DE GRUYTER OPEN
FASCICULI MATHEMATICI
Nr 56 2016
DOI:10.1515/fascmath-2016-0002

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COMMON FIXED POINT OF JUNGCK-KIRK-TYPE ITERATIONS FOR NON-SELF OPERATORS IN NORMED LINEAR SPACES

Abstract. In this paper, we introduce Jungck-Kirk-multistep and Jungck-Kirk-multistep-SP iterative schemes and use their strong convergences to approximate the common fixed point of nonself operators in a normed linear Space. The Jungck-Kirk-Noor, Jungck-Kirk-SP, Jungck-Kirk-Ishikawa, Jungck-Kirk-Mann and Jungck-Kirk iterative schemes follow our results as corollaries. We also study and prove stability results of these schemes in a normed linear space. Our results generalize and unify most approximation and stability results in the literature.

Key words: Jungck-Kirk-multistep, Jungck-Kirk-multistep-SP iterative schemes, strong convergence and stability results, non-self operators.

AMS Mathematics Subject Classification: 47H10.

1. Introduction and preliminary definitions

In [2], Akewe, Okeke and Olayiwola introduced the Kirk-multistep and Kirk-multistep-SP iterative schemes and prove their strong convergences and stabilities for contractive-type operators in a normed linear space. In this work, we extend the map T used in [2] to a pair of maps S, T by introducing Jungck-Kirk-multistep and Jungck-Kirk-multistep-SP iterative schemes and use their convergences to approximate the common fixed points of a pair of nonself maps using contractive-type operators. However, there are several iterative schemes in the literature for which the common fixed points of operators have been approximated over the years by various authors.The following schemes are some of them.

Definition 1 ([8])**.** *Let X be a Banach space and Y an arbitrary set. Let $S,T : Y \to X$ be two non self mappings such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, the Jungck iterative scheme is a sequence $\{Sx_n\}_n^\infty$ defined by*

$$Sx_{n+1} = Tx_n, \quad n = 0,1,2,\ldots \tag{1}$$

Olaleru and Akewe [10] introduced the Jungck-multistep iterative scheme and use the convergence to approximate the common fixed points of those pairs of generalized contractive-like operators without assuming the injectivity of any of the operators but rather they proved their results for a pair of weakly compatible maps S, T. They used the following definition:

Definition 2 ([10])**.** *Let X be a Banach space and Y an arbitrary set. Let $S,T : Y \to X$ be two non self mappings such that $T(Y) \subseteq S(Y)$. Let $x_0 \in Y$, the Jungck-multistep iterative scheme is the sequence $\{Sx_n\}_{n=0}^\infty$ defined by*

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Ty_n^1 \\ Sy_n^i &= (1-\beta_n^i)Sx_n + \beta_n^i Ty_n^{i+1}, \quad i = 1,2,\ldots,k-2, \\ Sy_n^{k-1} &= (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1}Tx_n, \quad k \geq 2, \end{aligned} \tag{2}$$

where $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n^i\}_{n=0}^\infty$, $i = 1,2,\ldots,k-1$ are real sequences in $[0,1)$ such that $\sum_{n=0}^\infty \alpha_n = \infty$.

Remark 1. The Jungck-multistep iterative scheme (2) generalizes the Jungck-Noor [13], Jungck-Ishikawa [12], Jungck-Mann [18] iterative schemes.

Definition 3 ([10])**.** *Let X be a Banach space and Y an arbitrary set. Let $S,T : Y \to X$ be two non self mappings such that $T(Y) \subseteq S(Y)$. A point $p \in X$ is called a coincident point of a pair of self maps S, T if there exist a point q (called a point of coincidence) in X such that $q = Sp = Tp$. Self maps S and T are said to be weakly compatible if they commute at their coincidence points, that is if $Sp = Tp$ for some $p \in X$, then $STp = TSp$.*

Definition 4 ([10])**.** *Let X be a Banach space and Y an arbitrary set. Let $S,T : Y \to X$ be two non self mappings such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X. For $x,y \in Y$ and $h \in (0,1)$ we have:*

$$\|Tx - Ty\| \leq h \max\left\{\|Sx - Sy\|, \frac{\|Sx - Tx\| + \|Sy - Ty\|}{2}, \frac{\|Sx - Ty\| + \|Sy - Tx\|}{2}\right\}. \tag{3}$$

$$\|Tx - Ty\| \leq h \max\left\{\|Sx - Sy\|, \frac{\|Sx - Tx\| + \|Sy - Ty\|}{2}, \|Sx - Ty\|, \|Sy - Tx\|\right\}. \tag{4}$$

There exists a real number $\delta \in [0,1)$ *and* $L > 0$ *such that for every* $x, y \in Y$*, we have*

$$\|Tx - Ty\| \leq \delta \|Sx - Sy\| + L\|Sx - Tx\|. \tag{5}$$

There exists a real number $\delta \in [0,1)$ *and a monotone increasing function* $\varphi : R^+ \to R^+$ *such that* $\varphi(0) = 0$ *and for every* $x, y \in Y$*, we have*

$$\|Tx - Ty\| \leq \frac{\delta \|Sx - Sy\| + \varphi(\|Sx - Tx\|)}{1 + M||Sx - Tx||}, \quad M \geq 0. \tag{6}$$

$$\|Tx - Ty\| \leq \delta \|Sx - Sy\| + \varphi(\|Sx - Tx\|). \tag{7}$$

Comparing (3) - (7), we have the following: (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) but the converses are not true. For details of proof, see (Proposition 1 [10]).

The Kirk and Kirk-type iterative schemes which are of interest in this work exist in literature, for example see ([4], [6] and [9]) for further study. Chugh and Kumar [4], introduced the Kirk-Noor and Jungck-Kirk-Noor iterative processes to obtain stability results in a Banach space.

Definition 5. *Let* $(X, \|.\|)$ *be a normed linear space and* $S, T : Y \to X$ *be nonself mappings and* z *a coincidence point of* S *and* T*, that is* $Sz = Tz = p$ *(say). For any* $x_0 \in Y$*, let the sequence* $\{Sx_n\}_{n=0}^{\infty}$*, generated by the iteration procedure (2) converge to* p*. Let sequence* $\{Su_n\}_{n=0}^{\infty}$ *be an arbitrary sequence and set* $\epsilon_n = \|Su_{n+1} - f(T, u_n)\|$*, for* $n \geq 0$*. Then, the iteration procedure (2) is* $(S,T)-$*stable if and only if* $\lim_{n\to\infty} \epsilon_n = 0$ *implies that* $\lim_{n\to\infty} Su_n = p$*.*

The first stability result for $T-$stable mapping was proved by Ostrowski [15]. Several other stability results exist in literature (for details see references [1] to [5], [7], [11], [12], [14] to [18]).

We shall need the following Lemma which appear in [2], [6] and [11], to prove our results.

Lemma 1 ([2])**.** *Let* δ *be a real number satisfying* $0 \leq \delta < 1$ *and* $\{\epsilon_n\}_{n=0}^{\infty}$ *a sequence of positive numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$*, then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying* $u_{n+1} \leq \delta u_n + \epsilon_n$*,* $n = 0, 1, 2, \ldots$*, we have* $\lim_{n\to\infty} u_n = 0$*.*

Lemma 2 ([6])**.** *Let* $(X, \|.\|)$ *be a normed linear space and* $T : X \to X$ *be a selfmap of* X *satisfying (3). Let* $\varphi : R^+ \to R^+$ *be a subadditive, monotone increasing function such that* $\varphi(0) = 0$*,* $\varphi(Lu) = L\varphi(u)$*,* $L \geq 0$*,* $u \in R^+$*. Then, for all* $i \in N$*,* $L \geq 0$*, and for all* $x, y \in X$*,*

$$\|T^i x - T^i y\| \leq a^i \|x - y\| + \sum_{j=0}^{i} \tbinom{i}{j} a^{i-j} \varphi(\|x - Tx\|). \tag{8}$$

Lemma 3 ([11]). *Let $(X, \|.\|)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting maps of X satisfying (3) such that $T(Y) \subseteq S(Y)$, $\|S^2x - T(Sx)\| \leq \|Sx - Tx\|$ for all $x \in Y$ and for all $x, y \in Y$, $\|S^2x - Sy\| \leq \|Sx - Sy\|$. Let $\varphi : R^+ \to R^+$ be a sublinear, monotone increasing function such that $\varphi(0) = 0$. Let w be the coincident point of S, T, S^i, T^i (i.e $Sw = Tw = p$ and $S^iw = T^iw = p$). Then, for all $i \in N$, $L \geq 0$, and for all $x, y \in Y$,*

$$\|T^ix - T^iy\| \leq a^i\|Sx - Sy\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j}\varphi(\|Sx - Tx\|). \tag{9}$$

We now define Jungck-Kirk-multistep and Jungck-Kirk-multistep-SP iterative schemes and use their convergences to approximate the common fixed points of a pair of nonself maps using contractive-type operators. We shall also prove stability results of these schemes in a normed linear space.

Let X be a Banach space, $S, T : Y \to X$ nonself commuting maps of Y with $T(Y) \subseteq S(Y)$ and $x_0 \in Y$. Then, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned}
Sx_{n+1} &= \alpha_{n,0}Sx_n + \sum_{i=1}^{k_1} \alpha_{n,i}T^iy_n^1, \quad \sum_{i=0}^{k_1} \alpha_{n,i} = 1 \\
Sy_n^j &= \beta_{n,0}^jSx_n + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^jT^iy_n^{j+1}, \\
&\sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1, \quad j = 1, 2, \ldots, q-2, \\
Sy_n^{q-1} &= \sum_{i=0}^{k_q} \beta_{n,i}^{q-1}T^ix_n, \quad \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1, \quad q \geq 2, n \geq 0
\end{aligned} \tag{10}$$

where $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_q$, for each j, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j}^j \geq 0$, $\beta_{n,0}^j \neq 0$, for each j, $\alpha_{n,i}, \beta_{n,i}^j \in [0, 1]$ for each j and k_1, k_j are fixed integers (for each j). (10) is called Jungck-Kirk- multistep iterative scheme.

Finally, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned}
Sx_{n+1} &= \alpha_{n,0}Sy_n^1 + \sum_{i=1}^{k_1} \alpha_{n,i}T^iy_n^1, \quad \sum_{i=0}^{k_1} \alpha_{n,i} = 1 \\
Sy_n^j &= \beta_{n,0}^jSy_n^{j+1} + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^jT^iy_n^{j+1}, \\
&\sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1, \quad j = 1, 2, \ldots, q-2,
\end{aligned} \tag{11}$$

$$Sy_n^{q-1} = \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} T^i x_n, \quad \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1, \quad q \geq 2, n \geq 0$$

where $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_q$, for each j, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j}^j \geq 0$, $\beta_{n,0}^j \neq 0$, for each j, $\alpha_{n,i}, \beta_{n,i}^j \in [0,1]$ for each j and k_1, k_j are fixed integers (for each j). (11) is called Jungck-Kirk- multistep-SP iterative scheme.

Remark 2. Jungck-Kirk-multistep (10) is a generalization of Jungck-Kirk-Noor, Jungck-Kirk-Ishikawa, Jungck-Kirk-Mann and Jungck-Kirk iterative schemes, infact if $q = 3$ in (10), we have Jungck-Kirk-Noor iterative scheme [10]. If $q = 2$ in (10), we obtain Jungck-Kirk- Ishikawa iterative scheme and if $q = 2$ and $k_2 = 0$ in (10), we obtain Jungck-Kirk-Mann iterative scheme.

2. Main results I

Theorem 1. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. Let w be the coincidence point of S, T, S^i, T^i (i.e $Sw = Tw = p$ and $S^i w = T^i w = p$) for each $x_0 \in Y$, the Jungck-Kirk-multistep iterative scheme (10) converges strongly to p.*

Further, if $Y = X$ and S, T commute at p (that is S and T are weakly compatible), then p is the unique common fixed point of S, T.

Proof. In view of (10) and Lemma 3, we have

$$\|Sx_{n+1} - p\| \leq \alpha_{n,0}\|Sx_n - p\| + \sum_{i=1}^{k_1} \alpha_{n,i}\|T^i y_n^1 - Tw\|. \tag{12}$$

Using (7), with $y = y_n^1$, gives

$$\|Tw - T^i y_n^1\| \leq a^i\|Sy_n^1 - Sw\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j}\varphi(\|Sw - Tw\|). \tag{13}$$

Substituting (13) in (12), we have

$$\|Sx_{n+1} - p\| \leq \alpha_{n,0}\|Sx_n - p\| + (\sum_{i=1}^{k_1} \alpha_{n,i}a^i)\|Sy_n^1 - p\|. \tag{14}$$

We note that $\beta^j_{n,i} \in [0,1]$ for each j and k_1, k_j are fixed integers (for each j), for $n = 1, 2, \ldots$ and $1 \leq j \leq q-1$.

$$
\begin{aligned}
(15)\qquad \|Sy^1_n - p\| &\leq \beta^1_{n,0}\|Sx_n - p\| + \sum_{i=1}^{k_2} \beta^1_{n,i}\|T^i y^2_n - Tw\| \\
&\leq \beta^1_{n,0}\|Sx_n - p\| + \sum_{i=1}^{k_2} \beta^1_{n,i} a^i \|Sy^2_n - Sw\| \\
&\quad + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi(\|Sw - Tw\|) \\
&\leq \beta^1_{n,0}\|Sx_n - p\| + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)[\beta^2_{n,0}\|Sx_n - p\| \\
&\quad + (\sum_{i=1}^{k_3} \beta^2_{n,i} a^i)\|Sy^3_n - p\|] \\
&\leq \beta^1_{n,0}\|Sx_n - p\| + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)\beta^2_{n,0}\|Sx_n - p\| \\
&\quad + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)(\sum_{i=1}^{k_3} \beta^2_{n,i} a^i)\beta^3_{n,0}\|Sx_n - p\| \\
&\quad + \ldots + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)(\sum_{i=1}^{k_3} \beta^2_{n,i} a^i)(\sum_{i=1}^{k_4} \beta^3_{n,i} a^i) \\
&\quad \ldots (\sum_{i=1}^{k_{q-1}} \beta^{q-2}_{n,i} a^i)(\sum_{i=1}^{k_q} \beta^{q-1}_{n,i} a^i)\beta^q_{n,0}\|Sx_n - p\|.
\end{aligned}
$$

(15) holds, since $Sw = Tw = p$ and $\varphi(0) = 0$. Substituting (15) in (4),

$$
\begin{aligned}
(16)\qquad \|Sx_{n+1} - p\| &\leq \alpha_{n,0}\|Sx_n - p\| \\
&\quad + (\sum_{i=1}^{k_1} \alpha_{n,i} a^i)\Big[\beta^1_{n,0}\|Sx_n - p\| \\
&\quad + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)\beta^2_{n,0}\|Sx_n - p\| \\
&\quad + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)(\sum_{i=1}^{k_3} \beta^2_{n,i} a^i)\beta^3_{n,0}\|Sx_n - p\| \\
&\quad + \ldots + (\sum_{i=1}^{k_2} \beta^1_{n,i} a^i)(\sum_{i=1}^{k_3} \beta^2_{n,i} a^i)(\sum_{i=1}^{k_4} \beta^3_{n,i} a^i)\beta^4_{n,0}
\end{aligned}
$$

$$\ldots (\sum_{i=1}^{k_{q-1}} \beta_{n,i}^{q-2} a^i)(\sum_{i=1}^{k_{q-1}} \beta_{n,i}^{q} a^i)\beta_{n,0}^{q}\|Sx_n - p\|\Big]$$

$$< \Big[\alpha_{n,0} + (1-\alpha_{n,0})\beta_{n,0}^1 + (1-\alpha_{n,0})(1-\beta_{n,0}^1)$$
$$+ (1-\alpha_{n,0})(1-\beta_{n,0}^1)(1-\beta_{n,0}^2)$$
$$+ (1-\alpha_{n,0})(1-\beta_{n,0}^1)(1-\beta_{n,0}^2)(1-\beta_{n,0}^3)$$
$$+ \ldots + (1-\alpha_{n,0})(1-\beta_{n,0}^1)(1-\beta_{n,0}^2)(1-\beta_{n,0}^3)$$
$$\ldots (1-\beta_{n,0}^{q-1})(1-\beta_{n,0}^{q})\Big]\|Sx_n - p\|.$$

Since $a^i \in [0,1)$ and $\sum_{i=1}^{k_1} \alpha_{n,i} = \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j = 1$ for $j = 1,2,3,\ldots,q-1$. Hence, $\lim_{n\to\infty} \|Sx_{n+1} - p\| = 0$. That is $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Next we show that p is unique. Suppose there exists another point of coincidence p^*, then there is an $w^* \in X$ such that $Tw^* = Sw^* = p^*$. Hence, using (7) we have $\|w - w^*\| = \|T^iw - T^iw^*\| \leq a^i\|Sx_n - Sw\| + \sum_{j=0}^{i}\binom{i}{j}a^{i-j}\varphi(\|Sw - Tw\|)$ hence $w = w^*$ and so p is unique. Since S, T are weakly compatible, then $TSw = STw$ and so $Tp = Sp$. Hence p is a coincidence point of S, T and since the coincidence point is unique, then $p = w$ and hence $Sp = Tp = p$ and therefore p is the unique common fixed point of S, T. This ends the proof. ∎

Theorem 1 leads to the following corollaries:

Corollary 1. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. Let w be the coincidence point of S, T, S^i, T^i (i.e $Sw = Tw = p$ and $S^iw = T^iw = p$) for each $x_0 \in Y$,*

(i) the Jungck-Kirk-Noor iterative scheme converges strongly to p;
(ii) the Jungck-Kirk-Ishikawa iterative scheme converges strongly to p;
(iii) the Jungck-Kirk-Mann iterative scheme converges strongly to p;
(iv) the Jungck-Kirk iterative scheme converges strongly to p.

Further, if $Y = X$ and S, T commute at p (that is S and T are weakly compatible), then p is the unique common fixed point of S, T.

Theorem 2. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. Let w be the coincidence point of S, T, S^i, T^i (i.e $Sw = Tw = p$ and $S^iw = T^iw = p$) for each $x_0 \in Y$, the Jungck-Kirk-multistep-SP iterative scheme (7) converges strongly to p.*

Further, if $Y = X$ and S, T commute at p (that is S and T are weakly compatible), then p is the unique common fixed point of S, T.

Proof. By similar approach in the proof of Theorem 1, the result of Theorem 2 follows. ■

Theorem 2 yields the following corollaries:

Corollary 2. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (3) holds with $T(Y) \subseteq S(Y)$. Let w be the coincidence point of S, T, S^i, T^i (i.e $Sw = Tw = p$ and $S^i w = T^i w = p$) for each $x_0 \in Y$, the*

(i) Jungck-Kirk-Noor-SP iterative scheme converges strongly to p.

(ii) Jungck-Kirk-Mann iterative scheme converges strongly to p.

(iii) Jungck-Kirk iterative scheme converges strongly to p.

Further, if $Y = X$ and S, T commute at p (that is S and T are weakly compatible), then p is the unique common fixed point of S, T.

2. Main results II

Theorem 3. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. For each $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Kirk-multi-step-SP iterative scheme (11) converging strongly to p (i.e $Sp = Tp = p$ and $S^i p = T^i p = p$) with $0 < \alpha < \alpha_{n,i}$, $0 < \beta < \beta_{n,i}^j$ for each $j = 1, 2, ..., q-1$, and all n. Then, the Jungck-Kirk-multistep-SP iterative scheme (11) is $S, T-$ stable.*

Proof. Let $\{Sz_n\}_{n=0}^{\infty}$, $\{Su_n^i\}_{n=0}^{\infty}$, for $i = 1, 2, \dots, k-1$ be real sequences in E. Let $\epsilon_n = \|Sz_{n+1} - \alpha_{n,0} Su_n^1 - \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1\|$, $n = 0, 1, 2, ...$, where $Su_n^j = \beta_{n,0}^j Su_n^{j+1} + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j T^i u_n^{j+1}$, $\sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1$, $j = 1, 2, ..., q-2$, $Su_n^{q-1} = \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} T^i z_n$, $\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1$, $q \geq 2$ and let $\lim_{n\to\infty} \epsilon_n = 0$. Then we shall prove that $\lim_{n\to\infty} Sz_n = p$ using the contractive mappings satisfying condition (7). That is,

$$
\begin{aligned}
(17)\quad \|Sz_{n+1} - p\| &\leq \|Sz_{n+1} - \alpha_{n,0} Su_n^1 \\
&\quad - \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1\| + \|\alpha_{n,0} Su_n^1 + \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 - p\| \\
&\leq \epsilon_n + \alpha_{n,0}\|Su_n^1 - p\| + \big(\sum_{i=1}^{k_1} \alpha_{n,i}\big) a^i \|Su_n^1 - Sp\| \\
&\quad + \sum_{j=0}^{i} \tbinom{i}{j} a^{i-j} \varphi(\|Sp - Tp\|)\|
\end{aligned}
$$

$$= \Big(\sum_{i=0}^{k_1} \alpha_{n,i} a^i\Big) \|Su_n^1 - p\| + \epsilon_n.$$

$$\begin{aligned}
(18)\ \|Su_n^1 - p\| &= \|\beta_{n,0} S u_n^2 + \sum_{i=1}^{k_2} \beta_{n,i} T^i u_n^2 - \sum_{i=0}^{k_2} \beta_{n,i} T^i p\| \\
&= \|\beta_{n,0}(Su_n^2 - p) + \sum_{i=1}^{k_2} \beta_{n,i}(T^i u_n^2 - T^i p)\| \\
&\leq \beta_{n,0}\|Su_n^2 - p\| + \big(\sum_{i=1}^{k_2} \beta_{n,i}\big)\Big[a^i \|Su_n^2 - Sp\| \\
&\quad + \sum_{j=0}^{i} \tbinom{i}{j} a^{i-j} \varphi(\|Sp - Tp\|)\Big] \\
&\leq \big(\sum_{i=0}^{k_2} \beta_{n,i} a^i\big)\Big[\beta_{n,0}\|Su_n^3 - p\| + \big(\sum_{i=1}^{k_3} \beta_{n,i}\big)[a^i \|Su_n^3 - Sp\| \\
&\quad + \sum_{j=0}^{i} \tbinom{i}{j} a^{i-j} \varphi(\|Sp - Tp\|)]\Big] \\
&\leq (\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i)(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i)(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i) \\
&\quad ... (\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2} a^i)(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} a^i)\|Sz_n - p\|.
\end{aligned}$$

(18) holds, since $Sp = Tp = p$ and $\varphi(0) = 0$. Substituting (18) in (17), we have

$$\begin{aligned}
(19)\quad \|Sz_{n+1} - p\| &\leq \big(\sum_{i=0}^{k_1} \alpha_{n,i} a^i\big)(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i)(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i)(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i) \\
&\quad \ldots (\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2} a^i)(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} a^i)\|Sz_n - p\| + \epsilon_n.
\end{aligned}$$

Since $a^i \in [0, 1)$ and $\sum_{i=1}^{k_1} \alpha_{n,i} = \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j = 1$ for $j = 1, 2, 3...q - 1$ and

$$\begin{aligned}
(20)\qquad & (\sum_{i=0}^{k_1} \alpha_{n,i} a^i)(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i)(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i)(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i) \\
&\quad \ldots (\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2} a^i)(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} a^i)
\end{aligned}$$

$$< (\sum_{i=0}^{k_1} \alpha_{n,i})(\sum_{i=0}^{k_2} \beta^1_{n,i})(\sum_{i=0}^{k_3} \beta^2_{n,i})(\sum_{i=0}^{k_4} \beta^3_{n,i})...(\sum_{i=0}^{k_{q-1}} \beta^{q-2}_{n,i})(\sum_{i=0}^{k_q} \beta^{q-1}_{n,i}) = 1.$$

Let

$$\delta = (\sum_{i=0}^{k_1} \alpha_{n,i}a^i)(\sum_{i=0}^{k_2} \beta^1_{n,i}a^i)(\sum_{i=0}^{k_3} \beta^2_{n,i}a^i)(\sum_{i=0}^{k_4} \beta^3_{n,i}a^i) \dots (\sum_{i=0}^{k_{q-1}} \beta^{q-2}_{n,i}a^i)(\sum_{i=0}^{k_q} \beta^{q-1}_{n,i}a^i)$$

then, $\delta < 1$. Hence

$$\|Sz_{n+1} - p\| \le \delta \|Sz_n - p\| + \epsilon_n. \tag{21}$$

Using Lemma 3 in (21), we have $\lim_{n\to\infty} Sz_n = p$. Conversely, let $\lim_{n\to\infty} z_n = p$, we show that $\lim_{n\to\infty} \epsilon_n = 0$ as follows:

$$\begin{aligned} \epsilon_n &= \|Sz_{n+1} - \alpha_{n,0}Su^1_n - \sum_{i=0}^{k_1} \alpha_{n,i}T^iu^1_n\| \\ &\le \|Sz_{n+1} - p\| + \|p - \alpha_{n,0}Su^1_n - \sum_{i=0}^{k_1} \alpha_{n,i}T^iu^1_n\| \\ &\le \|Sz_{n+1} - p\| + \alpha_{n,0}\|Su^1_n - p\| + \sum_{i=1}^{k_1} \alpha_{n,i}\|T^ip - T^iu^1_n\| \\ &\le \|Sz_{n+1} - p\| + \alpha_{n,0}\|Su^1_n - p\| + (\sum_{i=1}^{k_1} \alpha_{n,i})[a^i\|Su^1_n - Sp\| \\ &\quad + \sum_{j=0}^{i} \tbinom{i}{j} a^{i-j}\varphi(\|Sp - Tp\|)\|] = \|Sz_{n+1} - p\| \\ &\quad + (\sum_{i=0}^{k_1} \alpha_{n,i}a^i)\|Su^1_n - p\|. \end{aligned} \tag{22}$$

Substituting $\|Su^1_n - p\|$ that is (18) in (22), we have

$$\begin{aligned} \epsilon_n \le\ & \|Sz_{n+1} - p\| \\ & + (\sum_{i=0}^{k_1} \alpha_{n,i}a^i) \sum_{i=0}^{k_1} \alpha_{n,i}a^i)(\sum_{i=0}^{k_2} \beta^1_{n,i}a^i)(\sum_{i=0}^{k_3} \beta^2_{n,i}a^i)(\sum_{i=0}^{k_4} \beta^3_{n,i}a^i) \\ & \dots (\sum_{i=0}^{k_{q-1}} \beta^{q-2}_{n,i}a^i)(\sum_{i=0}^{k_q} \beta^{q-1}_{n,i}a^i)\|Sz_n - p\|. \end{aligned} \tag{23}$$

Using (20), (23) becomes $\epsilon_n \leq \|Sz_{n+1} - p\| + \delta\|Sz_n - p\|$. Hence, using $\lim_{n\to\infty} \|Sz_n - p\| = 0$ (by our assumption), we have $lim_{n\to\infty}\epsilon_n = 0$. Therefore the Jungck-Kirk- multistep-SP iterative scheme (11) is S, T-stable. This ends the proof. ■

Theorem 3 yields the following corollary:

Corollary 3. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. For each $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Kirk-SP, Jungck-Kirk-Mann and Jungck-Kirk iterative schemes respectively converging strongly to p (i.e. $Sp = Tp = p$ and $S^ip = T^ip = p$) with $0 < \alpha < \alpha_{n,i}$, $0 < \beta^j < \beta^j_{n,i}$ for each $j = 1, 2, ..., q-1$, and all n. Then,*

(i) the Jungck-Kirk-SP iterative scheme is $S, T-$ stable;

(ii) the Jungck-Kirk-Mann iterative scheme is $S, T-$ stable;

(iii) the Jungck-Kirk iterative scheme is $S, T-$ stable.

Theorem 4. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. For each $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Kirk-multistep iterative scheme (10) converging strongly to p (i.e. $Sp = Tp = p$ and $S^ip = T^ip = p$) with $0 < \alpha < \alpha_{n,i}$, $0 < \beta < \beta^j_{n,i}$ for each $j = 1, 2, ..., q-1$, and all n. Then, the Jungck-Kirk-multistep iterative scheme (10) is $S, T-$ stable.*

Proof. By similar approach in the proof of Theorem 3, the result of Theorem 4 follows. ■

Theorem 4 yields the following corollaries:

Corollary 4. *Let $(X, ||.||)$ be a normed linear space and $S, T : Y \to X$ be nonself commuting mappings for an arbitrary set Y such that (7) holds with $T(Y) \subseteq S(Y)$. For each $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Kirk-Noor iterative scheme converging strongly to p (i.e $Sp = Tp = p$ and $S^ip = T^ip = p$) with $0 < \alpha < \alpha_{n,i}$, $0 < \beta < \beta^j_{n,i}$ for each $j = 1, 2,$ and all n. Then,*

(i) the Jungck-Kirk-Noor iterative scheme is $S, T-$ stable;

(ii) the Jungck-Kirk-Ishikawa iterative scheme is $S, T-$ stable;

(iii) the Jungck-Kirk-Mann iterative scheme is $S, T-$ stable;

(iv) the Jungck-Kirk iterative scheme is $S, T-$ stable.

Acknowledgments. The authors wish to thank Prof. J.O. Olaleru for giving useful comments/suggestions leading to the improvement of this paper and for supervising their Ph.D. Thesis.

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Received on 23.05.2014 and, in revised form, on 02.02.2016.