamond b \in A(x, y)$ implies $a * b \in A(x, y)$. ■

4. Pseudo filters and upper sets

Definition 5. *A pseudo filter F is said to be normal, if for all $x, y \in X$*

$$x * y \in F \text{ if and only if } x \diamond y \in F.$$

By the following example we show that the class of normal pseudo filters and pseudo filters in pseudo BE-algebras are different.

Example 3. (i) Let $X = \{1, a, b, c, d\}$. Define operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	1	1
b	1	1	1	1	1
c	1	a	a	1	1
d	1	a	b	c	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	1	1
b	1	1	1	1	1
c	1	a	b	1	1
d	1	a	b	c	1

Then $(X; *, \diamond, 1)$ is a pseudo BE-algebra and $\{1, d\}$ is a normal pseudo filter.

(ii) Let $F = \{1, a, b, c\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	c
c	1	1	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	a
c	1	1	1	1

Then $(X;*,\diamond,1)$ is a pseudo BE-algebra and $F=\{1,b\}$ is a pseudo filter of X. Since $a*c=b\in F$ and $a\diamond c=a\notin F$, we see that F is not normal. Also, since $a*(b\diamond c)=a*a=1\neq b=1\diamond b=(a*b)\diamond(a*c)$, it is not distributive.

Theorem 9. *A subset F of distributive pseudo BE-algebra X is pseudo filter if and only if it is normal pseudo filter.*

Proof. Let F be a pseudo filter and $x*y\in F$. By Theorem 1, we have the following cases.

Case 1. if $x*y=x\diamond y$, then the proof is obvious.

Case 2. if $x*y=z$ and $x\diamond y=y$, then by Theorem 4(i), $z\leq y$. It means that $z*y=z\diamond y=1\in F$. Since F is a filter and $z\in F$, we have $x\diamond y=y\in F$.

Case 3. if $x*y=z$ and $x\diamond y=t$, then by Theorem 4(ii), $z\leq t$. It means that $z*t=z\diamond t=1\in F$. Since F is a filter and $z\in F$, we have $x\diamond y=t\in F$.

Conversely, let $x\diamond y\in F$. By Theorem 1, we have the following cases.

Case 1. if $x*y=x\diamond y$, then the proof is obvious.

Case 2. if $x*y=z$ and $x\diamond y=y$, then by Theorem 1(9), $y\leq z$. It means that $y*z=y\diamond z=1\in F$. Since F is a filter and $y=x\diamond y\in F$, we have $z\in F$ and so $x*y\in F$.

Case 3. if $x*y=z$ and $x\diamond y=t$, then by Theorem 4(ii), $t\leq z$. It means that $t*z=t\diamond z=1\in F$. Since F is a pseudo filter and $t\in F$, we have $x*y=z\in F$. ■

Proposition 6. *Pseudo upper set $A(x,y)$ is a pseudo filter, for all $x,y\in X$.*

Proof. Obviously, $1\in A(x,y)$. By Theorem 2, it is sufficient to prove that $a\in A(x,y)$ and $a*b\in A(x,y)$ implies $b\in A(x,y)$.

Let $a\in A(x,y)$ and $a*b\in A(x,y)$. Then by Theorem 8, $a\diamond b\in A(x,y)$. Now, by Corollary 1, $x*(y*(a\diamond b))=1$ and $x*(y*a)=1$. So by distributivity, we have

$$\begin{aligned} 1=x*(y*(a\diamond b)) &= x*((y*a)\diamond(y*b)) \\ &= (x*(y*a))\diamond(x*y*b) \\ &= 1\diamond(x*(y*b)) \\ &= x*(y*b). \end{aligned}$$

Then by Corollary 1, $b\in A(x,y)$. Therefore, $A(x,y)$ is a pseudo filter of X. ■

In the following example we show that the condition self distributivity in Proposition 6, is necessary.

Example 4. Let $X = \{1, a, b, c, d\}$. Define the operation "$*$" follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	b	c
c	1	a	1	1	b
d	1	a	1	1	1

.

Then $(X; *, 1)$ is a BE-algebra. Put "$\diamond := *$" and so $(X; *, \diamond, 1)$ is a pseudo BE-algebra. Since

$$c * (c \diamond d) = c * b = 1 \neq b = 1 \diamond b = (c * c) \diamond (c * d),$$

we can see that it is not distributive. By easy calculation we get $A(a, b) = \{1, a, b\}$. Since $b * c = b \diamond c = b \in A(a, b)$ and $b \in A(a, b)$ but $c \notin A(a, b)$, we can see that $A(a, b)$ is not a pseudo filter.

Theorem 10. *A non-empty subset F of pseudo BE-algebra X is a pseudo filter if and only if satisfy in:*
(i) $\forall x \in X$ and $\forall a \in F$ imply $x * a \in F$,
(ii) $\forall x \in X$ and $\forall a, b \in F$ imply $(a * (b * x)) \diamond x \in F$.

Proof. Let F be a pseudo filter, $a \in F$ and $x \in X$. Then by $(pBE4)$ and $(pBE3)$, $a \diamond (x * a) = x * (a \diamond a) = x * 1 = 1$. Using $(pF1)$, $a \diamond (x * a) \in F$. By Proposition 2, $x * a \in F$. Let $a, b \in F$ and $x \in X$. By $(pBE4)$ and $(pBE1)$, we have

$$\begin{aligned} a * (b * ((a * (b * x)) \diamond x)) &= a * ((a * (b * x) \diamond (b * x)) \\ &= (a * (b * x)) \diamond (a * (b * x)) = 1. \end{aligned}$$

By $(pF1)$, $a * (b * ((a * (b * x)) \diamond x)) \in F$. Since $a, b \in F$, by using $(pF2)$ we have $(a * (b * x)) \diamond x \in F$ and so (ii) is hold.

Conversely, let F satisfy in conditions (i) and (ii). Since F is a non-empty set, there is $a \in F$. By (i), $1 = a * a \in F$. Hence $(pF1)$ is hold. Let $x * y \in F$ and $x \in F$. By replacing $a = x * y$, $b = x$ and $x = y$ in (ii), $((x * y) * (x * y)) \diamond y \in F$. Thus $y = 1 \diamond y = ((x * y) * (x * y)) \diamond y \in F$ and so $(pF2)$ is hold. Therefore, F is a pseudo filter. ■

Corollary 2. *A non-empty subset F of pseudo BE-algebra X is a pseudo filter if and only if satisfy in:*
(i) $\forall x \in X$ and $\forall a \in F$ imply $x \diamond a \in F$,
(ii) $\forall x \in X$ and $\forall a, b \in F$ imply $(a \diamond (b \diamond x)) * x \in F$.

Theorem 11. *F is a pseudo filter if and only if for all $x, y \in F$, $A(x, y) \subseteq F$.*

Proof. Let F be a pseudo filter and $z \in A(x, y)$, for some $x, y \in F$. Then $x * (y \diamond z) = 1 \in F$. Thus $y \diamond z \in F$, and so $z \in F$. Therefore, $A(x, y) \subseteq F$, for all $x, y \in F$.

Conversely, let F be a non-empty set such that $A(x, y) \subseteq F$, for all $x, y \in F$. Since F is a non-empty set, there is $a \in F$. By using $(pBE1)$ and $(pBE3)$ we have $a * (a \diamond 1) = a * a = 1$ and so $1 \in A(a, a) \subseteq F$. Thus $(pF1)$ is hold. Now, let $x * y = z \in F$ and $x \in F$. Then by $(pBE1)$ and $(pBE4)$, $1 = z \diamond z = z \diamond (x * y) = x * (z \diamond y)$. Thus $y \in A(x, z)$. Since $x, z \in F$ and by assumption $A(x, z) \subseteq F$, we have $y \in F$. Hence $(pF2)$ is hold. Therefore, F is a pseudo filter of X. ■

Theorem 12. *Let F be a pseudo filter. Then $F = \bigcup_{u,v \in F} A(u, v)$.*

Proof. Let $x \in \bigcup_{u,v \in F} A(u, v)$. Then $x \in A(u, v)$, for some $u, v \in F$. By Theorem 11, $x \in F$.

Conversely, let $x \in F$. Since $x * (x \diamond x) = x * 1 = 1$, we see that

$$x \in A(x, x) \subseteq \bigcup_{u,v \in F} A(u, v).$$

■

If F is a pseudo filter of X and $a \in X$, put

$$F_a := \{x \in X : \ a * x \in F\}.$$

Note. Since by Theorem 9, every pseudo filter is normal, we have

$$F_a = \{x \in X : \ a * x \in F \ \text{ and } \ a \diamond x \in F\}.$$

Theorem 13. *Let F be a pseudo filter of X and $a \in X$. Then F_a is the least pseudo filter of X containing F and a.*

Proof. By $(pBE2)$ we have $a * 1 = 1$, for all $a \in X$, i.e., $1 \in F_a$. By using $(pBE1)$, we have $a * a = 1 \in F$, for any $a \in F$. Hence $a \in F_a$.

By Proposition 2, it is sufficient to prove that $x \in F_a$ and $x \diamond y \in F_a$ implies $y \in F_a$.

Let $x \in F_a$ and $x \diamond y \in F_a$. Then $a * x \in F$ and $a * (x \diamond y) \in F$. Since by the definition of distributivity, $a * (x \diamond y) = (a * x) \diamond (a * y) \in F$ and $a * x \in F$, we get that $a * y \in F$, and so $y \in F_a$. This proves that F_a is a pseudo filter of X.

Let $x \in F$. Since $x * (a \diamond x) = 1 \in F$ and F is a pseudo filter of X, we get that $a \diamond x \in F$. Since by Theorem 9, F is normal, we see that $a * x \in F$. Hence $x \in F_a$. Thus $F \subseteq F_a$.

Now, let G be a pseudo filter of X containing F and a. Let $x \in F_a$. Then $a * x \in F \subseteq G$. Since $a \in G$ and G is a pseudo filter of X, then $x \in G$. Therefore, $F \subseteq G$. ■

Example 5. (i) In Example 1(i), $F = \{1, e\}$ is a pseudo filter of X and $F_a = \{1, a, e\}$. Also, $\{1, a, e\}$ is the smallest pseudo filter of X containing F and a.

(ii) In Example 1(ii), $F = \{1, a\}$ is a pseudo filter but $F_b = \{1, a, b\}$ is not a pseudo filter, because $b * c = b \diamond c = b \in F_b$ and $b \in F_b$ but $c \notin F_b$.

Note. Let F be a pseudo filter of X and $a \in X$, if put

$$F_a := \{x \in X : \ a * x \in F \ \ and \ \ a \diamond x \in F\}$$

Then F_a is not a pseudo filter of X in general.

Example 6 ([1]). Let $X = \{1, a, b, c\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	c
c	1	1	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	a
c	1	1	1	1

.

Then $(X; *, \diamond, 1)$ is a pseudo BE-algebra. $F = \{1, b\}$ is a pseudo filter and $F_c = \{1, b, c\}$ is not a pseudo filter, because $c * a = b \in F_c$ and $c \diamond a = c \in F_c$ but $a \notin F_c$.

5. Some results on pseudo filters lattice

Let $pF(X)$ be the set of all pseudo filters and let A be a non-empty subset of X, then the set

$$< A > = \bigcap \{G \in pF(X) |\ A \subseteq G\}$$

is called the pseudo filter generated by A, written $< A >$. If $A = \{a\}$, we will denote $< \{a\} >$, briefly by $< a >$, and we call it a principal pseudo filter of X. For $F \in pF(X)$ and $a \in X$, we denote by $(F \cup \{a\}]$ the pseudo filter generated by $F \cup \{a\}$. For convenience, set $(\emptyset] = \{1\}$.

Example 7. In Example 1(i), $pF(X) = \{X, \{1, b, c, d\}, \{1, a, e\}, \{1, e\}, \{1\}\}$. Also, $< \{a, b\} > = X$ and $< \{b, c\} > = \{1, b, c, d\}$.

Proposition 7. *Let F and G are two subsets of X. Then the following statement holds:*
(i) $(1] = \{1\}, X] = X$,
(ii) *$F \subseteq G$ implies $(F] \subseteq (G]$,*
(iii) *if F is a pseudo filter of X, then $(F] = F$.*

Proposition 8. *Let $\emptyset \neq A \subseteq X$. Then*

$$\begin{aligned} < A > &= \{x \in X | a_n * (\ldots * (a_1 * x) \ldots) = 1 \; for \; some \; a_1, \ldots, a_n \in A\} \\ &= \{x \in X | a_n \diamond (\ldots * (a_1 * x) \ldots) = 1 \; for \; some \; a_1, \ldots, a_n \in A\} \\ &= \{x \in X | a_n * (\ldots \diamond (a_1 * x) \ldots) = 1 \; for \; some \; a_1, \ldots, a_n \in A\} \\ &= \{x \in X | a_n * *(a_1 \diamond x)) = 1 \; for \; some \; a_1, a_n \in A\} \\ &= \{x \in X | a_n \diamond (\ldots \diamond (a_1 \diamond x) \ldots) = 1 \; for \; some \; a_1, \ldots, a_n \in A\}. \end{aligned}$$

Proof. Set $F = \{x \in X | a_n*(...*(a_1*x)...) = 1 \; for \; some \; a_1, \ldots, a_n \in A\}$. Since $a * a = 1$, for all $a \in A$, we have $A \subseteq F$. Obviously, $1 \in F$. Let $x * y \in F$ and $x \in F$. Now, there are $a_1, \ldots, a_n, b_1, \ldots, b_m, \in A$ such that $a_n * (\ldots * (a_1 * (x * y) \ldots) = 1$ and $b_t * (\ldots * (b_1 * x) \ldots) = 1$. Hence $x \leq (a_n * (\ldots * (a_1 * y) \ldots)$. By definition,

$$1 = b_t * (\ldots * (b_1 * x) \ldots) \leq (b_t * (\ldots * (b_1 * (a_n * (\ldots * (a_1 * y) \ldots).$$

Therefore, $(b_t * (\ldots * (b_1 * \ldots * (a_n * (\ldots * (a_1 * y) \ldots) = 1$. Since F is a pseudo filter, we see that $y \in F$. Hence F is a pseudo filter.

Now, let G be a pseudo filter of X containing A. Let $x \in F$. Then there are $a_1, \ldots, a_n \in A$ such that $a_n * (... * (a_1 * x) \ldots) = 1$. Since $a_1, \ldots, a_n \in G$ and G is a pseudo filter of X, then $x \in G$. Therefore, $F \subseteq G$. ■

Corollary 3. *Let $F \in pF(X)$ and $a \in X$. Then $(F \cup \{a\}] = F_a$.*

Proof. The proof is clear by Theorem 13 and Proposition 8. ■

Theorem 14. *Pseudo filter F is maximal if and only if $x * y \in F$ or $y * x \in F$, for all $x, y \in X$.*

Proof. $(\Rightarrow)$ Let $x, y \in X$. If $x \in F$, then since $x \leq y * x$ and F is a pseudo filter, we have $y * x \in F$ and $y \diamond x \in F$. By a Similar way, if $y \in F$, then $x * y \in F$ and $x \diamond y \in F$. Finally, assume that $x, y \notin F$ and $x * y \notin F$. Then

$$F_{x*y} = \{z \in X : (x * y) * z \in F\}$$

is a pseudo filter containing F and $x * y$. Since F is a maximal filter, we see that $F_{x*y} = X$. Hence $(x * y) \diamond (y * x) \in F$. Now, by using $(pBE4)$ we have

$$y * ((x * y) \diamond x) \in F.$$

By Theorem 7, $(pBE3)$ and pseudo self distributivity X we have

$$\begin{aligned} y * x &= 1 \diamond (y * x) \\ &= (y * (x * y)) \diamond (y * x) \\ &= y * ((x * y) \diamond x) \in F. \end{aligned}$$

$(\Leftarrow)$ Let K be a pseudo filter, such that $F \subseteq K$. If $F \neq K$, then there is $x \in K$ such that $x \notin F$. Thus $(F \cup \{x\}] \subseteq K$. Let y be an arbitrary element of X. If $y \in F$, then $y \in (F \cup \{x\}] \subseteq K$. If $y \notin F$, then $x * y \in F$. By Corollary 3, $y \in (F \cup \{x\}]$. Thus $X = (F \cup \{x\}]$, and so $K = X$. Therefore, F is a maximal pseudo filter. ■

Theorem 15. *If $F, G \in pF(X)$, then*

$$\begin{aligned} (F \cup G] &= \{a \in X | x * (y \diamond a) = 1 \ for\ some\ x \in F,\ y \in G\} \\ &= \{a \in X | x \diamond (y \diamond a) = 1 \ for\ some\ x \in F,\ y \in G\} \\ &= \{a \in X | x * (y * a) = 1 for\ some\, x \in F,\ \ y \in G\}. \end{aligned}$$

Proof. By Proposition 8, the proof is clear . ■

Theorem 16. *Let $F \in pF(X)$ and $F \neq X$. Then the following statement are equivalent:*

(i) *F is a maximal pseudo filter,*

(ii) *if $x \notin F$, then $(F \cup \{x\}] = X$, for all $x \in X$.*

Proof. $(i) \Rightarrow (ii)$. We have that $F \subseteq (F \cup \{x\}]$ and $F \neq (F \cup \{x\}]$, since $x \notin F$. From the fact F is maximal it follows that $(F \cup \{x\}] = X$.

$(ii) \Rightarrow (i)$ Suppose that there is a proper pseudo filter G of X such that $F \subseteq G$ and $F \neq G$. Then there is $x \in G \setminus F$. By (ii), we have $(F \cup \{x\}] = X$. Since $(F \cup \{x\}] \subseteq G$, it follows that $G = X$. ■

Let $F, G \in pF(X)$. We define the meet of F and G (denoted by $F \wedge G$) by $F \wedge G = F \cap G$ and the join of the F and G (denoted by $F \vee G$) by $F \vee G = (F \vee G]$ which is, pseudo filter generated by F and G. Hence $(pF(X); \wedge, \vee)$ is a lattice. Moreover, by Theorem (2.11, [1]), we have

Theorem 17. *$(pF(X); \wedge, \vee)$ is a complete lattice.*

Theorem 18. *$(pF(X); \wedge, \vee)$ is a distributive lattice. (i.e., $I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K)$, for all $I, J, K \in pF(X)$).*

Proof. Let $x \in (I \wedge J) \vee (I \wedge K)$. Then by Theorem 15, $j * (k \diamond x) = 1$, for some $j \in I \cap J$ and for some $k \in I \cap K$. Since K is a pseudo filter and $j, k \in I$, it follows that $x \in I$. Also, since $j \in J$, $k \in K$ and $j*(k \diamond x) = 1$, by Theorem 15, we have $x \in J \vee K$. Thus $x \in I \wedge (J \vee K)$ and so $I \wedge (J \vee K) \supseteq (I \wedge J) \vee (I \wedge K)$.

Conversely, let $x \in I \wedge (J \vee K)$. Then $x \in I$ and $x \in J \vee K$. By Theorem 15, $j * (k \diamond x) = 1$, for some $j \in J$ and $k \in K$. Since $j * (k \diamond x) = 1 \in J$, $j \in J$ and J is a pseudo filter, it follows that $k \diamond x \in J$. By a similar way, $k \diamond (j * x) = 1 \in K$, $k \in K$ and K is a pseudo filter, implies $j * x \in K$. Now, by Definition 4 and ($pBE4$), we have

$$\begin{aligned}(j * x) * ((k \diamond x) \diamond x) &= ((j * x) * (k \diamond x)) \diamond ((j * x) * x)\\ &= (k \diamond ((j * x) * x)) \diamond ((j * x) * x)\\ &= (j * x) * ((k \diamond ((j * x) * x) \diamond x)).\end{aligned}$$

On the other hand, $k \diamond (j * x) = 1$ implies $(k \diamond (j * x) * x = 1 * x = x$ and so $((k \diamond (j * x) * x) \diamond x = x \diamond x = 1$. Hence

$$(j * x) * ((k \diamond x) \diamond x) = (j * x) * ((k \diamond ((j * x) * x) \diamond x)) = (j * x) * 1 = 1.$$

Since $j * x \in I \cap K$ and $k \diamond x \in I \cap J$, By Theorem 15, we see that $x \in (I \wedge J) \vee (I \wedge K)$. Therefore, $I \wedge (J \vee K) \subseteq (I \wedge J) \vee (I \wedge K)$ and so $I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K)$. ■

In the following Example we show that the lattice $(pF(X); \wedge, \vee)$ is not complemented in general.

Example 8. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	c	1	1
b	1	d	1	1	d	1
c	1	d	1	1	d	1
d	1	e	c	c	1	e
e	1	d	c	c	d	1

$\diamond$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	1	1
b	1	d	1	1	d	1
c	1	d	1	1	d	1
d	1	e	b	c	1	e
e	1	d	c	c	d	1

.

Then $(X; *, \diamond, 1)$ is a distributive pseudo BE-algebra. If set $I = \{1, a, d, e\}$, $J = \{1, b, c, e\}$, $K = \{1, d\}$, $L = \{1, e\}$, then $pF(X) = \{\{1\}, X, I, J, K, L\}$. We see that the complement of I dose not exists.

Let $F \in pF(X)$. Define $S(F) \subseteq pF(X)$ as follows: $S^* = \{I \in pF(X) \mid I \subseteq X/F \cup \{1\}\}$,

Theorem 19. *The complement of $F \in pF(X)$ exists if and only if there is $F' \in S(F)$ such that $F' \vee F = X$. Specially, $pF(X)$ is a boolean lattice if and only if for all $F \in pF(X)$, there is a unique $F' \in S(F)$ such that $F' \vee F = X$.*

Corollary 4. *If for all $F \in pF(X)$, $X/F \cup \{1\} \in pF(X)$, then $pF(X)$ is a complemented distributive lattice ($F' = X/F \cup \{1\}$, for all $F \in pF(X)$).*

Proof. By Theorem 19, the proof is clear. ■

Example 9. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	c	d	1
b	1	a	1	1	1	e
c	1	a	1	1	1	e
d	1	a	1	1	1	e
e	1	a	d	d	d	1

$\diamond$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	1	1
b	1	a	1	1	1	e
c	1	a	1	1	1	e
d	1	a	1	1	1	e
e	1	a	c	c	d	1

.

Then $(X; *, \diamond, 1)$ is a distributive pseudo BE-algebra and $pF(X) = \{\{1\}, X, I = \{1, a, e\}, J = \{1, b, c, d\}\}$. We can see that

$$X' = 1, \quad I' = X/I \cup \{1\} = J, \quad J' = X/J \cup \{1\} = I.$$

Therefore, $pF(X)$ is a boolean lattice.

6. Conclusion and future work

In this paper, the notion of distributive pseudo BE-algebra is defined. Normal pseudo filters can be thought of as a kind of generalization of pseudo filters, and also as a framework provides the foundation for the study of filter theory. We give some examples to show that this notions are different and consider the notion of normal pseudo filter and get some results in distributive pseudo BE-algebras.

In the future work, we are going to consider the notion of the pseudo filters on distributive pseudo BE-algebras to get a quotient algebras induced by this pseudo filters and define pseudo congruence relations on pseudo BE-algebras. Also, we try to define the other types of pseudo filters in pseudo BE-algebras and discuss on the relationship between them.

Acknowledgments. The authors wish to thank the reviewers for their excellent suggestions that have been incorporated into this paper.

References

[1] BORZOOEI R.A., BORUMAND SAEID A., REZAEI A., RADFAR A., AMERI R., On pseudo BE-algebras, *Discussiones Mathematica, General Algebra and Applicatins,* 33(2013), 95-108.

[2] GEORGESCU G., IORGULESCU A., Pseudo-MV algebras, *Multi. Val. Logic,* 6(2001), 95-135.

[3] GEORGESCU G., IORGULESCU A., Pseudo-BL algebras: a noncommutative extension of BL algebras, *in Abstracts of the Fifth International Conference FSTA 2000,* Slovakia, February 2000, 90-92.

[4] GEORGESCU G., IORGULESCU A., *Pseudo-BCK algebras: an extension of BCK-algebras,* Combinatorics, computability and logic, 97-114, Springer Ser. Discrete Math. Theor. Comput. Sci., Springer, London, 2001.

[5] KIM H.S., KIM Y.H., On BE-algebras, *Sci, Math, Jpn.,* 66(1)(2007), 113-117.

[6] RACHUNEK J., A non commutative generalization of MV-algebras, *Czehoslovak Math. J.,* 52(127)(2)(2002), 255-273.

[7] WALENDZIAK A., On axiom systems of pseudo-BCK algebras, *Bull. Malays. Math. Sci. Soc.,* 34(2)(2011), 287-293.

RAJAB ALI BORZOOEI
DEPARTMENT OF MATHEMATICI
SHAHID BEHESHTI UNIVERSITY OF TEHRAN
TEHRAN, IRAN
e-mail: borzooei@sbu.ac.ir

ARSHAM BORUMAND SAEID
DEPARTMENT OF PURE MATHEMATICS
FACULTY OF MATHEMATICS AND COMPUTER
SHAHID BAHONAR UNIVERSITY OF KERMAN
KERMAN, IRAN
e-mail: arsham@uk.ac.ir

AKBAR REZAEI[1] AND ATEFE RADFAR[2]
DEPARTMENT OF MATHEMATICS
PAYAME NOOR UNIVERSITY
P.O. BOX. 19395-3697, TEHRAN, IRAN
e-mail: [1]Rezaei@pnu.ac.ir
[2]ateferadfar@yahoo.com

REZA AMERI
DEPARTMENT OF MATHEMATICS
TEHRAN UNIVERSITY
TEHRAN, IRAN
e-mail: rameri@ut.ac.ir

Received on 18.10.2013 and, in revised form, on 01.07.2014.