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EXTENDED CONTINUOUS BLOCK BACKWARD DIFFERENTIATION FORMULA FOR STIFF SYSTEMS

Abstract. We present an Extended Continuous Block Backward Differentiation Formula (ECBBDF) of order k+1 for the numerical solution of stiff ordinary differential equations. This is achieved by constructing an Extended Continuous Backward Differentiation formula (ECBDF) together with the additional methods from its first derivative and are combined to form a single block of methods that simultaneously provide the approximate solutions for the stiff Initial Value Problems (IVPs). The error constant and stability property of the (ECBBDF) is discussed. We use the specific cases $k = 4$ and $k = 5$ to illustrate the process. The performance of the method is demonstrated on some numerical examples to show the accuracy and efficiency advantages of the method.

Key words: extended continuous backward differentiation formula, block method Stiff systems, stability.

AMS Mathematics Subject Classification: 65L05.

1. Introduction

The most popular class of multistep methods for solving stiff ODEs is the Backward Differentiation Formula (BDF). This method was first used for the solution of stiff problem by Curtis and Hirschfelder [6]. Over the years several methods have been developed and discussed extensively in literature (see [3], [4], [9], [15], [17], [18]).
The first order initial value problem (IVP) of the form

$$y' = f(t, y), \qquad y(t_0) = y_0, \tag{1}$$

on the interval $I = [t_0, T_n]$, where $t \in R$,$y : R \longrightarrow R^m$, $f : R \times R^m \longrightarrow R^m$, m is the dimensionality of the system, f satisfy the Lipschitz condition (see [11]), and the Jacobian $\frac{\partial f}{\partial t}$ whose negative real parts varies slowly([13]) is called stiff system. Previous works on block methods for solving (1) are given by (Brugnano et.al.[2], Rosser [16], Chu et.al. [5] and Fatunla[8]), among others. Continuous block methods produced numerical solutions with less computational effort as compared to non block method. This is due to the fact that they are self starting without the need for complicated sub-routings for starting values. More so block methods generate more than one solution simultaneously since they consist of more than one point in each block. Akinfenwa et.al. [1] derived the Continuous Block Backward Differentiation Formulas (CBBDFs) which were self starting and implemented using fixed step size with good accuracy. Our aim in this paper is to modify the method in [1] and to further improve its performance in terms of accuracy and computational effort. The rest of this paper is presented as follows. In Section 2 we construct the ECBBDF which consist of a main discrete method and additional methods combined as a single block method for solving (1), in Section 3 the order of accuracy and stability property of the methods are discussed, Section 4 is the computational aspect ECBBDF algorithm. Numerical examples are given in Section 5 to show the accuracy and efficiency advantages. Finally, the conclusion of the paper is discussed in Section 6.

2. Construction of ECBBDF

In this section, we construct the main method and additional methods derived from its first derivative and are combined to form the ECBBDF on the interval from t_n to $t_{n+k} = t_n + kh$ where h is the chosen step-length and k is the step number. We assume that the exact solution $y(t)$ on the interval $[t_n, t_{n+k}]$ is locally represented by $Y(t)$ in the form

$$Y(t) = \sum_{j=0}^{p+q-1} b_j \varphi_j(t), \tag{2}$$

b_j are unknown coefficients to be determined, and $\varphi_j(t)$ are polynomial basis function of degree $p + q - 1$. such that the number of interpolation points p and the number of distinct collocation points q are respectively chosen to satisfy $p = k$ and $q > 0$. The proposed class of methods is thus constructed by specifying the following parameters: $\varphi_j(t) = t^j_{n+i}$, $j = 0, \ldots, k$, $p = k$, $q = 2$, by imposing the following conditions

$$\sum_{j=0}^{p+q-1} b_j t^j_{n+i} = y_{n+i}, \quad i = 0, \ldots, k-1, \tag{3}$$

$$\sum_{j=0}^{q} m_j j t^j_{n+i} - 1 = f_{n+i}, \quad i = k-1, k \tag{4}$$

assuming that $y_{n+i} = Y(t_n+ih)$, denote the numerical approximation to the exact solution $y(t_{n+i})$ $f_{n+i} = Y'(t_n+ih), y_{n+j})$, denote the approximation to $y'(t_{n+i})$ n is the grid index. It should be noted that equation (3) and (4) lead to a system of $k+2$ equations which must be solved to obtain the coefficients b_j, $j = 0, \dots, p+q-1$. The main method is then obtained by substituting the values of b_j into equation (2). After some algebraic computation, the method yields the expression in the form (5)

$$Y(t) = \sum_{j=0}^{k-1} \alpha_j(t) y_{n+j} + h(\beta_{k-1}(t) f_{n+k-1} + \beta_k(t) f_{n+k}), \tag{5}$$

where $\alpha_j(t)$, $j = 0, \dots, k-1$, $\beta_{k-1}(t)$ and $\beta_k(t)$ are continuous coefficients. Equation (5) is then used to generate the main discrete method by evaluating at point $t = t_{n+k}$.

To obtain the additional methods, differentiate (5) with respect to t we have

$$Y'(t) = \frac{1}{h}\left[\sum_{j=0}^{k-1} \alpha'_j(t) y_{n+j} + h(\beta'_{k-1}(t) f_{n+k-1} + \beta'_k(t) f_{n+k})\right]. \tag{6}$$

The discrete additional methods are then obtained by evaluating (6) at points $t = [t_n, t_{n+1}, \dots, t_{n+k-2}]$. The methods (5) and (6) are then combined to produce the ECBBDF. The continuous coefficients $\alpha_j(t)$, $j = 0, \dots, k-1$, $\beta_{k-1}(t)$ and $\beta_k(t)$ are also expressed as a function of x for convenience where $x = \frac{t-t_{n+k-1}}{h}$. The methods (5), and (6), are combined and implemented as a one block ECBBDF for all k.

Next is the detailed discussion for the specific methods.

Case $k = 4$**.** Using (5) and (6) to obtain a continuous 4-step method, with the following specification $p = 4$, $q = 2$, $k = 4$, and $\varphi_j(t_{n+i}) = t^j_{n+i}$, $j = 0, 1, \dots, 5$, $i = 0, 1, \dots, 4$, also the expressions $\alpha_j(t)$, $\beta_3(t)$, $\beta_4(t)$, as functions of $x$for convenience, where $x = \frac{t-t_{n+k-1}}{h}$.

In what follows:

$$\begin{aligned}
\alpha_0(x) &= \frac{1}{222}(74x^4 - 68x^5), & \alpha_1(x) &= -\frac{2}{37}(48x^4 - 44x^5) \\
\alpha_2(x) &= \frac{3}{37}(136x^4 - 124x^5), & \alpha_3(x) &= \frac{2}{111}(3216x^3 - 6356x^4 - 3152x^5) \\
\beta_3(x) &= \frac{4}{37}(112x^4 - 100x^5), & \beta_4(x) &= \frac{12}{37}x^5).
\end{aligned}$$

The main method is obtained for $k = 4$ by evaluating (5) at $t = t_{n+4}$, which is equivalent to $t = 1$ we obtain the formula

$$\begin{aligned} y_{n+4} = {} & \frac{1}{37}y_n - \frac{8}{37}y_{n+1} + \frac{36}{37}y_{n+2} \\ & + \frac{8}{37}y_{n+3} + \frac{48h}{37}f_{n+3} + \frac{12h}{37}f_{n+4}. \end{aligned} \tag{7}$$

The following additional methods which are of the form (6) are then obtained from the first derivative of (5) by evaluating (6) at the points $t = t_n, t_{n+1}, t_{n+2}$

$$\begin{cases} hf_n &= -\frac{226}{111}y_n + \frac{216}{37}y_{n+1} - \frac{306}{37}y_{n+2} \\ & \quad + \frac{536}{111}y_{n+3} - \frac{112h}{37}f_{n+3} + \frac{9h}{37}f_{n+4} \\ hf_{n+1} &= -\frac{19}{111}y_n - \frac{48}{37}y_{n+1} + \frac{105}{37}y_{n+2} \\ & \quad - \frac{152}{111}y_{n+3} + \frac{29h}{37}f_{n+3} - \frac{2h}{37}f_{n+4} \\ hf_{n+2} &= \frac{10}{333}y_n - \frac{13}{37}y_{n+1} - \frac{34}{37}y_{n+2} \\ & \quad + \frac{413}{333}y_{n+3} - \frac{62h}{37}f_{n+3} + \frac{h}{37}f_{n+4} \end{cases} \tag{8}$$

the methods (7) and (8), are then combined to give the ECBBDF for $k = 4$.

Case $k = 5$**.** The method (5) is used to obtain a continuous 5-step method, with the following specification $p = 5, q = 2, k = 5$, and $\varphi_j(t_{n+i}) = t_{n+i}^j$, $j = 0, 1, \ldots, 6$, $i = 0, 1, \ldots, 5$. We also expressed $\alpha_j(t)$, $\beta_4(t)$, $\beta_5(t)$, as functions of $x$for convenience, where $x = \frac{t - t_{n+5}}{h}$.

In what follows:

$$\begin{aligned} \alpha_0(x) &= -\frac{1}{788}(197x^5 - 185x^6), \quad \alpha_1(x) = \frac{5}{1182}(485x^5 - 455x^6), \\ \alpha_2(x) &= -\frac{5}{197}(315x^5 - 295x^6), \quad \alpha_3(x) = \frac{5}{197}(895x^5 - 835x^6), \\ \alpha_4(x) &= \frac{5}{2364}(62080x^4 - 124665x^5 + 62525x^6), \\ \beta_4(x) &= \frac{3}{197}(745x^5 - 685x^6), \quad \beta_5(x) = \frac{60}{197}x^6). \end{aligned}$$

The main method is obtained for $k = 5$ by evaluating (5) at $t = t_{n+5}$, which is equivalent to $t = 1$ to obtain the formula

$$\begin{aligned} y_{n+5} = {} & -\frac{3}{197}y_n + \frac{25}{197}y_{n+1} - \frac{100}{197}y_{n+2} + \frac{300}{197}y_{n+3} \\ & - \frac{25}{197}y_{n+4} + \frac{300h}{197}f_{n+4} + \frac{60h}{197}f_{n+4}. \end{aligned} \tag{9}$$

.

The following additional methods which are of the form (6) are then obtained from the first derivative of (5) by evaluating (6) at the points $t = t_n, t_{n+1}, t_{n+2}, t_{n+3}$

$$
(10)\qquad \begin{cases}
hf_n &= -\frac{1490}{591}y_n + \frac{3880}{591}y_{n+1} - \frac{1890}{197}y_{n+2} \\
&\quad + \frac{7160}{591}y_{n+3} - \frac{3880}{591}y_{n+4} + \frac{745h}{197}f_{n+4} - \frac{48h}{197}f_{n+5} \\
hf_{n+1} &= -\frac{90}{197}y_n - \frac{826}{591}y_{n+1} + \frac{576}{197}y_{n+2} \\
&\quad - \frac{546}{197}y_{n+3} + \frac{826}{591}y_{n+4} - \frac{152h}{197}f_{n+4} + \frac{9h}{197}f_{n+5} \\
hf_{n+2} &= \frac{41}{1576}y_n - \frac{202}{591}y_{n+1} - \frac{315}{394}y_{n+2} \\
&\quad + \frac{374}{197}y_{n+3} - \frac{3703}{4728}y_{n+4} + \frac{157h}{394}f_{n+4} - \frac{4h}{197}f_{n+5} \\
hf_{n+3} &= -\frac{43}{4728}y_n + \frac{53}{591}y_{n+1} - \frac{207}{591}y_{n+2} \\
&\quad - \frac{349}{591}y_{n+3} + \frac{4895}{4728}y_{n+4} - \frac{167h}{394}f_{n+4} + \frac{3h}{197}f_{n+5}
\end{cases}
$$

the methods (9), and (10), are thus combined to give the ECBBDF for $k = 5$.

3. Analysis of the method

The extended continuous block backward differentiation formulae can be represented by a matrix finite difference equation in block form as

$$
(11)\qquad A^{(1)}Y_\omega = A^{(0)}Y_{\omega-1} + hB^{(1)}F_\omega + hB^{(0)}F_{\omega-1},
$$

where

$$
\begin{aligned}
Y_\omega &= (y_{n+1}, y_{n+2}, y_{n+3}, \dots, y_{n+k},)^T, \\
Y_{\omega-1} &= (y_{n-k+1}, y_{n-k+2}, y_{n-k+3}, \dots, y_n)^T \\
F_\omega &= (f_{n+1}, f_{n+2}, f_{n+3}, \dots, f_{n+k})^T, \\
F_{\omega-1} &= (f_{n-k+1}, f_{n-k+2}, f_{n-k+3}, \dots, f_n)^T
\end{aligned}
$$

for $\omega = 1, \dots$ and $n = 0, k, \dots, N-k$. And the matrices $A^{(1)}$, $A^{(0)}$, $B^{(1)}$ and $B^{(0)}$ are K by K matrices whose entries are given by the combined coefficients of the methods obtained in (5) and (6) as follows:

$$
A^{(1)} = \begin{pmatrix}
a_{11} & a_{12} & \dots & a_{1k-1} & 0 \\
a_{21} & a_{22} & \dots & a_{2k-1} & 0 \\
a_{31} & a_{32} & \dots & a_{3k-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{k1} & a_{k2} & \dots & a_{kk-1} & 1
\end{pmatrix}
$$

$$
A^{(0)} = \begin{pmatrix}
0 & 0 & 0 & \dots & \mu_{1k} \\
0 & 0 & 0 & \dots & \mu_{2k} \\
0 & 0 & 0 & \dots & \mu_{3k} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \mu_{kk}
\end{pmatrix}
$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \dots & b_{1k-1} & b_{1k} \\ -1 & 0 & 0 & \dots & b_{2k-1} & b_{2k} \\ 0 & -1 & 0 & \dots & b_{2k-1} & b_{2k} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & b_{k-1k-1} & b_{k-1k} \\ 0 & 0 & 0 & \dots & b_{k-1k} & b_{kk} \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Following Fatunla[8] and Lambert [15] the local truncation error associated with each of the method in the ECBBDF can be defined to be the linear difference operator

$$L[y(t);h] = \sum_{j=0}^{k-1} \alpha_j y_{n+j} - h(\beta_{k-1} f_{n+k-1} + \beta_k f_{n+k}). \tag{12}$$

Assuming that $y(t)$ is sufficiently differentiable,we can write the terms in (12) as a Taylor series expression of $y(t_{n+j})$ and $f(t_{n+j})$=$y'(t_{n+j})$ as

$$y(t_{n+j}) = \sum_{j=0}^{\infty} \frac{(jh)}{m!} y^{(m)}(t_n) \ \text{ and } \ y'(t_{n+j}) = \sum_{j=0}^{\infty} \frac{(jh)}{m!} y^{(m+1)}(t_n). \tag{13}$$

Substituting these into equations (13) and (12) we obtain the expression

$$L[y(t);h] = C_0 y(t) + C_1 h y'(t) + C_2 h^2 y''(t) + \ldots + C_p h^p y^p(t) + \ldots, \tag{14}$$

where the constant coefficients C_m, $m = 0, 1, 2, \ldots$, $l = 1, 2, \ldots, k$ are given as follows:

$$\begin{aligned} C_0 &= \sum_{j=0}^{k-1} \alpha_j \\ C_1 &= \sum_{j=1}^{k-1} j\alpha_j - \beta_{k-1} - \beta_k + \eta_l \\ C_2 &= \frac{1}{2!}(\sum_{j=1}^{k} j^2 \alpha_j - 2(k-1)\beta_{k-1} - 2k\beta_k + 2l\eta_l) \\ &\vdots \end{aligned}$$

$$C_m = \frac{1}{m!}(\sum_{j=1}^{k-1} j^m \alpha_j - m(k-1)^{m-1}\beta_{k-1} - mk^{m-1}\beta_k + ml^{m-1}\eta_l)$$

where $\eta_k = 0$ and $\eta_l = 1, l = 0, 1, \ldots k-2$.

According to Henrici [11], we say that the method in (11) to have a maximal order of accuracy m if

$$L[y(t); h] = \bigcirc(h^{m+1}), \quad C_0 = C_1 = \ldots = C_m = 0, \quad C_{m+1} \neq 0. \tag{15}$$

Therefore, C_{m+1} is the error constant and $C_{m+1}h^{m+1}y^{(m+1)}(t_n)$ the principal local truncation error at the point t_n.

3.1. Zero stability

The zero stability of the method is concerned with the stability of the difference system in the limit as h tends to zero (see [8]). Thus, as $h \longrightarrow 0$ the difference system (11) tends to

$$A^{(1)}Y_\omega = A^{(0)}Y_{\omega-1}.$$

And the first characteristics polynomial $\rho(R)$ is given as

$$\rho(R) = Det[RA^{(1)} - A^{(0)}] = \frac{u}{v}R^{k-1}(1-R), \tag{16}$$

where u and v are integers.

The block method (11) is zero stable for $\rho(R) = 0$ and satisfies $|R_j| \leq 1, j = 1, \ldots k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1. Hence the extended continuous block BDF is zero stable.

In particular taking $k = 4$ we have that

$$A^{(1)} = \begin{pmatrix} \frac{-216}{37} & \frac{306}{37} & \frac{-536}{111} & 0 \\ \frac{48}{37} & \frac{-105}{37} & \frac{152}{111} & 0 \\ \frac{13}{37} & \frac{34}{37} & -\frac{413}{333} & 0 \\ \frac{8}{37} & \frac{-36}{37} & -\frac{8}{37} & 1 \end{pmatrix}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{-266}{111} \\ 0 & 0 & 0 & \frac{-19}{111} \\ 0 & 0 & 0 & \frac{10}{333} \\ 0 & 0 & 0 & \frac{-1}{37} \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & \frac{-112}{37} & \frac{9}{37} \\ -1 & 0 & \frac{29}{37} & \frac{-2}{37} \\ 0 & -1 & \frac{-62}{111} & \frac{1}{37} \\ 0 & 0 & \frac{-48}{37} & \frac{12}{37} \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore from (15)the ECBBDF $k = 4$, $\rho(R) = 0$ implies $Det[RA^{(1)} - A^{(0)}] = \frac{240}{37}R^3(1 - R) = 0$ and $R_1 = R_2 = R_3 = 0$ and $R_4 = 1$, hence ECBBDF $k = 4$ is zero stable.

Similarly taking $k = 5$ we have that

$$A^{(1)} = \begin{pmatrix} \frac{-3880}{591} & \frac{1890}{197} & \frac{-7160}{591} & \frac{3880}{591} & 0 \\ \frac{826}{591} & \frac{-576}{197} & \frac{546}{197} & \frac{-826}{591} & 0 \\ \frac{202}{591} & \frac{315}{394} & \frac{-374}{197} & \frac{3703}{4728} & 0 \\ \frac{-53}{591} & \frac{207}{394} & \frac{349}{591} & \frac{-4895}{4728} & 0 \\ \frac{-25}{197} & \frac{100}{197} & \frac{-300}{197} & \frac{25}{197} & 1 \end{pmatrix}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{-1490}{591} \\ 0 & 0 & 0 & 0 & \frac{-30}{197} \\ 0 & 0 & 0 & 0 & \frac{41}{1576} \\ 0 & 0 & 0 & 0 & \frac{-43}{4728} \\ 0 & 0 & 0 & 0 & \frac{-3}{197} \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \frac{745}{197} & \frac{-48}{197} \\ -1 & 0 & 0 & \frac{-152}{37} & \frac{9}{197} \\ 0 & -1 & 0 & \frac{157}{394} & \frac{-4}{197} \\ 0 & 0 & -1 & \frac{-167}{394} & \frac{3}{197} \\ 0 & 0 & 0 & \frac{300}{197} & \frac{60}{197} \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus from (15)the ECBBDF $k = 5$, $\rho(R) = 0$ implies

$$Det[RA^{(1)} - A^{(0)}] = \frac{1800}{197}R^4(1 - R) = 0$$

and $R_1 = R_2 = R_3 = R_4 = 0$ and $R_5 = 1$, hence ECBBDF $k = 5$ is zero stable.

3.2. Consistency and convergence

We note that the new block method (11) is consistent as it has order $m > 1$ see Table 1. Since the block method (11) is zero stable. According to Henrici [11] . Convergence = zero stability + consistency. Hence the ECBBDF converges.

3.3. Linear stability

The linear stability properties of the Extended continuous block BDF is discussed in the spirit of Hairer and Wanner [10] and determined by expressing it in the form (11) and applying the test problem $y' = \lambda y$, $\lambda < 0$ to yield

$$Y_\omega = Q(z)Y_{\omega-1}, \quad z = \lambda h, \tag{17}$$

where the matrix $Q(z)$ is given by

$$Q(z) = (A^{(1)} - zB^{(1)})^{-1}(A^{(0)} + B^{(0)}).$$

The matrix Q(z) has eigenvalues $\{\xi_1, \xi_2, \xi_3, \dots, \xi_k\} = \{0, 0, 0, \dots, \xi_k\}$, where the dominant eigenvalue ξ_k is the stability function $\xi(z) : \mathbb{C} \to \mathbb{C}$ which is a rational function with real coefficients given by

$$\xi(z) = \frac{G(z)}{H(z)}. \tag{18}$$

Applying the test equation with $z = \lambda h$, from (17) we obtained the stability function for $k = 4$ and 5 as displayed in Table 1.

Table 1. Properties of ECBBDF for $k = 4$ and 5

k	Error Constant	Order$m$
4	$(-\frac{2}{185}, -\frac{4}{37}, \frac{41}{2220}, -\frac{13}{6660})$	$(5, 5, 5, 5)^T$
5	$(-\frac{10}{1379}, \frac{418}{4137}, -\frac{106}{6895}, \frac{31}{5910}, -\frac{227}{82740})$	$(6, 6, 6, 6, 6)^T$
k	$\xi(z)$	θ
4	$(\frac{60+120z+105z^2+50z^3+12z^4}{60-120z+105z^2-50z^3+12z^4})$	90
5	$(\frac{360+900z+1020z^2+675z^3+274z^4+60z^5}{360-900z+1020z^2-675z^3+274z^4-60z^5})$	90

It can be seen from Table 1 that the ECBBDF has order $k+1$, relatively small error constant and it is A- stable.

4. Computational aspect of the ECBBDF

The two newly derived schemes are implemented more efficiently as block numerical integrators for (1)) to simultaneously obtain the approximations $(y_{n+1}, y_{n+2}, \dots, y_{n+k})^T$ without requiring starting values and predictors, $n = 0, k, \dots, N-k$, over sub-intervals $[t_0, t_k], \dots, [t_{N-k}, t_N]$. For example $n = 0, \omega = 1$ $(y_{n+1}, y_{n+2}, \dots, y_{n+k})^T$, are simultaneously obtained over the sub-interval $[t_0, t_k]$, as y_0 is known from (1).

For $n = 1$, $\omega = 2$, $(y_k, y_{k+1}, \ldots, y_{2k})^T$ are simultaneously obtained over the sub-interval $[t_k, t_{2k}]$, as y_k is known from the previous block. Hence, the sub-intervals do not over-lap. The computations were carried out using our written code in Matlab. It should be noted that for linear problems, the code uses Gaussian elimination and uses the Newton's method for nonlinear problems.

5. Numerical examples

This section deals with some numerical experiments, executed in MATLAB language with double precision arithmetic, which illustrate the result derived in the previous sections.

Example 1. Consider the stiff system of initial value problem which has been solved by. Ismail et. al. [12]

$$y_1' = -2000y_1 + 1000y_2 + 1, \qquad y_1(0) = 0.$$

$$y_2' = y_1 - y_2, \qquad y_2(0) = 0.$$

The eigenvalues of the Jacobian are 2000.5 and -0.5. The theoretical solution is

$$y_1(t) - 4.97 \times 10^{-4} e^{-2000.5t} - 5.034 \times 10^{-4} e^{-0.5t} + 0.001$$

$$y_2(t) - 2.5 \times 10^{-7} e^{-2000.5t} - 1.007 \times 10^{-3} e^{-0.5t} + 0.001.$$

The system is integrated with $h = 0.0001$, for the purpose of comparison. Also, the results for $h = 0.01$ and $h = 0.1$ are tabulated at different values of t to show the A stability of the two new methods. The result of Ismail et. al. [12] is reproduced in Table 2 and compared with that obtained using the ECBBDF for $k = 5$. It can be seen in Table 2 that the result obtained for ECBBDF is superior to those of Ismail et. al. [12] and CBBDF [1] for the same number of steps

Table 2. Comparison of methods at the end points $t_\varepsilon$and $h = 0.0001$ for Example 1 $ERy_i = |y_i - y(t_i)|$

t	Ismail et-al [12] $\theta = 89$ ERy_1 ERy_2	CBBDF5 $\theta = 89.9$ ERy_1 ERy_2	ECBBDF5 $\theta = 90$ ERy_1 ERy_2
5	3.64920×10^{-7} 7.670023×10^{-7}	2.328953×10^{-7} 5.027468×10^{-7}	2.328953×10^{-7} 5.027468×10^{-7}
10	2.454035×10^{-7} 4.942995×10^{-7}	1.700858×10^{-8} 3.705176×10^{-8}	1.700858×10^{-8} 3.705176×10^{-8}

Table 3. Result of ECBBDF at the end points t_ε for Example 1 $ERy_i = |y_i - y(t_i)|$

h	t	ECBBDF4 $\theta = 90$ ERy_1 ERy_2	ECBBDF5 $\theta = 90$ ERy_1 ERy_2
0.01	5	2.3289534×10^{-7}	2.328953×10^{-7}
		5.027468×10^{-7}	5.027468×10^{-7}
	10	1.700857×10^{-8}	1.700858×10^{-8}
		3.705175×10^{-8}	3.705176×10^{-8}
0.1	5	4.924191×10^{-5}	3.163426×10^{-4}
		5.274904×10^{-7}	6.6107427×10^{-7}
	10	1.763763×10^{-4}	2.005234×10^{-4}
		1.252704×10^{-7}	1.373470×10^{-7}

Example 2. We consider another stiff system which has also been solved by Ezzeddine et. al. [3]:

$$y_1' = -y_1 - 30y_2 + 30e^{-t}, \quad y_1(0) = 1.$$

$$y_2' = 30y_1 - y_2 - 30e^{-t}, \quad y_2(0) = 1.$$

The stiffness ratio of this problem is 1:200 and the exact solution is

$$y_1(t) = e^{-t}, \qquad y_2(t) = e^{-t}$$

Table 4. Comparison of methods for Example 2, $h = 0.01$, error $y_i = |y_i - y(t_i)|$

t	y_i	Error in EBDF [7] $k = 4$	Error in HEBDF [7] $k = 4$	Error in ECBBDF $k = 4$
1.0	y_1	1.71×10^{-13}	8.15×10^{-15}	1.28×10^{-15}
	y_2	2.60×10^{-12}	8.48×10^{-13}	1.17×10^{-14}
10.0	y_1	5.03×10^{-17}	9.83×10^{-18}	1.08×10^{-19}
	y_2	3.36×10^{-16}	7.71×10^{-17}	1.62×10^{-18}
20.0	y_1	1.17×10^{-20}	1.29×10^{-21}	7.24×10^{-23}
	y_2	7.83×10^{-21}	2.79×10^{-21}	5.29×10^{-23}

Table 5. Comparison of methods for Example 2, $h = 0.01$, error $y_i = |y_i - y(t_i)|$

t	y_i	Error in CBBDF [1] $k = 5$	Error in ECBBDF $k = 5$
1.0	y_1	1.01×10^{-13}	4.07×10^{-16}
	y_2	1.15×10^{-13}	2.22×10^{-16}
10.0	y_1	1.52×10^{-17}	1.08×10^{-19}
	y_2	1.41×10^{-17}	4.07×10^{-20}
20.0	y_1	1.29×10^{-21}	2.07×10^{-24}
	y_2	7.82×10^{-23}	2.90×10^{-24}

Example 3. We consider the nonlinear stiff system proposed by Kaps [14], and compared with that of CBBDF [1] for different values of step size h. The results for $k = 4$ and 5 are reproduced in Table 4 and compared with CBBDF [1].

$$y_1' = -1002y_1 + 1000y_2^2, \quad y_1(0) = 1.$$

$$y_2' = y_1 - y_2(1 + y_2), \qquad y_2(0) = 1.$$

$0 \le t \le T$ the smaller is, the more serious the stiffness of the system. The exact solution is

$$y_1(t) = y_2^2(t), \qquad y_2(t) = e^{-t}.$$

In Table 6 the error at the end point $T = 10$ is shown and it is obvious that the ECBBDF improves those of CBBDF

Table 6. Comparison of methods at $T = 10$ for Example 3, error $i = |y - y(t)|$ $i = 1, 2$

h	CBBDF_4[1]	CBBDF_5[1]	ECBBDF_4	ECBBDF_5
0.02	4.88×10^{-16}	8.37×10^{-18}	2.48×10^{-19}	1.33×10^{-20}
	$5.39.01 \times 10^{-12}$	9.16×10^{-14}	3.75×10^{-16}	1.35×10^{-16}
0.01	3.13×10^{-17}	3.39×10^{-21}	2.68×10^{-19}	2.87×10^{-22}
	3.45×10^{-13}	1.23×10^{-17}	2.93×10^{-15}	2.93×10^{-19}
0.002	5.14×10^{-20}	4.64×10^{-21}	1.11×10^{-21}	2.32×10^{-21}
	5.67×10^{-14}	5.16×10^{-17}	1.09×10^{-17}	2.55×10^{-17}

Example 4. Finally we consider the stiff system of initial value problem.

$$y'(t) = \begin{bmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{bmatrix} y(t), \quad y(0) = (1, 0, -1)^T,$$

with theoretical solution

$$y_1(t) = \frac{1}{2}(e^{-2t} + e^{-40t}(\cos(40t) + \sin(40t)))$$
$$y_2(t) = \frac{1}{2}(e^{-2t} - e^{-40t}(\cos(40t) + \sin(40t)))$$
$$y_3(t) = -\frac{1}{2}(2e^{-40t}(\sin(40t) - \cos(40t))).$$

The main aim is to show the order and accuracy of the ECBBDF. For different choices of the constant step size h and the Rate Of Convergence (ROC) which is calculated using the formula $ROC = log_2(\frac{e^{2h}}{eh})$, where e^h is the maximum absolute error for h. In all cases the rate of convergence is consistent with the order of method.

Table 7. Maximum error for ECBBDF of order 5 and 6 for Example 4

h	$k = 4$ Max Error ECBBDF order $m = 5$	Rate	$k = 6$ Max Error ECBBDF order $m = 6$	Rate
0.01	3.08×10^{-4}	–	9.88×10^{-5}	–
0.005	7.77×10^{-6}	5.3	1.76×10^{-6}	5.8
0.0025	1.41×10^{-7}	5.7	2.69×10^{-8}	6.0
0.00125	2.31×10^{-9}	5.9	3.96×10^{-10}	6.1
0.00625	6.26×10^{-12}	6.0	6.26×10^{-12}	6.0

6. Conclusion

We have proposed an extended continuous block backward differentiation formula for case $k = 4$ and 5 with orders five and six for the solution of system of stiff IVPs. The algorithms are self-starting, provide good accuracy, and require k function evaluation per integration step in each block. Numerical examples using the ECBBDF showed that the method is accurate and efficient as evident in Table 2, 3, 4, 5, 6, 7 above.

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