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MAPPING THEOREMS ON SPACES WITH sn-NETWORK g-FUNCTIONS

Abstract. Let Δ be the sets of all topological spaces satisfying the following conditions.

(1) Each compact subset of X is metrizable;

(2) There exists an sn-network g-function g on X such that if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\{y_n\}$.

In this paper, we prove that if $X \in \Delta$, then each sequentially-quotient boundary-compact map on X is pseudo-sequence-covering; if $X \in \Delta$ and X has a point-countable sn-network, then each sequence-covering boundary-compact map on X is 1-sequence-covering. As the applications, we give that each sequentially-quotient boundary-compact map on g-metrizable spaces is pseudo-sequence-covering, and each sequence-covering boundary-compact on g-metrizable spaces is 1-sequence-covering.

Key words: sn-networks, sn-network g-functions, g-metrizable spaces, boundary-compact maps, sequentially-quotient maps, pseudo-sequence-covering maps, sequence-covering maps, 1-sequence-covering maps.

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1. Introduction and preliminaries

A study of images of topological spaces under certain sequence-covering maps is an important question in general topology. In 2001, S. Lin and P. Yan proved that each sequence-covering and compact map on metric spaces is 1-sequence-covering ([15]). Furthermore, S. Lin proved that each sequentially-quotient compact maps on metric spaces is pseudo-sequence-covering, and there exists a sequentially-quotient π-map on metric spaces is not pseudo-sequence-covering ([14]). In [1], T. V. An and L. Q. Tuyen proved that each sequence-covering π and s-map on metric spaces is 1-sequence-covering. After that, F. C. Lin and S. Lin proved that each sequence-covering and boundary-compact map on metric spaces is 1-sequence-covering ([10]). Recently, the authors proved that if X is an open image of metric spaces, then each sequentially-quotient boundary-compact map on X is pseudo-sequence-covering ([11]).

Let Δ be the sets of all topological spaces satisfying the following conditions.

(1) Each compact subset of X is metrizable;

(2) There exists an sn-network g-function g on X such that if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\{y_n\}$.

In this paper, we prove that if $X \in \Delta$, then each sequentially-quotient boundary-compact map on X is pseudo-sequence-covering; if $X \in \Delta$ and X has a point-countable sn-network, then each sequence-covering boundary-compact map on X is 1-sequence-covering. As the applications, we give that each sequentially-quotient boundary-compact map on g-metrizable spaces is pseudo-sequence-covering, and each sequence-covering boundary-compact on g-metrizable spaces is 1-sequence-covering.

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ be a collection of subsets of X, we denote $\bigcup \mathcal{P} = \bigcup \{P : P \in \mathcal{P}\}$.

Definition 1. *Let X be a space, $\{x_n\} \subset X$ and $P \subset X$.*

(1) *$\{x_n\}$ is called* eventually *in P, if $\{x_n\}$ converges to x, and there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$.*

(2) *$\{x_n\}$ is called* frequently *in P, if some subsequence of $\{x_n\}$ is eventually in P.*

(3) *P is called a* sequential neighborhood *of x in X [5], if whenever $\{x_n\}$ is a sequence converging to x in X, then $\{x_n\}$ is eventually in P.*

Definition 2. *Let $\mathcal{P}$ be a collection of subsets of X.*

(1) *$\mathcal{P}$ is* point-countable*, if each point $x \in X$ belongs to only countably many members of $\mathcal{P}$.*

(2) *$\mathcal{P}$ is* locally finite*, if for each $x \in X$, there exists a neighborhood V of x such that V meets only finite many members of $\mathcal{P}$.*

(3) *$\mathcal{P}$ is* σ-locally finite*, if $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is locally finite.*

(4) *$\mathcal{P}$ is a* network at x in X*, if $x \in P$ for every $P \in \mathcal{P}$, and whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.*

(5) *$\mathcal{P}$ is a* cs-cover *[19], if every convergent sequence is eventually in some $P \in \mathcal{P}$.*

Definition 3. *Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$.*

(1) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ *is a* σ-strong network *for* X *[8], if* $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ *is a network at each point* $x \in X$.

(2) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ *is a* σ-locally finite strong network consisting of cs-covers *for* X*, if it is a* σ*-strong network and each* $\mathcal{P}_n$ *is a locally finite* cs*-cover.*

Definition 4. *Let* $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ *be a cover of a space* X*. Assume that* $\mathcal{P}$ *satisfies the following (a) and (b) for every* $x \in X$*.*

(a) $\mathcal{P}_x$ *is a network at* x*.*

(b) *If* P_1*,* $P_2 \in \mathcal{P}_x$*, then there exists* $P \in \mathcal{P}_x$ *such that* $P \subset P_1 \cap P_2$*.*

(1) $\mathcal{P}$ *is a* weak base *of* X *[2], if for* $G \subset X$*,* G *is open in* X *if and only if for every* $x \in G$*, there exists* $P \in \mathcal{P}_x$ *such that* $P \subset G$*;* $\mathcal{P}_x$ *is said to be a* weak neighborhood base at x *in* X*.*

(2) $\mathcal{P}$ *is an* sn-network *for* X *[12], if each element of* $\mathcal{P}_x$ *is a sequential neighborhood of* x *for all* $x \in X$*;* $\mathcal{P}_x$ *is said to be an* sn-network at x *in* X*.*

Definition 5. *Let* X *be a space. Then,*

(1) X *is* gf-countable *[2] (resp.,* snf-countable *[7]), if* X *has a weak base (resp., sn-network)* $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ *such that each* $\mathcal{P}_x$ *is countable.*

(2) X *is* g-metrizable *[17], if* X *is regular and has a* σ*-locally finite weak base.*

(3) X *is* sequential *[5], if whenever* A *is a non closed subset of* X*, then there is a sequence in* A *converging to a point not in* A*.*

(4) X *is* strongly g-developable *[18], if* X *is sequential has a* σ*-locally finite strong network consisting of* cs*-covers.*

Remark 1. (1) Each strongly g-developable space is g-metrizable.

(2) A space X is gf-countable if and only if it is sequential and snf-countable.

Definition 6. *Let* $f : X \longrightarrow Y$ *be a map.*

(1) f *is a* compact *map [4], if each* $f^{-1}(y)$ *is compact in* X*.*

(2) f *is a* boundary-compact *map [4], if each* $\partial f^{-1}(y)$ *is compact in* X*.*

(3) f *is a* pseudo-sequence-covering *map [8], if for each convergent sequence* L *in* Y*, there is a compact subset* K *in* X *such that* $f(K) = \mathtt{cl}(L)$*.*

(4) f *is a* sequentially-quotient *map [3], if whenever* $\{y_n\}$ *is a convergent sequence in* Y*, there is a convergent sequence* $\{x_k\}$ *in* X *with each* $x_k \in f^{-1}(y_{n_k})$*.*

(5) f *is a* weak-open *map [21], if there exists a weak base* $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ *for* Y*, and for* $y \in Y$*, there exists* $x_y \in f^{-1}(y)$ *such that for each open neighborhood* U *of* x_y*,* $P_y \subset f(U)$ *for some* $P_y \in \mathcal{P}_y$*.*

(6) f *is an* 1-sequence-covering *map [12], if for each* $y \in Y$*, there is* $x_y \in f^{-1}(y)$ *such that whenever* $\{y_n\}$ *is a sequence converging to* y *in*

Y, there is a sequence $\{x_n\}$ converging to x_y in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.

(7) *f is a* sequence-covering *map [17], if every convergent sequence of Y is the image of some convergent sequence of X.*

Remark 2. (1) Each compact map is a compact-boundary map.

(2) Each 1-sequence-covering map is a sequence-covering map.

Definition 7 ([6])**.** *A function $g : \mathbb{N} \times X \longrightarrow \mathcal{P}(X)$ is called an* weak base g-function *on X, if it satisfies the following conditions.*

(1) *$x \in g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$.*

(2) *$g(n + 1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$.*

(3) *$\{g(n, x) : n \in \mathbb{N}\}$ is a weak neighborhood base at x for all $x \in X$.*

Note that a weak base g-functions were called CWC*-maps and* CWBC*-maps in [9] and [16], respectively.*

Definition 8. *A function $g : \mathbb{N} \times X \longrightarrow \mathcal{P}(X)$ is called an* sn-network g-function *on X, if it satisfies the following conditions.*

(1) *$x \in g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$.*

(2) *$g(n + 1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$.*

(3) *$\{g(n, x) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$.*

2. Main results

Let Δ be the sets of all topological spaces satisfying the following conditions.

(1) Each compact subset of X is metrizable;

(2) There exists an sn-network g-function g on X such that if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\{y_n\}$.

Theorem 1. *Let $f : X \longrightarrow Y$ be a boundary-compact map. If $X \in \Delta$, then f is a sequentially-quotient map if and only if it is a pseudo-sequence-covering map.*

Proof. Necessity. Let f be a sequentially-quotient map and $\{y_n\}$ be a non-trivial sequence converging to y in Y. Since $X \in \Delta$, there exists an sn-network g-function g on X satisfying that if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\{y_n\}$. For $n \in \mathbb{N}$, let

$$U_{y,n} = \bigcup\{g(n, x) : x \in \partial f^{-1}(y)\} \quad \text{and} \quad P_{y,n} = f(U_{y,n}).$$

It is obvious that $\{P_{y,n} : n \in \mathbb{N}\}$ is a decreasing sequence in X. Furthermore, $P_{n,y}$ is a sequential neighborhood of y in Y for all $n \in \mathbb{N}$. If not, there exists $n \in \mathbb{N}$ such that $P_{y,n}$ is not a sequential neighborhood of y in Y. Thus, there exists a sequence L converges to y in Y such that $L \cap P_{y,n} = \emptyset$. Since f is sequentially-quotient, there exists a sequence S converges to $x \in \partial f^{-1}(y)$ such that $f(S)$ is a subsequence of L. On the other hand, because $g(n,x)$ is a sequential neighborhood of x in X, S is eventually in $g(n,x)$. Thus, S is eventually in $U_{y,n}$. Therefore, L is frequently in $P_{y,n}$. This contradicts to $L \cap P_{y,n} = \emptyset$.

Then for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $y_i \in P_{y,n}$ for all $i \geq i_n$. So $f^{-1}(y_i) \cap U_{y,n} \neq \emptyset$. We can suppose that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, we take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if} \quad j < i_1, \\ f^{-1}(y_j) \cap U_{y,n}, & \text{if} \quad i_n \leq j < i_{n+1}. \end{cases}$$

Let $K = \partial f^{-1}(y) \cup \{x_j : j \in \mathbb{N}\}$. Clearly, $f(K) = \{y\} \cup \{y_n : n \in \mathbb{N}\}$. Furthermore, K is a compact subset in X. In fact, let $\mathcal{U}$ be an open cover for K in X. Since $\partial f^{-1}(y)$ is a compact subset in X, there exists a finite subfamily $\mathcal{H} \subset \mathcal{U}$ such that $\partial f^{-1}(y) \subset \bigcup \mathcal{H}$. Then there exists $m \in \mathbb{N}$ such that $U_{n,y} \subset \bigcup \mathcal{H}$ for all $n \geq m$. If not, for each $n \in \mathbb{N}$, there exists $v_n \in U_{y,n} - \bigcup \mathcal{H}$. It implies that $v_n \in g(n, u_n) - \bigcup \mathcal{H}$ for some $u_n \in \partial f^{-1}(y)$. Since $\{u_n\} \subset \partial f^{-1}(y)$ and each compact subset of X is metrizable, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to x \in \partial f^{-1}(y)$. Now, for each $i \in \mathbb{N}$, we put

$$a_i = \begin{cases} u_{n_1}, & \text{if} \quad i \leq n_1 \\ u_{n_{k+1}}, & \text{if} \quad n_k < i \leq n_{k+1}; \end{cases}$$

$$b_i = \begin{cases} v_{n_1}, & \text{if} \quad i \leq n_1 \\ v_{n_{k+1}}, & \text{if} \quad n_k < i \leq n_{k+1}. \end{cases}$$

Then $a_i \to x$. Because $g(n+1, x) \subset g(n,x)$ for all $x \in X$ and $n \in \mathbb{N}$, it implies that $b_i \in g(i, a_i)$ for all $i \in \mathbb{N}$. By property of g, it implies that x is a cluster point of $\{b_i\}$. Thus, x is a cluster point of $\{v_{n_k}\}$. This contradicts to $\bigcup \mathcal{H}$ is a neighborhood of x and $v_{n_k} \notin \bigcup \mathcal{H}$ for all $k \in \mathbb{N}$.

Because $P_{y,i+1} \subset P_{y,i}$ for all $i \in \mathbb{N}$, it implies that $\partial f^{-1}(y) \cup \{x_i : i \geq m\} \subset \bigcup \mathcal{H}$. For each $i < m$, take $V_i \in \mathcal{U}$ such that $x_i \in V_i$. Put $\mathcal{V} = \mathcal{U} \cup \{V_i : i < m\}$. Then $\mathcal{V} \subset \mathcal{U}$ and $K \subset \bigcup \mathcal{V}$. Therefore, K is compact in X, and f is pseudo-sequence-covering.

Sufficiency. Suppose that f is a pseudo-sequence-covering map. If $\{y_n\}$ is a convergent sequence in Y, then there is a compact subset K in X such that $f(K) = \mathtt{cl}(\{y_n\})$. For each $n \in \mathbb{N}$, take a point $x_n \in f^{-1}(y_n) \cap K$. Since K is compact and metrizable, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, and $\{f(x_{n_k})\}$ is a subsequence of $\{y_n\}$. Therefore, f is sequentially-quotient. ■

By Theorem 2.6 [20] and Theorem 1, we have

Corollary 1. *Let $f : X \longrightarrow Y$ be a boundary-compact map. If X is g-metrizable or strongly g-developable, then f is a sequentially-quotient map if and onlyif it is a pseudo-sequence-covering map.*

Corollary 2. *Let $f : X \longrightarrow Y$ be a compact map. If X is g-metrizable or strongly g-developable, then f is a sequentially-quotient map if and only if it is a pseudo-sequence-covering map.*

Lemma 1. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a point-countable sn-network for X, and K be a compact metrizable subset of X. If $x \in K$, then $x \in \mathtt{Int}_K(P \cap K)$ for all $P \in \mathcal{P}_x$.*

Proof. Let $P \in \mathcal{P}_x$ and $\{V_n : n \in \mathbb{N}\}$ be a local base at the point x in K. Then $x \in V_n \subset P \cap K$ for some $n \in \mathbb{N}$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in V_n - (P \cap K)$. It implies that the sequence $\{x_n\}$ converges to x in X. Since P is a sequential neighborhood of x in X, $\{x_n\}$ is eventually in P. This contradicts to $x_n \notin P$ for all $n \in \mathbb{N}$.

Therefore, $V_n \subset P \cap K$ for some $n \in \mathbb{N}$, and $x \in \mathtt{Int}_K(P \cap K)$. ∎

Lemma 2. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a point-countable sn-network for X. If K is a compact metrizable subset of X, then $\bigcup\{\mathcal{P}_x : x \in K\}$ is countable.*

Proof. Let $D \subset K$ be a countable subset of K such that $K = \mathtt{cl}_K(D)$, and $P \in \bigcup\{\mathcal{P}_x : x \in K\}$. Then $P \in \mathcal{P}_x$ for some $x \in K$. By Lemma 1, $x \in \mathtt{Int}_K(P \cap K)$. Therefore, $D \cap \mathtt{Int}_K(P \cap K) \neq \emptyset$, it implies that $P \cap D \neq \emptyset$. This follows that

$$\bigcup\{\mathcal{P}_x : x \in K\} \subset \{P \in \mathcal{P} : P \cap D \neq \emptyset\}.$$

Finally, since $\mathcal{P}$ is point-countable and D is countable, it implies that $\bigcup\{\mathcal{P}_x : x \in K\}$ is countable. ∎

Theorem 2. *Let $f : X \longrightarrow Y$ be a boundary-compact map and $X \in \Delta$. If X has a point-countable sn-network, then f is a sequence-covering map if and only if it is a 1-sequence-covering map.*

Proof. Necessity. Let $f : X \longrightarrow Y$ be a sequence-covering boundary-compact map, and $X \in \Delta$. Firstly, we prove that Y is snf-countable. In fact, since $X \in \Delta$, there exists an sn-network g-function g on X such that if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\{y_n\}$. For each $y \in Y$ and $n \in \mathbb{N}$, we put

$$P_{y,n} = f\Big(\bigcup\{g(n,x) : x \in \partial f^{-1}(y)\}\Big), \quad \text{and} \quad \mathcal{P}_y = \{P_{y,n} : n \in \mathbb{N}\}.$$

Then each $\mathcal{P}_y$ is countable and $P_{y,n+1} \subset P_{y,n}$ for all $y \in Y$ and $n \in \mathbb{N}$. Furthermore, we have

(1) $\mathcal{P}_y$ is a network at y. Let $y \in U$ with U open in Y. Then there exists $n \in \mathbb{N}$ such that

$$\bigcup\{g(n,x) : x \in \partial f^{-1}(y)\} \subset f^{-1}(U).$$

If not, for each $n \in \mathbb{N}$, there exist $x_n \in \partial f^{-1}(y)$ and $z_n \in X$ such that $z_n \in g(n, x_n) - f^{-1}(U)$. Since $X \in \Delta$, it follows that each compact subset of X is metrizable. On the other hand, since $\{x_n\} \subset \partial f^{-1}(y)$ and f is a boundary-compact map, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in \partial f^{-1}(y)$. Now, for each $i \in \mathbb{N}$, we put

$$a_i = \begin{cases} x_{n_1}, & \text{if } \ i \leq n_1 \\ x_{n_{k+1}}, & \text{if } \ n_k < i \leq n_{k+1}; \end{cases}$$

$$b_i = \begin{cases} z_{n_1}, & \text{if } \ i \leq n_1 \\ z_{n_{k+1}}, & \text{if } \ n_k < i \leq n_{k+1}. \end{cases}$$

Then $a_i \to x$. Because $g(n+1, x) \subset g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$, it implies that $b_i \in g(i, a_i)$ for all $i \in \mathbb{N}$. By the property of g, it implies that x is a cluster point of $\{b_i\}$. Thus, x is a cluster point of $\{z_{n_k}\}$. This contradicts to $f^{-1}(U)$ is a neighborhood of x and $z_{n_k} \notin f^{-1}(U)$ for all $k \in \mathbb{N}$.

Therefore, $P_{y,n} \subset U$, and $\mathcal{P}_y$ is a network at y.

(2) Let $P_{y,m}$, $P_{y,n} \in \mathcal{P}_y$. If we take $k = \max\{m, n\}$, then $P_{y,k} \subset P_{y,m} \cap P_{y,n}$.

(3) Each element of $\mathcal{P}_y$ is a sequential neighborhood of y. Let $P_{y,n} \in \mathcal{P}_y$ and L be a sequence converging to y in Y. Since f is sequence-covering, L is an image of some sequence S converges to $x \in \partial f^{-1}(y)$. On the other hand, since $g(n,x)$ is a sequential neighborhood of x, S is eventually in $g(n,x)$. This implies that L is eventually in $P_{y,n}$. Therefore, $P_{y,n}$ is a sequential neighborhood of y.

Therefore, $\bigcup\{\mathcal{P}_y : y \in Y\}$ is an sn-network for X, and Y is an snf-countable space.

Next, let $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ be a point-countable sn-network for X. We prove that each non-isolated point $y \in Y$, there exists $x_y \in \partial f^{-1}(y)$ such that for each $B \in \mathcal{B}_{x_y}$, there exists $P \in \mathcal{P}_y$ satisfying $P \subset f(B)$. Otherwise, there exists a non-isolated point $y \in Y$ so that for each $x \in \partial f^{-1}(y)$, there exists $B_x \in \mathcal{B}_x$ such that $P \not\subset f(B_x)$ for all $P \in \mathcal{P}_y$. Since $\mathcal{P}_y$ is an sn-network at y, we can choose a decreasing countable network $\{P_{y,n} : n \in \mathbb{N}\} \subset \mathcal{P}_y$ at y. Furthermore, since $X \in \Delta$, f is a boundary-compact map and $\mathcal{B}$ is a point-countable sn-network for X, it follows from Lemma 2 that

$\bigcup\{\mathcal{B}_x : x \in \partial f^{-1}(y)\}$ is countable. Thus, $\{B_x : x \in \partial f^{-1}(y)\}$ is countable. Assume that

$$\{B_x : x \in \partial f^{-1}(y)\} = \{B_m : m \in \mathbb{N}\}.$$

Hence, for each $m, n \in \mathbb{N}$, there exists $x_{n,m} \in P_{y,n} - f(B_m)$. For $n \geq m$, we denote $y_k = x_{n,m}$ with $k = m + n(n-1)/2$. Since $\{P_{y,n} : n \in \mathbb{N}\}$ is a decreasing network at y, $\{y_k\}$ is a sequence converging to y in Y. On the other hand, because f is a sequence-covering map, $\{y_k\}$ is an image of some sequence $\{x_n\}$ converging to $x \in \partial f^{-1}(y)$ in X. Furthermore, since $B_x \in \{B_m : m \in \mathbb{N}\}$, there exists $m_0 \in \mathbb{N}$ such that $B_x = B_{m_0}$. Because B_{m_0} is a sequential neighborhood of x, $\{x\} \cup \{x_k : k \geq k_0\} \subset B_{m_0}$ for some $k_0 \in \mathbb{N}$. Thus, $\{y\} \cup \{y_k : k \geq k_0\} \subset f(B_{m_0})$. But if we take $k \geq k_0$, then there exists $n \geq m_0$ such that $y_k = x_{n,m_0}$, and it implies that $x_{n,m_0} \in f(B_{m_0})$. This contradicts to $x_{n,m_0} \in P_{y,n} - f(B_{m_0})$.

We now prove that f is an 1-sequence-covering map. Suppose $y \in Y$, by the above proof there is $x_y \in \partial f^{-1}(y)$ such that whenever $B \in \mathcal{B}_{x_y}$, there exists $P \in \mathcal{P}_y$ satisfying $P \subset f(B)$. Let $\{y_n\}$ be an any sequence in Y, which converges to y. Since $\mathcal{B}_{x_y}$ is an sn-network at x_y, we can choose a decreasing countable network $\{B_{y,n} : n \in \mathbb{N}\} \subset \mathcal{B}_{x_y}$ at x_y. We choose a sequence $\{z_n\}$ in X as follows.

Since $B_{y,n} \in \mathcal{B}_{x_y}$, by the above argument, there exists $P_{y,k_n} \in \mathcal{P}_y$ satisfying $P_{y,k_n} \subset f(B_{y,n})$ for all $n \in \mathbb{N}$. On the other hand, since each element of $\mathcal{P}_y$ is a sequential neighborhood of y, it follows that for each $n \in \mathbb{N}$, $f(B_{y,n})$ is a sequential neighborhood of y in Y. Hence, for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $y_i \in f(B_{y,n})$ for every $i \geq i_n$. Assume that $1 < i_n < i_{n+1}$ for each $n \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, we take

$$z_j = \begin{cases} z_j \in f^{-1}(y_j), & \text{if} \quad j < i_1 \\ z_{j,n} \in f^{-1}(y_j) \cap B_{y,n}, & \text{if} \quad i_n \leq j < i_{n+1}. \end{cases}$$

If we put $S = \{z_j : j \geq 1\}$, then S converges to x_y in X, and $f(S) = \{y_n\}$. Therefore, f is 1-sequence-covering.

Sufficiency. By Remark 2. ■

Corollary 3. *Let $f : X \longrightarrow Y$ be a boundary-compact map and $X \in \Delta$. If X has a point-countable weak base, then f is a sequence-covering map if and only if it is a 1-sequence-covering map.*

Corollary 4. *Let $f : X \longrightarrow Y$ be a boundary-compact map. If X is g-metrizable or strongly g-developable, then f is a sequence-covering map if and only if it is a 1-sequence-covering map.*

Corollary 5. *Let $f : X \longrightarrow Y$ be a boundary-compact map. If X is g-metrizable or strongly g-developable, then f is a sequence-covering quotient map if and only if it is a weak-open map.*

Example 1. Let Ω be the sets of all topological spaces such that, for each compact subset $K \subset X \in \Omega$, K is metrizable and also has a countably neighborhood base in X (see [11]). Put $X = \mathbb{N} \cup \{p\}$ with $p \in \beta\mathbb{N} - \mathbb{N}$. Then X is a subspace of $\beta\mathbb{N}$ and $X \in \Delta - \Omega$. In fact, by Remark 1.5 [13], each compact subset of X is metrizable but it is not sequential. Thus, $X \notin \Omega$. Furthermore, for each $n \in \mathbb{N}$ and $x \in X$, if we put $g(n,x) = \{x\}$, then $g : \mathbb{N} \times X \longrightarrow \mathcal{P}(X)$ is an sn-network g-function on X such that if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\{y_n\}$. Therefore, $X \in \Delta - \Omega$.

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