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A NOTE ON $q-$CALCULUS

Abstract. In this article, we let $\mathcal{PC}_q$ denote the class of $q-$convex functions. Certain analytic properties of the class $\mathcal{PC}_q$ are studied. The maximum of the absolute value of the Fekete-Szegö functional is briefly determined.

Key words: univalent, starlike, close-to-convex, $q-$integral operator, convex, $q-$difference operator, $q-$starlike, $q-$close-to-convex, and $q-$convex functions.

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1. Introduction and preliminaries

Let $\mathcal{A}_k$ denote the class of functions analytic in the unit disc $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$. In other words, the functions $f(z)$ in $\mathcal{A}_k$ have the power series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathcal{U}).$$

The class $\mathcal{S}^{\star}(\alpha)$ of starlike functions of order α consists of $f \in \mathcal{A}_k$ that satisfies

$$\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathcal{U}),$$

if $g \in \mathcal{S}^{\star}(0)$, (that is $g \in \mathcal{S}^{\star}$) the class $\mathcal{K}$ of close-to-convex functions consists of $f \in \mathcal{A}_k$ that satisfies

$$\mathcal{R}e\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad (z \in \mathcal{U})$$

and the class $\mathcal{C}$ of convex functions consists of $f \in \mathcal{A}_k$ that satisfies

$$\mathcal{R}e\{1 + \frac{zf''(z)}{f'(z)}\} > 0, \quad (z \in \mathcal{U}).$$

In [10], [11], Jackson defined the $q-$derivative operator $\mathcal{D}_{z,q}$ as follows:

Definition 1.

$$\begin{cases} \mathcal{D}_{z,q}f(z) = \frac{f(z)-f(qz)}{z(1-q)}, & (z \in \mathbb{C} - \{0\}; 0 < q < 1), \\ \mathcal{D}_{z,q}f(z) \mid_{z=0} = f'(0). \end{cases} \tag{1}$$

From (1), we have

$$\mathcal{D}_{z,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad (z \neq 0),$$

where $[k]_q = \frac{1-q^k}{1-q}$, *and as* $q \to 1$, $[k]_q \to k$.

The q-shift factorial, the multiple q-shift factorial and the q-binomial coefficients are defined by

(2)
$$(a_1, a_2, \cdots, a_n; q)_k = \begin{cases} 1, & (k = 0, j = 1, a_1 = a) \\ \prod_{n=0}^{k-1} (1 - aq^n), & (j = 1, k \neq 0, a_1 = a, n \in \mathbb{N}) \\ \prod_{j=1}^{n} (a_j; q)_n, & (j = 1, 2, \ldots, n; n \in \mathbb{N}, k \in \mathbb{Z}) \end{cases}$$

and

$$\left[\begin{array}{c} a \\ k \end{array} \right]_q = \begin{cases} 1, & (k = 0), \\ \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-k+1})}{(q;q)_k}, & (k \in \mathbb{N}), \end{cases} \tag{3}$$

where $a, q \in \mathbb{C}$.

Let ${}_r\Phi_s$ denote the q-Hypergeometric series

$${}_r\Phi_s \left(\begin{array}{ccc|c} a_1, & \cdots, & a_r & \\ & & & q, z \\ b_1, & \cdots, & b_s & \end{array} \right) = {}_r\Phi_s(a_1, \ldots, a_r; b_1, \cdots, b_s; q, z) \tag{4}$$

$$= \left[\begin{array}{c} a \\ 0 \end{array} \right]_q + \sum_{k=1}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} z^k (-q^{\frac{(k-1)}{2}})^{k(s+1-r)}. \tag{5}$$

Sofonea [20], derived the nth other of q-derivative in Theorem 1 as follows:

Theorem 1. *Assume f has $q-$derivatives up to order n in $\mathcal{U}$, $n \in \mathbb{N}$ then,*

$$\begin{aligned} (\mathcal{D}_{z,q}^n f)(z) &= (-1)^n (1-q)^{-n} z^{-n} \; q^{\frac{-n(n-1)}{2}} \\ &\quad \times \sum_{k=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{\frac{k(k-1)}{2}} f(q^{n-k} z). \end{aligned}$$

The q-integration from 0 to a was defined in [9] by

$$\mathcal{I}_{z,q}f(z) = \int_0^a f(z)d_qz = (1-q)a\sum_{k=0}^{\infty} f(aq^k)q^k,$$

provided the sum converges absolutely.

Based on the principles of calculus and $q-$calculus, we define the $q-$integral of function $f(g(z))$, from 0 to a as

$$\mathcal{I}_{z,q}f(g(z)) = \int_0^a f(g(z))d_qz = \frac{1}{\mathcal{D}_{z,q}g(z)}\int_0^a f(\zeta)d_q\zeta, \quad (\zeta = g(z)),$$

provided the sum converges absolutely.

Agrawal and Sahoo [2] defined and studied the class $\mathcal{PS}_q^\star(\alpha)$ of functions $q-$starlike of order alpha as follows:

$$\left|\frac{z(\mathcal{D}_qf)(z)}{f(z)} - \frac{1-\alpha q}{1-q}\right| \leq \frac{1-\alpha}{1-q}, \quad (0<q<1 \ z\in\mathcal{U}).$$

Previously, Ismail, Merkes and Styer [8] defined and studied some important properties of functions f in the class $\mathcal{PS}_q^\star$. They applied the technique used by Merkes and Scott [13] involving the starlikeness of Gaussian hypergeometric function

$$F(a,b,c;z) = 1 + \frac{ab}{c} + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \cdots,$$

to establish some interesting results.

Recently Sahoo and Sharma [18] (see also [17]) defined and studied the class $\mathcal{PK}_q$ of $q-$close-to-convex functions. In 1989, Srivastava [21] proposed the study of the class, $\mathcal{PC}_q$ of $q-$convex functions in $\mathcal{U}$, defined by

$$\left|\frac{z\mathcal{D}_{z,q}^2f(z)}{\mathcal{D}_{z,q}f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \quad (0<q<1, \ z\in\mathcal{U}). \tag{6}$$

Motivated by the work of Srivastava[21], we focus our article on certain analytic properties of the class $\mathcal{PC}_q$ and the Fekete-Szegö problems. For more studies on $q-$calculus, see [3], [4], [14] and their references.

Now, we state a new lemma and known lemmas that may require in the next section.

Lemma 1. *For $f\in\mathcal{A}_k$ and $0<q<1$, we have*

$$\mathcal{I}_{z,q}^2\left\{\frac{\mathcal{D}_{z,q}^2f(z)}{\mathcal{D}_{z,q}f(z)}\right\} = \frac{(1-q)^4}{\ln q}\frac{qz^3\sum_{k=0}^{\infty}Log(zq^k)q^k}{qf(z)-(q+1)f(qz)+f(q^2z)}.$$

Proof. Let $\mathcal{D}_{z,q}f(z) = \vartheta(z)$,

$$\mathcal{D}_{z,q}\left\{Log\left(\mathcal{D}_{z,q}f(z)\right)\right\} = \mathcal{D}_{z,q}Log\left[\nu(z)\right]\mathcal{D}^2_{z,q}f(z),$$

but

$$\mathcal{D}_{z,q}\left[Log\left[\nu(z)\right]\right] = \frac{Log[\nu(z)] - Log[q\nu(z)]}{(1-q)\nu(z)}.$$

Hence,

$$\mathcal{D}_{z,q}\left[Log\left(\mathcal{D}_{z,q}f(z)\right)\right] = \frac{\ln q}{(q-1)}\frac{\mathcal{D}^2_{z,q}f(z)}{\mathcal{D}_{z,q}f(z)}.$$

Since

$$\begin{aligned}\mathcal{I}_{z,q}\left[Log\left(\mathcal{D}_{z,q}f(z)\right)\right] &= \mathcal{I}_{z,q}\left[Log\left(\vartheta(z)\right)\right]\\ &= \frac{1}{\mathcal{D}_{z,q}\vartheta(z)}(1-q)z\sum_{k=o}^{\infty}Log(zq^k)q^k,\end{aligned}$$

hence

$$\mathcal{I}^2_{z,q}\left\{\frac{\mathcal{D}^2_{z,q}f(z)}{\mathcal{D}_{z,q}f(z)}\right\} = \frac{(1-q)^4}{\ln q}\frac{qz^3\sum_{k=0}^{\infty}Log(zq^k)q^k}{qf(z)-(q+1)f(qz)+f(q^2z)}.$$

Next, we define the following two sets, similar to the one defined in [8] and [2]:

We set

$$\mathcal{B}_q = \{\eta : \eta \in \mathcal{A}_k, \eta(0) = q \textit{ and } \eta : \mathcal{U} \to \mathcal{U}\} \tag{7}$$

and

$$\mathcal{B}^{\imath}_q = \left\{\vartheta : \vartheta \in \mathcal{B}_q \textit{ and } \vartheta(q^{\imath}z) \neq \vartheta(q^{\jmath}z), \quad (\imath = \jmath + 1,\ \imath \in \mathbb{N})\right\}. \tag{8}$$

■

Then, we give a slight modification of the results in [[8], Lemma 2.1 and Lemma 2.2] as the following lemmas:

Lemma 2. *If $\eta \in \mathcal{B}^{\imath}_q$, then*

$$\prod_{k=0}^{\infty}\left\{\frac{\eta(zq^k)}{q}\right\}$$

converges uniformly on compact subsets of $\mathcal{U}$.

Lemma 3. *If $\sigma \in \mathcal{B}_q^{\imath}$, then $\prod_{k=0}^{\infty}\left\{\frac{\sigma(zq^k)}{q}\right\}$ converges uniformly on compact subsets of $\mathcal{U}$ to a nonzero function in $\mathcal{A}_k$ with no zeros. Furthermore the function*

$$f(z) := \frac{z}{\prod_{k=0}^{\infty}\left\{\frac{\sigma(zq^k)}{q}\right\}}$$

belongs to $\mathcal{PC}_q$ and $\sigma(z) = \frac{f(q^2z)-f(qz)}{f(qz)-f(z)}$.

For the technique of the prove of Lemmas 2 and 3, see [8] or [2].

Lemma 4. *[5]A function $f(z)$ is in the class $\mathcal{PC}_q$ if and only if*

$$\left|\frac{f(q^2z)-f(qz)}{f(qz)-f(z)}\right| \leq q, \quad (f(qz) \neq f(z)\ z \in \mathcal{U}). \tag{9}$$

2. Analytic properties of the class $\mathcal{PC}_q$

In this section, we determine certain analytic properties of functions f in the class $\mathcal{PC}_q$.

We shall prove the second part of Theorem 2 by using the $q-$analogue of the results of [[15], page 168], see also [19].

Theorem 2. *Let $|z| \in \mathcal{U}$, with $0 < q < 1$, then class $\mathcal{PC}_q$ satisfies the inclusion relations*

$$\bigcap_{0<q<1} \mathcal{PC}_q = \mathcal{C}.$$

Proof. Assume $\omega = \frac{z\mathcal{D}^2_{z,q}f(z)}{\mathcal{D}f(z)}$, if $f \in \mathcal{PC}_q$, with $0 < q < 1$, then as $q \to 1^-$, the close disc

$$\left|\omega - \frac{1}{1-q}\right| \leq \frac{1}{1-q}$$

becomes the right-half plane, and $\mathcal{R}e\{1 + \frac{f''(z)}{f'(z)}\} \geq 0$, hence $f \in \mathcal{C}$. Thus

$$\mathcal{C} \supseteq \bigcap_{0<q<1} \mathcal{PC}_q.$$

Conversely, if $f \in \mathcal{C}$, by the hypothesis of Theorem 2, the analytic function $f(zq)$ is in $\mathcal{C}$, hence $f(zq))$ is bounded in $\mathcal{U}$, and

$$g(z) = \frac{f(zq) - f(zq^2)}{1 - \overline{f(zq^2)}f(zq)}, \quad 0 < q < 1$$

is also bounded in $\mathcal{U}$ and $g(z)$ vanishes for $q = 1$. The function

$$h(z) = \frac{g(z)}{\frac{zq-zq^2}{1-z\overline{z}q^2}} = \frac{(f(zq)-f(zq^2))(1-z\overline{z}q^2)}{(zq-zq^2)(1-\overline{f(zq^2)}f(zq))} \tag{10}$$
$$= \left\{\frac{\mathcal{D}_{z,q}f(z)}{(1-q)z} - \mathcal{D}^2_{z,q}f(z)\right\} \frac{(1-z\overline{z}q^2)}{(1-\overline{f(zq^2)})f(zq)}$$

is regular when $q = 1$ and also at all other points of $\mathcal{U}$. Furthermore, $h(z)$ is bounded in $\mathcal{U}$, and

$$\lim_{|z|\to 1} |g(z)| \leq 1, \quad \text{and} \quad \left|\frac{zq-zq^2}{1-z\overline{z}q^2}\right| = 1, \text{ for } |z| = 1,$$

hence by maximum principle, $|h(z)| \leq M$, $M > 0$ throughout $|z| < 1$.

Let $\omega = \frac{z\mathcal{D}^2_{z,q}f(z)}{\mathcal{D}_{z,q}f(z)}$, then from (10)

$$\left|\frac{\mathcal{D}_{z,q}f(z)}{z}\right| \left|\frac{1}{(1-q)} - w\right| \left(\frac{1-|z|^2}{1-|f(z)|^2}\right) \leq M.$$

By [[19], Lemma 6] $\left|\frac{\mathcal{D}_{z,q}f(z)}{z}\right| \left(\frac{1-|z|^2}{1-|f(z)|^2}\right) \leq 1$, then $\left|\frac{1}{1-q} - \omega\right| \leq M$. Let $M = \frac{1}{1-q}$, then

$$\bigcap_{0<q<1} \mathcal{PC}_q = \mathcal{C}.$$

■

Theorem 3. *The mapping*

$$\varrho : \mathcal{PC}_q \to \mathcal{B}^{\imath}_q$$

defined by

$$\varrho(f) = \frac{f(q^2z) - f(qz)}{f(qz) - f(z)} \quad (z \in \mathcal{U}), \tag{11}$$

is bijective.

Proof. By Lemmas 2 and 3, the mapping ϱ is onto. We need to show that ϱ is one-to-one. Let $\rho(z) = \frac{f_1(qz)-f_1(z)}{f_2(qz)-f_2(z)}$, assume

$$\frac{f_1(q^2z) - f_1(qz)}{f_1(qz) - f_1(z)} = \frac{f_2(q^2z) - f_2(qz)}{f_2(qz) - f_2(z)},$$

then

$$\rho(z) = \frac{f_1(qz) - f_1(z)}{f_2(qz) - f_2(z)} = \frac{f_1(q^2z) - f_1(qz)}{f_2(q^2z) - f_2(qz)} = \rho(qz).$$

It follows that the function $\rho(z)$ satisfies the recurrence relation

$$\rho(z) = \rho(zq^n), \quad (n \in \mathbb{N}, \ z \in \mathcal{U}).$$

Hence, $\rho(z)$ must be a constant, since $f_1'(z) = f_2'(z) = 1$, then the constant must be equal to 1. ■

3. Fekete-Szegö problem for a class of $q-$calculus

In this section, we calculate the maximum of the absolute value of the Fekete-Szegö functional of functions f in $\mathcal{PC}_q$.

Let $\mathcal{P}$ be the class of all analytic functions p given by $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ with $\mathcal{R}e\{p(z)\} > 0$, for $z \in \mathcal{U}$. For more study on Fekete-Szegö problems, see [12], [1], [6] and [7] to mention but a few.

Lemma 5 ([16])**.** *If* $\delta(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is in* $\mathcal{P}$, *then*
(i) $|c_k| \le 2$ *for* $k \ge 1$,
(ii) $|c_2 - \frac{1}{2}c_1^2| \le 2 - \frac{|c_1|^2}{2}$.

Lemma 6. *If* $f(z) = z + \sum_{k=1}^{\infty} a_k z^k$ *is in* $\mathcal{PC}_q$, *and* $0 < \lambda \le 1$, *then for some series* $\sum_{k=0}^{\infty} c_k z^k$, *with* $c_0 = 1$
(i) $|a_2| = \frac{2\lambda}{|q^2-1|}$
(ii) $|a_3| \le \begin{cases} \frac{8\lambda}{|q^3-1|}, & \lambda \le \frac{|q-1|}{q+1}; \\ \frac{8\lambda^2(q+1)}{|q-1||q^3-1|}, & \lambda > \frac{|q-1|}{q+1}. \end{cases}$

Proof. Let $f \in \mathcal{PC}_q$, then $t_f(z)$ define by

$$t_f(z) := \frac{1}{q}\frac{f(q^2z) - f(qz)}{f(qz) - f(z)} \tag{12}$$

is an analytic function in $\mathcal{U}$, and by Lemma 4, $|t_f(z)| < 1$ for $z \in \mathcal{U}$. Hence the function $t_f(z)$ can be represented as $t_f(z) = \left(1 + \sum_{k=1}^{\infty} c_k z^k\right)^{\lambda}$ and equation (12) can be rewritten as

$$q\left(1 + c_1 z + c_2 z^2 + \ldots\right)^{\lambda} \left\{(q-1)z + \sum_{k=2}^{\infty} a_k (q^k - 1) z^k\right\} = q(q-1)z + \sum_{k=2}^{\infty} a_k q^k (q^k - 1) z^k. \tag{13}$$

Set ${}_i\mathcal{C}_j = a_i(q^i - j)$, $i \in \mathbb{N} \cup \{0\}$, $j \in \mathbb{N}$, $a_1 = 1$, then equation (13) can be written as

$$(1 + \lambda c_1 z + [\lambda c_2 + \frac{\lambda(\lambda-1)c_1^2}{2!}]z^2 + \ldots) \times \left\{ ({}_1\mathcal{C}_0)({}_1\mathcal{C}_1)z + \sum_{k=2}^{\infty} ({}_1\mathcal{C}_0)({}_k\mathcal{C}_1)z^k \right\} = \left\{ ({}_1\mathcal{C}_0)({}_1\mathcal{C}_1)z + \sum_{k=2}^{\infty} (\frac{{}_k\mathcal{C}_0}{a_k})({}_k\mathcal{C}_1)z^k \right\}. \tag{14}$$

This implies

$$\begin{aligned} {}_1\mathcal{C}_0 \{ {}_1\mathcal{C}_1 z &+ [({}_2\mathcal{C}_1) + ({}_1\mathcal{C}_1)\lambda c_1]z^2 \\ &+ [({}_3\mathcal{C}_1) + \lambda c_1({}_2\mathcal{C}_1) + ({}_1\mathcal{C}_1)(\lambda c_2 + \frac{\lambda(\lambda-1)}{2}c_1^2)]z^3 + \ldots \} \\ &= ({}_1\mathcal{C}_0)({}_1\mathcal{C}_1)z + \frac{({}_2\mathcal{C}_0)}{a_2}({}_2\mathcal{C}_1)z^2 + \frac{({}_3\mathcal{C}_0)}{a_3}({}_3\mathcal{C}_1)z^3 + \ldots. \end{aligned} \tag{15}$$

Equating the coefficients of z^2 in equation (15), we get

$$a_2 = \frac{\lambda c_1}{(q^2-1)}, \tag{16}$$

and by Lemma 5 (i), we have [

$$|a_2| = \frac{2\lambda}{|q^2-1|}.$$

Equating the coefficients of z^3 in equation (15) we get

$$a_3 = \frac{\lambda}{(q^3-1)} \left\{ \left(c_2 - \frac{c_1^2}{2} \right) + \frac{\lambda c_1^2(q+1)}{2(q-1)} \right\}, \tag{17}$$

and by Lemma 5 (ii), we have

$$|a_3| \leq \frac{2\lambda}{|q^3-1|} \left\{ 4 - |c_1|^2 + \lambda |c_1^2| \frac{q+1}{|q-1|} \right\}.$$

This inequality proves the result. ∎

Theorem 4. *Let $f \in \mathcal{PC}_q$ and $\mu \in \mathbb{C}$, then*

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{\lambda}{(q^2-1)^2 |q^3-1|} \max \{1, \Theta(q, \lambda, \mu)\},$$

where $\Theta(q,\lambda,\mu) = \{2(q^2-1)^2 + \lambda q^3(q+2) + 2|\mu| - (\lambda[1 + 2q(|\mu|q^2+1)] + (q^2-1)^2)\}$.

Proof. From (16) and (17)

$$a_3 - \mu a_2^2 = \frac{\lambda}{(q^3-1)}\left\{\left(c_2 - \frac{c_1^2}{2}\right) + \lambda c_1^2 \frac{q^3(q+2) - [2q+1+2\mu(q^3-1)]}{2(q^2-1)^2}\right\}. \tag{18}$$

Hence

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\lambda}{|q^3-1|}\left\{\left|c_2 - \frac{c_1^2}{2}\right| + |c_1^2| \frac{\lambda q^3(q+2) + 2|\mu| - \lambda[1+2q(|\mu|q^2+1)]}{2(q^2-1)^2}\right\}.$$

By Lemma 5 (i), we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\lambda}{4|q^3-1|(q^2-1)^2} \times \left\{8(q^2-1)^2 + |c_1|^2\left[\lambda q^3(q+2) + 2|\mu| - \left(\lambda[1+2q(|\mu|q^2+1)] + (q^2-1)^2\right)\right]\right\}.$$

If

$$\lambda q^3(q+2) + 2|\mu| \leq \left(\lambda[1+2q(|\mu|q^2+1)] + (q^2-1)^2\right),$$

then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\lambda 2}{|q^3-1|}.$$

Furthermore, if

$$\lambda q^3(q+2) + 2|\mu| \geq \left(\lambda[1+2q(|\mu|q^2+1)] + (q^2-1)^2\right),$$

and by Lemma 5 (ii), we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\lambda}{(q^2-1)^2|q^3-1|}\left\{2(q^2-1)^2 + \lambda q^3(q+2) + 2|\mu| - \left(\lambda[1+2q(|\mu|q^2+1)] + (q^2-1)^2\right)\right\}.$$

■

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