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## ON OSTROWSKI TYPE INEQUALITIES

**ABSTRACT.** In this paper, new forms of Ostrowski type inequalities are established for a general class of fractional integral operators. The main results are used to derive Ostrowski type inequalities involving the familiar Riemann-Liouville fractional integral operators and other important integral operators. We further obtain similar types of inequalities for the integral operators whose kernels are the Fox-Wright generalized hypergeometric function. Several consequences and special cases of some of the results including applications to Stolarsky's means are also pointed out.

**KEY WORDS:** Ostrowski's inequality,  $s$ -convex function, Riemann-Liouville fractional integral operator, Fox-Wright function, Stolarsky's means.

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### 1. Introduction

In 1938, Ostrowski [10] proved the following integral inequality.

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$ . If  $|f'(x)| \leq M$  for all  $x \in [a, b]$ , then the following inequality holds:*

$$(1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right],$$

for all  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible in the sense that it cannot be replaced by a smaller constant.

After Ostrowski's original paper, first papers with applications in numerical integration were appeared about 40 years later (see [8] and [9]), but much later an explosion on this subject has happened. In recent years, many generalizations, improvements and variants of the Ostrowski inequality have appeared in the literature (see, for example [1], [2], [4] and [19]).

In the present investigation, we focus on some new variants of the Ostrowski inequality (1), that is, the Ostrowski type inequalities for the fractional integral operators. The Ostrowski type inequalities for the Riemann-Liouville fractional integrals have been considered by many authors and for such related results, we refer the reader to the recent works in [12], [13] and [14]. The inequalities involving more general fractional integral operators have also been considered in [20]. Since work in this direction has gained much attention, we attempt to establish a general formulation in this article such that the essential facts covered by different fractional integrals become more clear and the implications yield certain new inequalities.

We first give here some definitions and properties of a class of new fractional integral operators which will serve as the fundamental tool for our investigation. In [11], Raina studied a class of functions defined formally by

$$(2) \quad \mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\dots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbf{R}),$$

where the coefficients  $\sigma(k)$  ( $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ) is a bounded sequence of positive real numbers and  $\mathbf{R}$  is the set of real numbers. With the help of (2), Raina defined the following left-sided fractional integral operator:

$$(3) \quad (\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}\varphi)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(x-t)^{\rho}] \varphi(t) dt \quad (x > a),$$

where  $\lambda, \rho > 0$ ,  $\omega \in \mathbb{R}$ , and  $\varphi(t)$  is such that the integral on the right side exists. In this paper, we define the correspondingly right-sided fractional integral operator by

$$(4) \quad (\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}\varphi)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-x)^{\rho}] \varphi(t) dt \quad (x < b),$$

where  $\lambda, \rho > 0$ ,  $\omega \in \mathbb{R}$ , and  $\varphi(t)$  is such that the integral on the right side exists.

It is easy to verify that  $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}$  and  $\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}$  are bounded integral operators on  $L(a, b)$ , if

$$(5) \quad \mathfrak{M} := \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(b-a)^{\rho}] < \infty.$$

In fact, for  $\varphi \in L(a, b)$ , we have

$$(6) \quad \|\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}\varphi\|_1 \leq \mathfrak{M}(b-a)^{\lambda} \|\varphi\|_1$$

and

$$(7) \quad \|\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}\varphi\|_1 \leq \mathfrak{M}(b-a)^{\lambda} \|\varphi\|_1,$$

where

$$\|\varphi\|_p := \left( \int_a^b |\varphi(t)|^p dt \right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient  $\sigma(k)$ . Here, we just point out that the classical Riemann-Liouville fractional integrals  $I_{a+}^\alpha$  and  $I_{b-}^\alpha$  of order  $\alpha$  defined by (see, [7, p. 69])

$$(8) \quad (I_{a+}^\alpha \varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t) dt \quad (x > a; \alpha > 0)$$

and

$$(9) \quad (I_{b-}^\alpha \varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \varphi(t) dt \quad (x < b; \alpha > 0)$$

follow easily by setting

$$(10) \quad \lambda = \alpha, \quad \sigma(0) = 1, \quad \text{and} \quad \omega = 0$$

in (3) and (4), and the boundedness of (8) and (9) on  $L(a, b)$  is also inherited from (6) and (7). Other useful fractional integral operators and their related Ostrowski type inequalities will be considered in Section 3.

## 2. Osrowski type inequalities for $\mathcal{J}_{\rho, \lambda, a+; \omega}^\sigma$ and $\mathcal{J}_{\rho, \lambda, b-; \omega}^\sigma$

In order to prove the main theorems, we need the following lemma.

**Lemma 1.** *Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$  such that  $|\varphi'(x)| \leq M$  for every  $x \in [a, b]$  and  $\lambda > 0$ . Then*

$$(11) \quad \begin{aligned} & [\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)] \varphi(x) \\ & \quad - [(\mathcal{J}_{\rho, \lambda, x+; \omega}^\sigma \varphi)(b) + (\mathcal{J}_{\rho, \lambda, x-; \omega}^\sigma \varphi)(a)] \\ & = (\mathcal{J}_{\rho, \lambda+1, x-; \omega}^\sigma \varphi')(a) - (\mathcal{J}_{\rho, \lambda+1, x+; \omega}^\sigma \varphi')(b) \end{aligned}$$

and

$$(12) \quad \begin{aligned} & [\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)] \varphi(x) \\ & \quad - [(\mathcal{J}_{\rho, \lambda, x+; \omega}^\sigma \varphi)(b) + (\mathcal{J}_{\rho, \lambda, x-; \omega}^\sigma \varphi)(a)] \\ & = (x-a)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] \varphi'(tx + (1-t)a) dt \\ & \quad - (b-x)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(b-x)^\rho t^\rho] \varphi'(tx + (1-t)b) dt, \end{aligned}$$

where, and in what follows, we define

$$(13) \quad \mathcal{K}_\lambda^\sigma(z-y) := (z-y)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(z-y)^\rho].$$

**Proof.** It is fairly easy to verify that

$$-\frac{d}{dt} \left\{ (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-t)^\rho] \right\} = (b-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [\omega (b-t)^\rho].$$

Consequently, by integration by parts, we have

$$\begin{aligned} (14) \quad (\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma \varphi)(b) &= \int_x^b (b-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [\omega (b-t)^\rho] \varphi(t) dt \\ &= - \int_x^b \varphi(t) d \left\{ (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-t)^\rho] \right\} \\ &= - \varphi(t) (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-t)^\rho] \Big|_x^b \\ &\quad + \int_x^b (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-t)^\rho] \varphi'(t) dt \\ &= \varphi(x) (b-x)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-x)^\rho] \\ &\quad + \int_x^b (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-t)^\rho] \varphi'(t) dt \\ &= \varphi(x) (b-x)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (b-x)^\rho] \\ &\quad + (\mathcal{J}_{\rho,\lambda+1,x+;\omega}^\sigma \varphi')(b). \end{aligned}$$

Similarly, using again integration by parts, and noting that

$$\frac{d}{dt} \left\{ (t-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (t-a)^\rho] \right\} = (t-a)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [\omega (t-a)^\rho],$$

we find that

$$\begin{aligned} (15) \quad (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma \varphi)(a) &= \int_a^x (t-a)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [\omega (t-a)^\rho] \varphi(t) dt \\ &= \int_a^x \varphi(t) d \left\{ (t-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (t-a)^\rho] \right\} \\ &= \varphi(x) (x-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (x-a)^\rho] \\ &\quad - \int_a^x (t-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (t-a)^\rho] \varphi'(t) dt \\ &= \varphi(x) (x-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega (x-a)^\rho] \\ &\quad - (\mathcal{J}_{\rho,\lambda+1,x-;\omega}^\sigma \varphi')(a). \end{aligned}$$

From (14) and (15), we obtain (11) immediately.

The proof of (12) is analogous to that of (11). By applying integrating by parts, we have

$$\begin{aligned}
(16) \quad & \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] \varphi'(tx + (1-t)a) dt \\
&= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] d \left\{ \frac{\varphi(tx + (1-t)a)}{x-a} \right\} \\
&= t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] \frac{\varphi(tx + (1-t)a)}{x-a} \Big|_0^1 \\
&\quad - \frac{1}{x-a} \int_0^1 \varphi(tx + (1-t)a) t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(x-a)^\rho t^\rho] dt \\
&= \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho] \frac{\varphi(x)}{x-a} \\
&\quad - \frac{1}{x-a} \int_a^x \frac{(u-a)^{\lambda-1}}{(x-a)^{\lambda-1}} \varphi(u) \mathcal{F}_{\rho, \lambda}^\sigma [\omega(u-a)^\rho] \frac{du}{x-a} \\
&= \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho] \frac{\varphi(x)}{x-a} \\
&\quad - \frac{1}{(x-a)^{\lambda+1}} \int_a^x (u-a)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(u-a)^\rho] \varphi(u) du \\
&= \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho] \frac{\varphi(x)}{x-a} - \frac{1}{(x-a)^{\lambda+1}} (\mathcal{J}_{\rho, \lambda, x-; \omega}^\sigma \varphi)(a),
\end{aligned}$$

which (after rearrangement) gives the following identity:

$$\begin{aligned}
(17) \quad & (x-a)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] \varphi'(tx + (1-t)a) dt \\
&= (x-a)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(x-a)^\rho] \varphi(x) - (\mathcal{J}_{\rho, \lambda, x-; \omega}^\sigma \varphi)(a).
\end{aligned}$$

The same above evaluation of (17) in similar manner also gives

$$\begin{aligned}
(18) \quad & (b-x)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(b-x)^\rho t^\rho] \varphi'(tx + (1-t)a) dt \\
&= -(b-x)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega(b-x)^\rho] \varphi(x) + (\mathcal{J}_{\rho, \lambda, x+; \omega}^\sigma \varphi)(b).
\end{aligned}$$

Now adding (17) and (18) and using the notation (13), we obtain the desired result. This completes the proof.  $\blacksquare$

We also need the following convexity concept (see [1] and [14]).

**Definition 1.** A function  $f : [0, \infty) \rightarrow \mathbb{R}$  is said to be  $s$ -convex in the second sense if

$$(19) \quad f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all  $x, y \in [0, \infty)$ ,  $\lambda \in [0, 1]$  and for some fixed  $s \in (0, 1]$ .

Evidently, the  $s$ -convexity reduces to the ordinary convexity for  $s = 1$ .

Using Lemma 1 and the  $s$ -convexity, we obtain the following Ostrowski type fractional integral inequalities.

**Theorem 2.** *Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$  such that  $|\varphi'| \leq M$ , then for all  $x \in [a, b]$  and  $\lambda > 0$ :*

$$(20) \quad \begin{aligned} & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\ & \quad - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]| \\ & \leq M [\mathcal{K}_{\lambda+1}^\sigma(x-a) + \mathcal{K}_{\lambda+1}^\sigma(b-x)]. \end{aligned}$$

If, in addition,  $|\varphi'|$  is  $s$ -convex in the second sense on  $[a, b]$  for some  $s \in (0, 1]$ , then the following inequality for fractional integral operators with  $\lambda > 0$  holds:

$$(21) \quad \begin{aligned} & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\ & \quad - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]| \\ & \leq |\varphi'(x)| \left[ \frac{\mathcal{K}_{\lambda+s+1}^{\sigma_1}(x-a)}{(x-a)^s} + \frac{\mathcal{K}_{\lambda+s+1}^{\sigma_1}(b-x)}{(b-x)^s} \right] \\ & \quad + \Gamma(s+1) \left[ |\varphi'(a)| \frac{\mathcal{K}_{\lambda+s+1}^\sigma(x-a)}{(x-a)^s} + |\varphi'(b)| \frac{\mathcal{K}_{\lambda+s+1}^\sigma(b-x)}{(b-x)^s} \right], \end{aligned}$$

where  $\sigma_1$  is given by

$$(22) \quad \sigma_1 \equiv \sigma_1(k) := \sigma(k) \frac{\Gamma(\rho k + \lambda + s + 1)}{\Gamma(\rho k + \lambda + 1)} \quad (k \in \mathbb{N}_0).$$

**Proof.** We first prove the inequality (20). By using (11) and the property that  $|\varphi'| \leq M$ , we have

$$(23) \quad \begin{aligned} & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\ & \quad - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]| \\ & \leq |(\mathcal{J}_{\rho,\lambda+1,x-;\omega}^\sigma\varphi')(a)| + |(\mathcal{J}_{\rho,\lambda+1,x+;\omega}^\sigma\varphi')(b)| \\ & \leq M \left\{ \int_a^x (t-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(t-a)^\rho] dt \right. \\ & \quad \left. + \int_x^b (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-t)^\rho] dt \right\}, \end{aligned}$$

where the positivity of  $\mathcal{F}_{\rho,\lambda}^\sigma[\omega(t-a)^\rho]$  follows from the constraints imposed in (2).

The last two integrals on the right side of (23) can be evaluated by term-wise integration, and we have

$$(24) \quad \int_a^x (t-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(t-a)^\rho] dt = (x-a)^{\lambda+1} \mathcal{F}_{\rho,\lambda+2}^\sigma [\omega(x-a)^\rho] \\ = \mathcal{K}_{\lambda+1}^\sigma(x-a)$$

and

$$(25) \quad \int_x^b (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(b-t)^\rho] dt = (b-x)^{\lambda+1} \mathcal{F}_{\rho,\lambda+2}^\sigma [\omega(b-x)^\rho] \\ = \mathcal{K}_{\lambda+1}^\sigma(b-x).$$

On substituting (24) and (25) into (23), we obtain the desired inequality (20).

The proof of (21) is similar. By using (12) and the  $s$ -convexity of  $|\varphi'|$ , we have

$$(26) \quad |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)] \varphi(x) \\ - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma \varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma \varphi)(a)]| \\ \leq (x-a)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] |\varphi'(tx + (1-t)a)| dt \\ + (b-x)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(b-x)^\rho t^\rho] |\varphi'(tx + (1-t)b)| dt \\ \leq (x-a)^{\lambda+1} |\varphi'(x)| \int_0^1 t^{\lambda+s} \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] dt \\ + (x-a)^{\lambda+1} |\varphi'(a)| \int_0^1 t^\lambda (1-t)^s \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] dt \\ + (b-x)^{\lambda+1} |\varphi'(x)| \int_0^1 t^{\lambda+s} \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(b-x)^\rho t^\rho] dt \\ + (b-x)^{\lambda+1} |\varphi'(b)| \int_0^1 t^\lambda (1-t)^s \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(b-x)^\rho t^\rho] dt.$$

It follows upon using the term-wise integration that

$$(27) \quad \int_0^1 t^{\lambda+s} \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(x-a)^\rho t^\rho] dt = \sum_{k=0}^{\infty} \frac{\sigma(k) \omega^k (x-a)^{\rho k}}{\Gamma(\rho k + \lambda + 1)} \int_0^1 t^{\rho k + \lambda + s} dt \\ = \sum_{k=0}^{\infty} \frac{\sigma(k) \omega^k (x-a)^{\rho k} \Gamma(\rho k + \lambda + s + 1)}{\Gamma(\rho k + \lambda + 1) \Gamma(\rho k + \lambda + s + 2)} \\ = \sum_{k=0}^{\infty} \frac{\sigma_1(k) \omega^k (x-a)^{\rho k}}{\Gamma(\rho k + \lambda + s + 2)} \\ = \mathcal{F}_{\rho,\lambda+s+2}^{\sigma_1} [\omega(x-a)^\rho],$$

where the coefficient  $\sigma_1$  is given by (22).

$$\begin{aligned}
(28) \quad & \int_0^1 t^\lambda (1-t)^s \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega (x-a)^\rho t^\rho] dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k) \omega^k (x-a)^{\rho k}}{\Gamma(\rho k + \lambda + 1)} \int_0^1 t^{\rho k + \lambda} (1-t)^s dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k) \omega^k (x-a)^{\rho k}}{\Gamma(\rho k + \lambda + 1)} \frac{\Gamma(\rho k + \lambda + 1) \Gamma(s+1)}{\Gamma(\rho k + \lambda + s + 2)} \\
&= \Gamma(s+1) \sum_{k=0}^{\infty} \frac{\sigma(k) \omega^k (x-a)^{\rho k}}{\Gamma(\rho k + \lambda + s + 2)} \\
&= \Gamma(s+1) \mathcal{F}_{\rho, \lambda+s+2}^\sigma [\omega (x-a)^\rho].
\end{aligned}$$

In an analogous manner by using (2), we are easily lead to

$$(29) \quad \int_0^1 t^{\lambda+s} \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega (b-x)^\rho t^\rho] dt = \mathcal{F}_{\rho, \lambda+s+2}^{\sigma_1} [\omega (b-x)^\rho]$$

and

$$\begin{aligned}
(30) \quad & \int_0^1 t^\lambda (1-t)^s \mathcal{F}_{\rho, \lambda+1}^\sigma [\omega (b-x)^\rho t^\rho] dt \\
&= \Gamma(s+1) \mathcal{F}_{\rho, \lambda+s+2}^\sigma [\omega (b-x)^\rho],
\end{aligned}$$

where  $\sigma_1$  (as before) is given by (22).

From (26) to (30), we get

$$\begin{aligned}
& |[\mathcal{K}_\lambda(b-x) + \mathcal{K}_\lambda(x-a)] \varphi(x) - [(\mathcal{J}_{\rho, \lambda, x+; \omega}^\sigma \varphi)(b) + (\mathcal{J}_{\rho, \lambda, x-; \omega}^\sigma \varphi)(a)]| \\
&\leq (x-a)^{\lambda+1} |\varphi'(x)| \mathcal{F}_{\rho, \lambda+s+2}^{\sigma_1} [\omega (x-a)^\rho] \\
&\quad + (x-a)^{\lambda+1} |\varphi'(a)| \Gamma(s+1) \mathcal{F}_{\rho, \lambda+s+2}^\sigma [\omega (x-a)^\rho] \\
&\quad + (b-x)^{\lambda+1} |\varphi'(x)| \mathcal{F}_{\rho, \lambda+s+2}^{\sigma_1} [\omega (b-x)^\rho] \\
&\quad + (b-x)^{\lambda+1} |\varphi'(b)| \Gamma(s+1) \mathcal{F}_{\rho, \lambda+s+2}^\sigma [\omega (b-x)^\rho] \\
&= |\varphi'(x)| \frac{\mathcal{K}_{\lambda+s+1}^{\sigma_1}(x-a)}{(x-a)^s} + \Gamma(s+1) |\varphi'(a)| \frac{\mathcal{K}_{\lambda+s+1}^\sigma(x-a)}{(x-a)^s} \\
&\quad + |\varphi'(x)| \frac{\mathcal{K}_{\lambda+s+1}^{\sigma_1}(b-x)}{(b-x)^s} + \Gamma(s+1) |\varphi'(b)| \frac{\mathcal{K}_{\lambda+s+1}^\sigma(b-x)}{(b-x)^s},
\end{aligned}$$

which is (21). This completes the proof of our theorem. ■



**Theorem 3.** Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  such that  $|\varphi'| \leq M$  and let  $p_i > 1$  ( $i = 1, 2, 3$ ) with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ . Then the following Ostrowski type fractional integral inequality holds:

$$(31) \quad \begin{aligned} & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\ & \quad - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]| \\ & \leq \frac{(x-a)^{\lambda+\frac{1}{p_2}+\frac{1}{p_3}}}{(\lambda p_2+1)^{\frac{1}{p_2}}} \mathfrak{A}(x; p_3) \|\varphi'\|_{p_1, [a,x]} \\ & \quad + \frac{(b-x)^{\lambda+\frac{1}{p_2}+\frac{1}{p_3}}}{(\lambda p_2+1)^{\frac{1}{p_2}}} \mathfrak{B}(x; p_3) \|\varphi'\|_{p_1, [x,b]}, \end{aligned}$$

where

$$(32) \quad \mathfrak{A}(x; p) = \left( \int_0^1 \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(x-a)^\rho t^\rho]^p dt \right)^{\frac{1}{p}}$$

and

$$(33) \quad \mathfrak{B}(x; p) = \left( \int_0^1 \mathcal{F}_{\rho,\lambda+1}^\sigma [\omega(b-x)^\rho t^\rho]^p dt \right)^{\frac{1}{p}}.$$

Moreover, if we require that  $|\varphi'|^q$  is  $s$ -convex in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1]$  and  $q > 1$ , then we have the following inequality:

$$(34) \quad \begin{aligned} & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\ & \quad - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]| \\ & \leq \frac{(x-a)^{\lambda+1}}{(1+\lambda p_1)^{\frac{1}{p_1}}} \left( \frac{1}{1+s} \right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(a)|^{p_3}]^{\frac{1}{p_3}} \mathfrak{A}(x; p_2) \\ & \quad + \frac{(b-x)^{\lambda+1}}{(1+\lambda p_1)^{\frac{1}{p_1}}} \left( \frac{1}{1+s} \right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(b)|^{p_3}]^{\frac{1}{p_3}} \mathfrak{B}(x; p_2), \end{aligned}$$

where  $\mathfrak{A}(x; p)$  and  $\mathfrak{B}(x; p)$  are defined, respectively, by (32) and (33).

**Proof.** The proofs of inequalities (31) and (34) mainly depend on the Hölder inequality for three functions, viz.

$$(35) \quad \|fgh\|_1 \leq \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3},$$

where  $p_i \in (1, \infty)$  ( $i = 1, 2, 3$ ) and

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.$$

We first prove the inequality (31). From the identity (12), it is easy to find that

$$\begin{aligned}
(36) \quad & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\
& - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)] \\
& \leq |(\mathcal{J}_{\rho,\lambda+1,x-;\omega}^\sigma\varphi')(a)| + |(\mathcal{J}_{\rho,\lambda+1,x+;\omega}^\sigma\varphi')(b)| \\
& \leq \int_a^x (t-a)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(t-a)^\rho] |\varphi'(t)| dt \\
& \quad + \int_x^b (b-t)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-t)^\rho] |\varphi'(t)| dt \\
& \leq \left( \int_a^x |\varphi'(t)|^{p_1} dt \right)^{\frac{1}{p_1}} \left( \int_a^x (t-a)^{\lambda p_2} dt \right)^{\frac{1}{p_2}} \\
& \quad \times \left( \int_a^x \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(t-a)^\rho]^{p_3} dt \right)^{\frac{1}{p_3}} + \left( \int_x^b |\varphi'(t)|^{p_1} dt \right)^{\frac{1}{p_1}} \\
& \quad \times \left( \int_x^b (b-t)^{\lambda p_2} dt \right)^{\frac{1}{p_2}} \left( \int_x^b \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-t)^\rho]^{p_3} dt \right)^{\frac{1}{p_3}} \\
& = \frac{(x-a)^{\lambda+\frac{1}{p_2}}}{(\lambda p_2+1)^{\frac{1}{p_2}}} \|\varphi'\|_{p_1,[a,x]} \left( \int_a^x \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(t-a)^\rho]^{p_3} dt \right)^{\frac{1}{p_3}} \\
& \quad + \frac{(b-x)^{\lambda+\frac{1}{p_2}}}{(\lambda p_2+1)^{\frac{1}{p_2}}} \|\varphi'\|_{p_1,[x,b]} \left( \int_x^b \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-t)^\rho]^{p_3} dt \right)^{\frac{1}{p_3}}.
\end{aligned}$$

By applying the substitution  $u = (t-a)/(x-a)$ , we get

$$\begin{aligned}
(37) \quad & \left( \int_a^x \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(t-a)^\rho]^{p_3} dt \right)^{\frac{1}{p_3}} \\
& = (x-a)^{\frac{1}{p_3}} \left( \int_0^1 \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(x-a)^\rho u^\rho]^{p_3} dt \right)^{\frac{1}{p_3}} \\
& = (x-a)^{\frac{1}{p_3}} \mathfrak{A}(x; p_3),
\end{aligned}$$

and (in similar manner)

$$(38) \quad \left( \int_x^b \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-t)^\rho]^{p_3} dt \right)^{\frac{1}{p_3}} = (b-x)^{\frac{1}{p_3}} \mathfrak{B}(x; p_3).$$

Using (37) and (38) in (36), we obtain the inequality (31).

Next, we prove the inequality (34). The use of identity (12) and inequality (35) gives

$$\begin{aligned}
(39) \quad & |[\mathcal{K}_\lambda^\sigma(b-x) + \mathcal{K}_\lambda^\sigma(x-a)]\varphi(x) \\
& - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]| \\
& \leq (x-a)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(x-a)^\rho t^\rho] |\varphi'(tx + (1-t)a)| dt \\
& \quad + (b-x)^{\lambda+1} \int_0^1 t^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-x)^\rho t^\rho] |\varphi'(tx + (1-t)b)| dt \\
& \leq (x-a)^{\lambda+1} \left( \int_0^1 t^{\lambda p_1} dt \right)^{\frac{1}{p_1}} \left( \int_0^1 \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(x-a)^\rho t^\rho]^{p_2} dt \right)^{\frac{1}{p_2}} \\
& \quad \times \left( \int_0^1 |\varphi'(tx + (1-t)a)|^{p_3} dt \right)^{\frac{1}{p_3}} \\
& \quad + (b-x)^{\lambda+1} \left( \int_0^1 t^{\lambda p_1} dt \right)^{\frac{1}{p_1}} \left( \int_0^1 \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(b-x)^\rho t^\rho]^{p_2} dt \right)^{\frac{1}{p_2}} \\
& \quad \left( \int_0^1 |\varphi'(tx + (1-t)b)|^{p_3} dt \right)^{\frac{1}{p_3}}.
\end{aligned}$$

Since  $|\varphi'|^q$  ( $q > 1$ ) is  $s$ -convex in the second sense, we have

$$\begin{aligned}
(40) \quad & \left( \int_0^1 |\varphi'(tx + (1-t)a)|^{p_3} dt \right)^{\frac{1}{p_3}} \\
& \leq \left( \int_0^1 [t^s |\varphi'(x)|^{p_3} + (1-t)^s |\varphi'(a)|^{p_3}] dt \right)^{\frac{1}{p_3}} \\
& \leq \left( \frac{1}{1+s} \right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(a)|^{p_3}]^{\frac{1}{p_3}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(41) \quad & \left( \int_0^1 |\varphi'(tx + (1-t)b)|^{p_3} dt \right)^{\frac{1}{p_3}} \\
& \leq \left( \frac{1}{1+s} \right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(b)|^{p_3}]^{\frac{1}{p_3}}.
\end{aligned}$$

Hence, we finally get

$$\begin{aligned}
(42) \quad & |[\mathcal{K}_\lambda(b-x) + \mathcal{K}_\lambda(x-a)]\varphi(x) \\
& - [(\mathcal{J}_{\rho,\lambda,x+;\omega}^\sigma\varphi)(b) + (\mathcal{J}_{\rho,\lambda,x-;\omega}^\sigma\varphi)(a)]|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{(x-a)^{\lambda+1}}{(1+\lambda p_1)^{\frac{1}{p_1}}} \left(\frac{1}{1+s}\right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(a)|^{p_3}]^{\frac{1}{p_3}} \mathfrak{A}(x; p_2) \\ &\quad + \frac{(b-x)^{\lambda+1}}{(1+\lambda p_1)^{\frac{1}{p_1}}} \left(\frac{1}{1+s}\right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(b)|^{p_3}]^{\frac{1}{p_3}} \mathfrak{B}(x; p_2). \end{aligned}$$

This completes the proof. ■

### 3. Some consequences and applications

In this section, we consider some consequences of Theorems 2 and 3 and also point out applications of Theorems 4 and 5 to Stolarsky's means defined in [17].

It may be observed that many known results involving the familiar Riemann-Liouville fractional integrals are direct consequences of our main theorems. Thus, if we set  $\lambda = \alpha$ ,  $\sigma(0) = 1$  and  $\omega = 0$  in (3), (4) and (13), and make use of (2), then we get the following relations:

$$(\mathcal{J}_{\rho, \alpha, a+; 0}^{\sigma} \varphi)(x) = (I_{a+}^{\alpha} \varphi)(x), \quad (\mathcal{J}_{\rho, \alpha, b-; 0}^{\sigma} \varphi)(x) = (I_{b-}^{\alpha} \varphi)(x),$$

$$\mathcal{K}_{\alpha}^{\sigma}(z-y) = \frac{(z-y)^{\alpha}}{\Gamma(\alpha+1)}, \quad \mathcal{K}_{\alpha+s+1}^{\sigma}(z-y) = \frac{(z-y)^{\alpha+s+1}}{\Gamma(\alpha+s+2)}$$

and

$$\mathcal{K}_{\alpha+s+1}^{\sigma_1}(z-y) = \frac{(z-y)^{\alpha+s+1}}{(\alpha+s+1)\Gamma(\alpha+1)}.$$

Hence, as a consequence of above relations, we get the following corollary from Theorem 2.

**Corollary 1.** *Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$  such that  $|\varphi'| \leq M$ , then for all  $x \in [a, b]$  and  $\alpha > 0$ :*

$$(43) \quad \begin{aligned} &|[(b-x)^{\alpha} + (x-a)^{\alpha}] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^{\alpha} \varphi)(b) + (I_{x-}^{\alpha} \varphi)(a)]| \\ &\leq \frac{M}{\alpha+1} [(x-a)^{\alpha+1} + (b-x)^{\alpha+1}]. \end{aligned}$$

*If, in addition,  $|\varphi'|$  is  $s$ -convex in the second sense on  $[a, b]$  for some  $s \in (0, 1]$ , then the following inequality for fractional integrals with  $\lambda > 0$  holds:*

$$(44) \quad \begin{aligned} &|[(b-x)^{\alpha} + (x-a)^{\alpha}] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^{\alpha} \varphi)(b) + (I_{x-}^{\alpha} \varphi)(a)]| \\ &\leq |\varphi'(x)| \left[ \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+s+1} \right] \\ &\quad + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \left[ \frac{|\varphi'(a)|(x-a)^{\alpha+1} + |\varphi'(b)|(b-x)^{\alpha+1}}{\alpha+s+1} \right]. \end{aligned}$$

**Remark 1.** The inequality (43) was earlier given by Sarikaya and Filiz [12, p. 188, Theorem 2]. Further, if we impose the condition that  $|\varphi'(x)| \leq M$  for all  $x \in [a, b]$ , then the inequality (44) can be expressed as

$$(45) \quad \left| [(b-x)^\alpha + (x-a)^\alpha] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^\alpha \varphi)(b) + (I_{x-}^\alpha \varphi)(a)] \right| \\ \leq M \left[ 1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right] \left[ \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+s+1} \right],$$

which was established by Set [14, p. 1150, Theorem 7].

We note that if we choose  $\alpha = 1$  in (43), then it reduces to Ostrowski's inequality (1).

On the other hand, for  $\alpha = 1$ , the inequality in (45) reduces to

$$\left| \varphi(x) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{s+1} \right],$$

which is due to M. Alomari et al. in [1, p. 1072, Theorem 2.].

**Corollary 2.** Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  such that  $|\varphi'| \leq M$  and  $p_i > 1$  ( $i = 1, 2$ ) with  $\frac{1}{p_1} + \frac{1}{p_2} = 1$ . Then, the following Ostrowski type fractional integral inequality holds:

$$(46) \quad \left| [(b-x)^\alpha + (x-a)^\alpha] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^\alpha \varphi)(b) + (I_{x-}^\alpha \varphi)(a)] \right| \\ \leq \frac{(x-a)^{\alpha+\frac{1}{p_2}}}{(\alpha p_2 + 1)^{\frac{1}{p_2}}} \|\varphi'\|_{p_1, [a, x]} + \frac{(b-x)^{\alpha+\frac{1}{p_2}}}{(\alpha p_2 + 1)^{\frac{1}{p_2}}} \|\varphi'\|_{p_1, [x, b]}.$$

Moreover, if we require that  $|\varphi'|^q$  is  $s$ -convex in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1]$ ,  $q > 1$ , then we have the following inequality:

$$(47) \quad \left| [(b-x)^\alpha + (x-a)^\alpha] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^\alpha \varphi)(b) + (I_{x-}^\alpha \varphi)(a)] \right| \\ \leq \frac{(x-a)^{\alpha+1}}{(1+\alpha p_1)^{\frac{1}{p_1}}} \left( \frac{1}{1+s} \right)^{\frac{1}{p_2}} [|\varphi'(x)|^{p_2} + |\varphi'(a)|^{p_2}]^{\frac{1}{p_2}} \\ + \frac{(b-x)^{\alpha+1}}{(1+\alpha p_1)^{\frac{1}{p_1}}} \left( \frac{1}{1+s} \right)^{\frac{1}{p_2}} [|\varphi'(x)|^{p_2} + |\varphi'(a)|^{p_2}]^{\frac{1}{p_2}}.$$

**Proof.** By setting  $\lambda = \alpha$ ,  $\sigma(0) = 1$  and  $\omega = 0$  in (31) and using (3), (4) and (13), we have

$$(48) \quad \left| [(b-x)^\alpha + (x-a)^\alpha] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^\alpha \varphi)(b) + (I_{x-}^\alpha \varphi)(a)] \right| \\ \leq \frac{(x-a)^{\alpha+\frac{1}{p_2}+\frac{1}{p_3}}}{(\alpha p_2 + 1)^{\frac{1}{p_2}}} \|\varphi'\|_{p_1, [a, x]} + \frac{(b-x)^{\alpha+\frac{1}{p_2}+\frac{1}{p_3}}}{(\alpha p_2 + 1)^{\frac{1}{p_2}}} \|\varphi'\|_{p_1, [x, b]}.$$

On letting  $p_3 \rightarrow \infty$ , we immediately get (46). The proof of the inequality (47) is analogous to (46), and hence we omit its details.  $\blacksquare$

**Remark 2.** The inequality (46) is actually a result given in [12, p. 189, Theorem 3]. If we impose an added constraint that  $|\varphi'(x)| \leq M$ ,  $x \in [a, b]$  in (47), then

$$\begin{aligned} & \left| [(b-x)^\alpha + (x-a)^\alpha] \varphi(x) - \Gamma(\alpha+1) [(I_{x+}^\alpha \varphi)(b) + (I_{x-}^\alpha \varphi)(a)] \right| \\ & \leq \frac{M}{(1+\alpha p_1)^{\frac{1}{p_1}}} \left( \frac{2}{1+s} \right)^{\frac{1}{p_2}} \left[ (x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right], \end{aligned}$$

which was proved in [14, p. 1150, Theorem 8].

By considering a very general case, let us put

$$(49) \quad \sigma(k) = \frac{\Gamma(\rho + \lambda) \prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{k! \prod_{j=1}^q \Gamma(b_j + \beta_j k)}$$

in (2), then  $\mathcal{F}_{\rho, \lambda}^\sigma(x)$  becomes the Fox-Wright function defined by (see [7, pp. 56–57]; see also [6])

$$(50) \quad {}_p\Psi_q[x] \equiv {}_p\Psi_q \left[ \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}; x \right] := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k) x^k}{\prod_{j=1}^q \Gamma(b_j + \beta_j k) k!},$$

( $x, a_i, b_j \in \mathbb{C}$ ,  $\alpha_i, \beta_j \in \mathbb{R}_+$  ( $i = 1, \dots, p$ ,  $j = 1, \dots, q$ ),  $\Delta := \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i \geq -1$ ) where the equality in the convergence condition holds true for suitably bounded values of  $|x|$  given by

$$|x| < \delta := \prod_{i=1}^p |\alpha_i|^{-\alpha_i} \prod_{j=1}^q |\beta_j|^{\beta_j},$$

and  $|x| = \delta$  when

$$\mu := \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2} > \frac{1}{2}.$$

Then, the left-sided and right-sided fractional integral operators obtainable from (3) and (4) are given by

$$(51) \quad \left( \mathcal{H}_{\omega, a+; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right) (x) = \int_a^x (x-t)^{\lambda-1} {}_p\Psi_q[\omega(x-t)^\rho] \varphi(t) dt$$

and

$$(52) \quad \left( \mathcal{H}_{\omega, b-; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right) (x) = \int_x^b (t-x)^{\lambda-1} {}_p\Psi_q[\omega(t-x)^\rho] \varphi(t) dt,$$

The existence conditions of (51) and (52) follows directly from the convergence conditions of the Fox-Wright function stated above. The integral operator (51) has been discussed in [11, Section 4].

It may be noted that by employing the substitutions (49), we have in view of (13) that

$$\begin{aligned} \mathcal{K}_\lambda^\sigma(z-y) &= (z-y)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(z-y)^\rho] \\ &= (z-y)^\lambda \sum_{k=0}^{\infty} \frac{\Gamma(\rho k + \lambda)}{\Gamma(\rho k + \lambda + 1)} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{\omega^k (z-y)^{\rho k}}{k!} \\ &= (z-y)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(z-y)^\rho \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_{\lambda+s+1}^{\sigma_1}(z-y) &= (z-y)^{\lambda+s+1} \\ &\quad \times {}_{p+2}\Psi_{q+2} \left[ \begin{matrix} (\lambda + s + 1, \rho), (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (\lambda + s + 2, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(z-y)^\rho \right]. \end{aligned}$$

Here and throughout below, it is assumed that  $a_i > 0, \alpha_i > 0$  ( $i = 1, \dots, p$ ) and  $b_j > 0, \beta_j > 0$  ( $j = 1, \dots, q$ ).

Thus, in view of (49) – (52), Theorems 2 and 3 yield the following inequalities.

**Corollary 3.** *Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$  such that  $|\varphi'| \leq M$ , then for all  $x \in [a, b]$  and  $\lambda > 0$ :*

$$\begin{aligned} (53) \quad & \left| \left\{ (b-x)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(b-x)^\rho \right] \right. \right. \\ & \quad \left. \left. + (x-a)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(x-a)^\rho \right] \right\} \varphi(x) \right. \\ & \quad \left. - \left[ \left( \mathcal{H}_{\omega, x+:(b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right)(b) + \left( \mathcal{H}_{\omega, x-:(b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right)(a) \right] \right| \\ & \leq M \left\{ (x-a)^{\lambda+1} {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 2, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(x-a)^\rho \right] \right. \\ & \quad \left. + (b-x)^{\lambda+1} {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 2, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(b-x)^\rho \right] \right\}. \end{aligned}$$

If, in addition,  $|\varphi'|$  is  $s$ -convex in the second sense on  $[a, b]$  for some  $s \in (0, 1]$ , then the following inequality for fractional integrals with  $\lambda > 0$  holds:

$$(54) \quad \left| \left\{ (b-x)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix} ; \omega(b-x)^\rho \right] \right. \right.$$

$$\begin{aligned}
& + (x-a)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(x-a)^\rho \right] \Big\} \varphi(x) \\
& - \left[ \left( \mathcal{H}_{\omega, x+; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right) (b) + \left( \mathcal{H}_{\omega, x-; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right) (a) \right] \Big| \\
\leq & |\varphi'(x)| \left\{ (x-a)^{\lambda+1} {}_{p+2}\Psi_{q+2} \right. \\
& \times \left[ \begin{matrix} (\lambda+s+1, \rho), (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+1, \rho), (\lambda+s+2, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(x-a)^\rho \right] \\
& + (b-x)^{\lambda+1} {}_{p+2}\Psi_{q+2} \\
& \times \left[ \begin{matrix} (\lambda+s+1, \rho), (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+1, \rho), (\lambda+s+2, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(b-x)^\rho \right] \Big\} \\
& + \Gamma(s+1) \left\{ |\varphi'(a)| (x-a)^{\lambda+1} {}_{p+1}\Psi_{q+1} \right. \\
& \times \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+s+2, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(x-a)^\rho \right] \\
& \left. + |\varphi'(b)| (b-x)^{\lambda+1} {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+s+2, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(b-x)^\rho \right] \right\}.
\end{aligned}$$

**Corollary 4.** *Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  such that  $|\varphi'| \leq M$  and  $p_i > 1$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ . Then, the following Ostrowski type fractional integral inequality holds:*

$$\begin{aligned}
(55) \quad & \left\{ (b-x)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(b-x)^\rho \right] \right. \\
& \left. + (x-a)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(x-a)^\rho \right] \right\} \varphi(x) \\
& - \left[ \left( \mathcal{H}_{\omega, x+; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right) (b) + \left( \mathcal{H}_{\omega, x-; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right) (a) \right] \Big| \\
\leq & \frac{(x-a)^{\lambda+\frac{1}{p_2}+\frac{1}{p_3}}}{(\lambda p_2 + 1)^{\frac{1}{p_2}}} \tilde{\mathfrak{A}}(x; p_3) \|\varphi'\|_{p_1, [a, x]} \\
& + \frac{(b-x)^{\lambda+\frac{1}{p_2}+\frac{1}{p_3}}}{(\lambda p_2 + 1)^{\frac{1}{p_2}}} \tilde{\mathfrak{B}}(x; p_3) \|\varphi'\|_{p_1, [x, b]},
\end{aligned}$$

where

$$(56) \quad \tilde{\mathfrak{A}}(x; p) = \left( \int_0^1 {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda+1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(b-x)^\rho t^\rho \right]^p dt \right)^{\frac{1}{p}}$$



and

$$(57) \quad \tilde{\mathfrak{B}}(x; p) = \left( \int_0^1 {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(b-x)^\rho t^\rho \right] dt \right)^{\frac{1}{p}}.$$

Moreover, if we require that  $|\varphi'|^q$  is  $s$ -convex in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1]$  ( $q > 1$ ), then we have the following inequality:

$$(58) \quad \left| \left\{ (b-x)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(b-x)^\rho \right] \right. \right. \\ \left. \left. + (x-a)^\lambda {}_{p+1}\Psi_{q+1} \left[ \begin{matrix} (\lambda, \rho), (a_i, \alpha_i)_{1,p} \\ (\lambda + 1, \rho), (b_j, \beta_j)_{1,q} \end{matrix}; \omega(x-a)^\rho \right] \right\} \varphi(x) \right. \\ \left. - \left[ \left( \mathcal{H}_{\omega, x+; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right)(b) + \left( \mathcal{H}_{\omega, x-; (b_q, \beta_q)}^{\lambda, \rho; (a_p, \alpha_p)} \varphi \right)(a) \right] \right| \\ \leq \frac{(x-a)^{\lambda+1}}{(1+\lambda p_1)^{\frac{1}{p_1}}} \left( \frac{1}{1+s} \right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(a)|^{p_3}]^{\frac{1}{p_3}} \tilde{\mathfrak{A}}(x; p_2) \\ + \frac{(b-x)^{\lambda+1}}{(1+\lambda p_1)^{\frac{1}{p_1}}} \left( \frac{1}{1+s} \right)^{\frac{1}{p_3}} [|\varphi'(x)|^{p_3} + |\varphi'(b)|^{p_3}]^{\frac{1}{p_3}} \tilde{\mathfrak{B}}(x; p_2),$$

where  $\tilde{\mathfrak{A}}(x; p)$  and  $\tilde{\mathfrak{B}}(x; p)$  are, respectively, defined by (56) and (57).

**Remark 3.** By choosing parameters suitably in (51) and (52), many important fractional integral operators including those involving the generalized Mittag-Leffler function as kernel (see [15], [16] and [18]) can be easily obtained and corresponding to these integral operators related inequalities from the Corollaries 3 and 4 can be deduced.

Finally, let us consider a pair of integral operators which are not evidently reducible from (3) and (4).

In [20], Yildirim and Kirtay prove new generalizations for Ostrowski type inequalities by using the following fractional integral operators:

$$(59) \quad (I_{a+}^{\alpha, \eta} f)(x) := \frac{(1+\eta)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{\eta+1} - t^{\eta+1})^{\alpha-1} t^\eta f(t) dt \quad (x > a)$$

and

$$(60) \quad (I_{b-}^{\alpha, \eta} f)(x) := \frac{(1+\eta)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{\eta+1} - x^{\eta+1})^{\alpha-1} t^\eta f(t) dt \quad (b > x),$$

where  $\alpha > 0$  and  $\eta \geq 0$ . The integral operator (59) was considered by Katugampola in [5]. It may be pointed out here that (59) and (60) are special

cases of the operators discussed in [7, Section 2.5]. When  $\eta = 0$  in (59) and (60), we get the left-sided and right-sided familiar Riemann-Liouville fractional integral operators.

We now examine how (3) and (4) can reduce to (59) and (60).

**Lemma 2.** *Corresponding to (10), the integral operators (3) and (4) in view of (2) yield the following relationships:*

$$(61) \quad \left( \mathcal{J}_{\rho, \alpha, a^{\eta+1+}; 0}^{\sigma} \right) (x^{\eta+1}) = (\eta + 1)^{\alpha} (I_{a+}^{\alpha, \eta} \varphi \circ g) (x)$$

and

$$(62) \quad \left( \mathcal{J}_{\rho, \alpha, b^{\eta+1-}; 0}^{\sigma} \right) (x^{\eta+1}) = (\eta + 1)^{\alpha} (I_{b-}^{\alpha, \eta} \varphi \circ g) (x),$$

where  $g(t) = t^{\eta+1}$  and  $\varphi$  is chosen such that the right-hand sides of (61) and (62) exist.

**Proof.** From the definition of  $\mathcal{J}_{\rho, \lambda, a+; \omega}^{\sigma}$  given by (3) and using (10) in conjunction with (2), we have

$$(63) \quad \left( \mathcal{J}_{\rho, \lambda, a^{\eta+1+}; \omega}^{\sigma} \right) (x^{\eta+1}) = \frac{1}{\Gamma(\alpha)} \int_{a^{\eta+1}}^{x^{\eta+1}} (x^{\eta+1} - t)^{\alpha-1} \varphi(t) dt.$$

The substitution  $t = u^{\eta+1}$  then leads to

$$(64) \quad \begin{aligned} \left( \mathcal{J}_{\rho, \lambda, a^{\eta+1+}; \omega}^{\sigma} \right) (x^{\eta+1}) &= \frac{\eta + 1}{\Gamma(\alpha)} \int_a^x (x^{\eta+1} - u^{\eta+1})^{\alpha-1} u^{\eta} \varphi(u^{\eta+1}) du \\ &= (\eta + 1)^{\alpha} \frac{(\eta + 1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{\eta+1} - u^{\eta+1})^{\alpha-1} u^{\eta} \tilde{\varphi}(u) du \\ &= (\eta + 1)^{\alpha} (I_{a+}^{\alpha, \eta} \tilde{\varphi}) (x), \end{aligned}$$

where  $\tilde{\varphi}(u) := (\varphi \circ g)(u) = \varphi(u^{\eta+1})$ . This completes the proof of the first relationship. The second relationship can be proved similarly.  $\blacksquare$

By using Lemma 2 and performing elementary calculations, we obtain the following results.

**Theorem 4.** *Let  $\varphi : [a^{\eta+1}, b^{\eta+1}] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a^{\eta+1}, b^{\eta+1})$  with  $a < b$  such that  $|(\varphi' \circ g)(x)| \leq M$ , then for all  $x \in [a, b]$  and  $\alpha > 0$ :*

$$(65) \quad \left| \frac{(b^{\eta+1} - x^{\eta+1})^{\alpha} + (x^{\eta+1} - a^{\eta+1})^{\alpha}}{(\eta + 1)^{\alpha}} (\varphi \circ g)(x) \right. \\ \left. - \Gamma(\alpha + 1) [(I_{x+}^{\alpha, \eta} \varphi \circ g)(b) + (I_{x-}^{\alpha, \eta} \varphi \circ g)(a)] \right| \\ \leq \frac{M}{\alpha + 1} \left[ \frac{(x^{\eta+1} - a^{\eta+1})^{\alpha+1} + (b^{\eta+1} - x^{\eta+1})^{\alpha+1}}{(\eta + 1)^{\alpha}} \right].$$

If, in addition,  $|\varphi'|$  is  $s$ -convex in the second sense on  $[a^{\eta+1}, b^{\eta+1}]$  for some  $s \in (0, 1]$ , then the following inequality for fractional integrals with  $\lambda > 0$  holds:

$$(66) \quad \left| \frac{(b^{\eta+1} - x^{\eta+1})^\alpha + (x^{\eta+1} - a^{\eta+1})^\alpha}{(\eta + 1)^\alpha} (\varphi \circ g)(x) \right. \\ \left. - \Gamma(\alpha + 1) [(I_{x+}^{\alpha, \eta} \varphi \circ g)(b) + (I_{x-}^{\alpha, \eta} \varphi \circ g)(a)] \right| \\ \leq |(\varphi' \circ g)(x)| \left[ \frac{(x^{\eta+1} - a^{\eta+1})^{\alpha+1} + (b^{\eta+1} - x^{\eta+1})^{\alpha+1}}{(\alpha + s + 1)(\eta + 1)^\alpha} \right] \\ + \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)} \left[ \frac{|(\varphi' \circ g)(a)| (x^{\eta+1} - a^{\eta+1})^{\alpha+1}}{(\alpha + s + 1)(\eta + 1)^\alpha} \right. \\ \left. + \frac{|(\varphi' \circ g)(b)| (b^{\eta+1} - x^{\eta+1})^{\alpha+1}}{(\alpha + s + 1)(\eta + 1)^\alpha} \right].$$

**Theorem 5.** Let  $\varphi : [a^{\eta+1}, b^{\eta+1}] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a^{\eta+1}, b^{\eta+1})$  such that  $|(\varphi' \circ g)(x)| \leq M$  for  $x \in [a, b]$  and  $p_i > 1$  ( $i = 1, 2$ ) with  $\frac{1}{p_1} + \frac{1}{p_2} = 1$ . Then, the following Ostrowski type fractional integral inequality holds:

$$(67) \quad \left| \frac{(b^{\eta+1} - x^{\eta+1})^\alpha + (x^{\eta+1} - a^{\eta+1})^\alpha}{(\eta + 1)^\alpha} (\varphi \circ g)(x) \right. \\ \left. - \Gamma(\alpha + 1) [(I_{x+}^{\alpha, \eta} \varphi \circ g)(b) + (I_{x-}^{\alpha, \eta} \varphi \circ g)(a)] \right| \\ \leq \frac{(x^{\eta+1} - a^{\eta+1})^{\alpha + \frac{1}{p_2}} \|\varphi'\|_{p_1, [a^{\eta+1}, x^{\eta+1}]} }{(\eta + 1)^\alpha (\alpha p_2 + 1)^{\frac{1}{p_2}}} \\ + \frac{(b^{\eta+1} - x^{\eta+1})^{\alpha + \frac{1}{p_2}} \|\varphi'\|_{p_1, [x^{\eta+1}, b^{\eta+1}]} }{(\eta + 1)^\alpha (\alpha p_2 + 1)^{\frac{1}{p_2}}}.$$

Moreover, if we require that  $|\varphi'|^q$  is  $s$ -convex in the second sense on  $[a^{\eta+1}, b^{\eta+1}]$  for some fixed  $s \in (0, 1]$ ,  $q > 1$ , then we have the following inequality:

$$(68) \quad \left| \frac{(b^{\eta+1} - x^{\eta+1})^\alpha + (x^{\eta+1} - a^{\eta+1})^\alpha}{(\eta + 1)^\alpha} (\varphi \circ g)(x) \right. \\ \left. - \Gamma(\alpha + 1) [(I_{x+}^{\alpha, \eta} \varphi \circ g)(b) + (I_{x-}^{\alpha, \eta} \varphi \circ g)(a)] \right|$$

$$\begin{aligned}
&\leq \frac{(x^{\eta+1} - a^{\eta+1})^{\alpha+1}}{(\eta+1)^\alpha (1 + \alpha p_1)^{\frac{1}{p_1}}} \\
&\quad \times \left( \frac{1}{1+s} \right)^{\frac{1}{p_2}} [ |(\varphi' \circ g)(x)|^{p_2} + |(\varphi' \circ g)(a)|^{p_2} ]^{\frac{1}{p_2}} \\
&+ \frac{(b^{\eta+1} - x^{\eta+1})^{\alpha+1}}{(\eta+1)^\alpha (1 + \alpha p_1)^{\frac{1}{p_1}}} \\
&\quad \times \left( \frac{1}{1+s} \right)^{\frac{1}{p_2}} [ |(\varphi' \circ g)(x)|^{p_2} + |(\varphi' \circ g)(b)|^{p_2} ]^{\frac{1}{p_2}}.
\end{aligned}$$

The structure of the inequalities stated in Theorems 4 and 5 indicates their connection with Stolarsky's means defined by (see [17, p. 88, Eq. (7)]; see also [3, p. 519]):

$$(69) \quad S_p(x, y) := \left[ \frac{x^p - y^p}{p(x-y)} \right]^{\frac{1}{p-1}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

By setting  $p = \eta + 1$  in (69) and rearranging the resulting equation, we get a more convenient form given by

$$(70) \quad \left[ \frac{x^{\eta+1} - y^{\eta+1}}{\eta+1} \right]^{\frac{1}{\eta}} = (x-y)^{\frac{1}{\eta}} S_{\eta+1}(x, y), \quad x \geq y.$$

We will just consider here the inequality (65) and similar analysis can easily be applied to other inequalities (66)–(68). If we set  $\alpha = \frac{1}{\eta}$  in (65), we get

$$\begin{aligned}
(71) \quad &\left| \left[ \frac{(b^{\eta+1} - x^{\eta+1})^{\frac{1}{\eta}}}{(\eta+1)^{\frac{1}{\eta}}} + \frac{(x^{\eta+1} - a^{\eta+1})^{\frac{1}{\eta}}}{(\eta+1)^{\frac{1}{\eta}}} \right] (\varphi \circ g)(x) \right. \\
&\quad \left. - \Gamma \left( \frac{1}{\eta} + 1 \right) \left[ \left( I_{x+}^{\frac{1}{\eta}, \eta} \varphi \circ g \right)(b) + \left( I_{x-}^{\frac{1}{\eta}, \eta} \varphi \circ g \right)(a) \right] \right| \\
&\leq \frac{M\eta}{\eta+1} \left[ \frac{(x^{\eta+1} - a^{\eta+1})^{\frac{1}{\eta}}}{(\eta+1)^{\frac{1}{\eta}}} (x^{\eta+1} - a^{\eta+1}) \right. \\
&\quad \left. + \frac{(b^{\eta+1} - x^{\eta+1})^{\frac{1}{\eta}}}{(\eta+1)^{\frac{1}{\eta}}} (b^{\eta+1} - x^{\eta+1}) \right].
\end{aligned}$$

Applying now (70), we get the following inequality:

$$(72) \quad \left| \left[ (b-x)^{\frac{1}{\eta}} S_{\eta+1}(b, x) + (x-a)^{\frac{1}{\eta}} S_{\eta+1}(x, a) \right] (\varphi \circ g)(x) \right. \\ \left. - \Gamma\left(\frac{1}{\eta} + 1\right) \left[ \left( I_{x+}^{\frac{1}{\eta}, \eta} \varphi \circ g \right)(b) + \left( I_{x-}^{\frac{1}{\eta}, \eta} \varphi \circ g \right)(a) \right] \right| \\ \leq \frac{M\eta}{\eta+1} \left[ (x-a)^{\frac{1}{\eta}} S_{\eta+1}(x, a) (x^{\eta+1} - a^{\eta+1}) \right. \\ \left. + (b-x)^{\frac{1}{\eta}} S_{\eta+1}(b, x) (b^{\eta+1} - x^{\eta+1}) \right].$$

If we set

$$x = 2^{\frac{1}{\eta+1}} a \quad \text{and} \quad b = 3^{\frac{1}{\eta+1}} a$$

in (72), we have

$$(73) \quad \left| \left[ \left( 3^{\frac{1}{\eta+1}} - 2^{\frac{1}{\eta+1}} \right)^{\frac{1}{\eta}} a^{\frac{1}{\eta}} S_{\eta+1} \left( 3^{\frac{1}{\eta+1}} a, 2^{\frac{1}{\eta+1}} a \right) \right. \right. \\ \left. \left. + \left( 2^{\frac{1}{\eta+1}} - 1 \right)^{\frac{1}{\eta}} a^{\frac{1}{\eta}} S_{\eta+1} \left( 2^{\frac{1}{\eta+1}} a, a \right) \right] (\varphi \circ g) \left( 2^{\frac{1}{\eta+1}} a \right) \right. \\ \left. - \Gamma\left(\frac{1}{\eta} + 1\right) \left[ \left( I_{2^{\frac{1}{\eta+1}} a+}^{\frac{1}{\eta}, \eta} \varphi \circ g \right) \left( 3^{\frac{1}{\eta+1}} a \right) + \left( I_{2^{\frac{1}{\eta+1}} a-}^{\frac{1}{\eta}, \eta} \varphi \circ g \right) (a) \right] \right| \\ \leq \frac{M\eta}{\eta+1} \left[ \left( 3^{\frac{1}{\eta+1}} - 2^{\frac{1}{\eta+1}} \right)^{\frac{1}{\eta}} a^{\frac{1}{\eta}} S_{\eta+1} \left( 3^{\frac{1}{\eta+1}} a, 2^{\frac{1}{\eta+1}} a \right) \right. \\ \left. + \left( 2^{\frac{1}{\eta+1}} - 1 \right)^{\frac{1}{\eta}} a^{\frac{1}{\eta}} S_{\eta+1} \left( 2^{\frac{1}{\eta+1}} a, a \right) \right] a^{\eta+1}.$$

Simplifying this inequality by using the homogeneous property of Stolarsky's means that

$$S_p(\lambda x, \lambda y) = \lambda S_p(x, y) \quad (\lambda \geq 0),$$

we finally get

$$(74) \quad \left| (\varphi \circ g) \left( 2^{\frac{1}{\eta+1}} a \right) \right. \\ \left. - \frac{\Gamma\left(\frac{1}{\eta} + 1\right) \left[ \left( I_{2^{\frac{1}{\eta+1}} a+}^{\frac{1}{\eta}, \eta} \varphi \circ g \right) \left( 3^{\frac{1}{\eta+1}} a \right) + \left( I_{2^{\frac{1}{\eta+1}} a-}^{\frac{1}{\eta}, \eta} \varphi \circ g \right) (a) \right]}{a^{\frac{1}{\eta}+1} \left[ \left( 3^{\frac{1}{\eta+1}} - 2^{\frac{1}{\eta+1}} \right)^{\frac{1}{\eta}} S_{\eta+1} \left( 3^{\frac{1}{\eta+1}}, 2^{\frac{1}{\eta+1}} \right) + \left( 2^{\frac{1}{\eta+1}} - 1 \right)^{\frac{1}{\eta}} S_{\eta+1} \left( 2^{\frac{1}{\eta+1}}, 1 \right) \right]} \right| \\ \leq \frac{M\eta}{\eta+1} a^{\eta+1}.$$

Since

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi},$$

if we put  $\eta = 2$  in (74), then we obtain the following inequality:

$$(75) \left| (\varphi \circ g)\left(2^{\frac{1}{3}}a\right) - \frac{\sqrt{\pi} \left[ \left( I_{2^{\frac{1}{3}}a+}^{\frac{1}{2},2} \varphi \circ g \right)\left(3^{\frac{1}{3}}a\right) + \left( I_{2^{\frac{1}{3}}a-}^{\frac{1}{2},2} \varphi \circ g \right)(a) \right]}{2a^{\frac{3}{2}} \left[ \left(3^{\frac{1}{3}} - 2^{\frac{1}{3}}\right)^{\frac{1}{2}} S_3\left(3^{\frac{1}{3}}, 2^{\frac{1}{3}}\right) + \left(2^{\frac{1}{3}} - 1\right)^{\frac{1}{2}} S_3\left(2^{\frac{1}{3}}, 1\right) \right]} \right| \leq \frac{2}{3}a^3 M.$$

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