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Renukadevi S. Dyavanal and Madhura M. Mathai

ENTIRE AND MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

Abstract. In this paper, we shall investigate the existence of finite order entire and meromorphic solutions of linear difference equation of the form

$$f^n(z) + p(z)f^{n-2}(z) + L(z,f) = h(z)$$

where $L(z,f)$ is linear difference polynomial in $f(z)$, $p(z)$ is non-zero polynomial and $h(z)$ is a meromorphic function of finite order. We also consider finite order entire solution of linear difference equation of the form

$$f^n(z) + p(z)L(z,f) = r(z)e^{q(z)}$$

where $r(z)$ and $q(z)$ are polynomials.

Key words: finite order, linear difference equation, meromorphic function, Nevanlinna theory.

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1. Introduction and main results

In this paper, a meromorphic function always mean it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notations in the Nevanlinna theory of entire and meromorphic functions as explained in ([5], [6], [14]). The values $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ and $T(r,f)$ denote the proximity function, the counting function, the reduced counting function and the characteristic function of $f(z)$, respectively

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta,$$

$$N(r,f) := \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r,$$

$$\begin{aligned}\overline{N}(r,f) &:= \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt + \overline{n}(0,f)\log r,\\ T(r,f) &= m(r,f)+N(r,f),\end{aligned}$$

where $\log^+|x| = \max(\log x, 0)$ for all $x \geq 0$, $n(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$, counting multiplicities; and $\overline{n}(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$, ignoring multiplicities.

Let $f(z)$ and $\alpha(z)$ be two mermorphic functions. We say that $\alpha(z)$ is a small function with respect to $f(z)$, if $T(r,\alpha(z)) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$, outside of a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{(1,r]\cap E}\frac{dt}{t} < \infty$. The order $\rho(f)$ of a meromorphic function $f(z)$ is defined as

$$\rho(f) = \overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log r}.$$

The study of the existence and uniqueness of entire solutions of finite order of non-linear differential equation of the form

$$L(f) - p(z)f^n(z) = h(z)$$

was started by Yang [11] in 2001, where $L(f)$ is a linear differential polynomial in $f(z)$ with polynomial coefficients, $p(z)$ is a non-vanishing polynomial, $h(z)$ is an entire function and $n \geq 3$ is an integer.

Later on, In 2010, Yang and Laine [12] proved the following Theorem.

Theorem A ([12])**.** *Let $n \geq 4$ be an integer, $M(z,f)$ be a linear differential-difference polynomial of $f(z)$, not vanishing identically, and $h(z)$ be a meromorphic function of finite order. Then the differential-difference equation*

$$f^n + M(z,f) = h(z)$$

possesses atmost one admissible transcendental entire solution of finite order such that all coefficients of $M(z,f)$ are small functions of $f(z)$. If such a solution $f(z)$ exists, then $f(z)$ is of the same order as $h(z)$.

Next, X. Qi and L. Yang [9] in 2013 proved the following result for the existence of finite order meromorphic solution of the difference equation of the form

$$f^n(z) + L(z,f) = h(z). \tag{1}$$

Theorem B ([9])**.** *Let $L(z,f) = a_0f(z)+a_1f(z+c_1)+\cdots+a_kf(z+c_k)$ be a linear difference polynomial in $f(z)$ with small meromorphic functions as the coefficients and c_i are constants, $i = 1,2,\cdots,k$ and $h(z)$ be a meromorphic function of finite order. If $f(z)$ is a finite order meromorphic solution* *of the difference equation (1) satisfying $N(r,f)=S(r,f)$ and $n\geq 4$ be an integer, then one of the following statements hold:*

(a) *Equation (1) has $f(z)$ as its unique transcendental meromorphic solution with finite order such that $N(r,f)=S(r,f)$;*

(b) *Equation (1) has exactly n transcendental meromorphic solutions f_j $(j=1,2,\cdots,n)$, with finite order such that $N(r,f_j)=S(r,f_j)$.*

Later, In the year 2014, X. Qi, J. Dou and L. Yang [10] obtained the following result for the non-linear difference equation of the form

$$f(z)^n+p(z)\left(\Delta_c f\right)^m=r(z)e^{q(z)} \tag{2}$$

where $\Delta_c f=f(z+c)-f(z)$ and c is a non-zero constant.

Theorem C ([10]). *Consider the non-linear difference equation of the form (2), where $p(z)\not\equiv 0$, $q(z)$, $r(z)$ are polynomials, n and m are positive integers. Suppose that $f(z)$ is a transcendental entire function of finite order, not of period c. If $n>m$, then $f(z)$ cannot be a solution of (2).*

In this paper, we prove the following theorems for linear difference equations.

Theorem 1. *Let $n\geq 4$ be an integer, $L(z,y)=a_0y(z)+a_1y(z+c_1)+\cdots+a_ky(z+c_k)$ be a non-zero linear difference polynomial of $y(z)$, $h(z)$ be a meromorphic function of finite order and $p(z)$ be a non-zero polynomial. Then, there exists at most one transcendental entire function $f(z)$ of finite order such that*

$$f^n(z)+p(z)f^{n-2}+L(z,f)=h(z),\quad L(z,f)\not\equiv 0 \tag{3}$$

and such that all coefficients a_λ of $L(z,y)$ are small functions of $f(z)$ in the sense that $T\left(r,a_\lambda\right)=S(r,f)$ and c_i, $i=1,2,\cdots,k$ are constants. If such solution $f(z)$ exists than $f(z)$ has same order as $h(z)$.

Theorem 2. *Let $n\geq 4$ be an integer, $L(z,y)$ be defined as in Theorem 1, $h(z)$ be a meromorphic function of finite order and $p(z)$ be a non-zero polynomial. If there exists a finite order transcendental meromorphic solution $f(z)$ of (3) satisfying $N(r,f)=S(r,f)$, then $f(z)$ is an unique solution.*

Theorem 3. *Let $n>1$ be an integer. Let $L(z,y)$ be defined as in Theorem 1 and $p(z)\not\equiv 0$, $q(z)$ and $r(z)$ be polynomials. Consider the linear difference equation of the form*

$$f(z)^n+p(z)L(z,f)=r(z)e^{q(z)}. \tag{4}$$

Then a transcendental entire function $f(z)$ of finite order cannot be a solution of (4).

2. Some lemmas

We need the following lemmas to prove our results.

Lemma 1 ([3]). *Let $f(z)$ be a non-constant meromorphic function of finite order, and $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{\delta}}\right)$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2 ([1]). *Let $f(z)$ be a finite order meromorphic function, then for each $\epsilon > 0$,*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho(f)-1+\epsilon}) + O(\log r)$$

and

$$\rho(f(z+c)) = \rho(f(z)).$$

Thus, if $f(z)$ is a transcendental meromorphic function with finite order, then

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Lemma 3 ([12]). *Let $f(z)$ be a transcendental meromorphic solution of finite order $\rho(f)$ of a difference equation of the form*

$$H(z,f)P(z,f) = Q(z,f),$$

where $H(z,f)$, $P(z,f)$ and $Q(z,f)$ are difference polynomials in $f(z)$ such that the total degree of $H(z,f)$ in $f(z)$ and its shifts is n, and that the corresponding total degree of $Q(z,f)$ is $\leq n$. If $H(z,f)$ contains just one term of maximal total degree, then for any $\epsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho(f)-1+\epsilon}) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Yang and Laine in ([12]) further pointed out the following.

Remark 1. If in the above lemma, $H(z,f) = f^n$, then a similar conclusion holds, if $P(z,f)$, $Q(z,f)$ are differential-difference polynomials in f.

Lemma 4 ([14]). *Let $f_j(z)(j = 1,2,3)$ be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j(z) \equiv 1.$$

If $f_1(z)$ is not a constant, and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ and $T(r) = \max_{1\leq j\leq 3}\{T(r, f_j)\}$, then either $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

3. Proof of the theorems

Proof. of Theorem 1. We first prove that $\rho(h) = \rho(f)$ for all entire solutions of finite order of (3). Since the inequality $\rho(h) \leq \rho(f)$ trivially holds, now suppose that $\rho(h) < \sigma < \rho(f) = \rho$. (3) can be written as

$$f^{n-1} = \frac{h}{f} - p(z)f^{n-3} - \frac{L(z, f)}{f}. \tag{5}$$

Consider,

$$(n-1)T(r, f) = (n-1)m(r, f) + (n-1)N(r, f) = (n-1)m(r, f) \tag{6}$$

Using (5), (6) reduces to

$$(n-1)T(r, f) \leq m\left(r, \frac{h}{f}\right) + m\left(r, p(z)f^{n-3}\right) + m\left(r, \frac{L(z, f)}{f}\right) + S(r, f)$$

From Lemma 1, we deduce

$$\begin{aligned}(n-1)T(r, f) &\leq m(r, h) + T(r, f) + (n-3)T(r, f) + O\left(r^{\rho-1+\epsilon}\right) + S(r, f)\\ &\leq T(r, h) + T(r, f) + (n-3)T(r, f) + O\left(r^{\rho-1+\epsilon}\right) + S(r, f)\end{aligned}$$

$$\Rightarrow T(r, f) \leq T(r, h) + O\left(r^{\rho-1+\epsilon}\right) + S(r, f) \tag{7}$$

Since $\rho(h) < \sigma < \rho(f) = \rho$, by definition of order, we have $T(r, h) < r^{\sigma}$. Thus, (7) reduces to

$$\begin{aligned}T(r, f) &\leq r^{\sigma} + O\left(r^{\rho-1+\epsilon}\right) + S(r, f)\\ &\leq r^{\sigma+\epsilon} + O\left(r^{\rho-1+2\epsilon}\right) + S(r, f)\end{aligned}$$

for all r sufficiently large, outside of an exceptional set of finite logarithmic measure, provided ϵ has been chosen small enough and removing the exceptional set, we get

$$\rho(f) \leq \max\{\rho - 1 + 2\epsilon, \sigma + \epsilon\} < \rho \quad \Rightarrow \quad \rho(f) < \rho$$

which is a contradiction. Hence, we have proved that $\rho(f) = \rho(g) = \rho(h)$. Next, we prove that (3) possesses at most one admissible transcendental entire solution of finite order. Now, assume that f and g are two distinct finite order transcendental entire solutions of (3). Thus, we have

$$f^n + p(z)f^{n-2} + L(z, f) = g^n + p(z)g^{n-2} + L(z, g). \tag{8}$$

Clearly, $\rho(f) = \rho(g)$. Since the difference polynomial L is linear, (8) can be written as

$$(f^n - g^n) + p(z)\left(f^{n-2} - g^{n-2}\right) = L(z, g) - L(z, f) = L(z, g - f)$$

Let

$$\begin{aligned} F &= \frac{f^n - g^n}{f - g} + p(z)\frac{f^{n-2} - g^{n-2}}{f - g} = \frac{-L(z, f - g)}{f - g} \\ \Rightarrow F &= \frac{f^n - g^n}{f - g} + p(z)\frac{f^{n-2} - g^{n-2}}{f - g} \\ &= \prod_{p=1}^{n-1}(f - \eta_p g) + p(z)\prod_{m=1}^{n-3}(f - \gamma_m g) = \frac{-L(z, f - g)}{f - g} \end{aligned} \tag{9}$$

is an entire function, here $\eta_1, \eta_2, \cdots, \eta_{n-1}$ are the distinct roots $\neq 1$ of the equation $z^n = 1$ and $\gamma_1, \gamma_2, \cdots, \gamma_{n-3}$ are distinct roots $\neq 1$ of the equation $z^{n-2} = 1$. From (9) and Lemma 1, we obtain

$$\begin{aligned} T(r, F) &= m(r, F) + N(r, F) = m(r, F) = m\left(r, \frac{L(z, g - f)}{f - g}\right) \\ &= O\left(r^{\rho(f-g)-1+\epsilon}\right) + S(r, f) + S(r, g) \\ &\leq O\left(r^{\rho(f)-1+\epsilon}\right) + S(r, f) = S_\rho(r, f) \end{aligned}$$

where $\epsilon > 0$ is arbitrary and sufficiently small.

$$\Rightarrow N\left(r, \frac{1}{F}\right) = S_\rho(r, f)$$

$$\Rightarrow \sum_{p=1}^{n-1} N\left(r, \frac{1}{f - \eta_p g}\right) = S_\rho(r, f) \quad \text{and} \quad \sum_{m=1}^{n-3} N\left(r, \frac{1}{f - \gamma_m g}\right) = S_\rho(r, f)$$

$$\Rightarrow N\left(r, \frac{1}{f - \eta_p g}\right) = S_\rho(r, f) \quad \text{and} \quad N\left(r, \frac{1}{f - \gamma_m g}\right) = S_\rho(r, f)$$

holds for all $p = 1, 2, \cdots, n-1$ and $m = 1, 2, \cdots, n-3$. Since $\frac{1}{\frac{f}{g} - \eta_p} = g\frac{1}{f - \eta_p g}$ and $\frac{1}{\frac{f}{g} - \gamma_m} = g\frac{1}{f - \gamma_m g}$, we obtain

$$N\left(r, \frac{1}{\frac{f}{g} - \eta_p}\right) = S_\rho(r, f) \quad \text{and} \quad N\left(r, \frac{1}{\frac{f}{g} - \gamma_m}\right) = S_\rho(r, f)$$

holds for all $p = 1, 2, \cdots, n-1$ and $m = 1, 2, \cdots, n-3$. Using $n \geq 4$ and applying the Second main theorem ([14], Theorem 1.8) for $\psi = \frac{f}{g}$, we get

$$\begin{aligned}
(n-3)T\left(r, \frac{f}{g}\right) &\leq \sum_{p=1}^{n-1} N\left(r, \frac{1}{\frac{f}{g} - \eta_p}\right) + S\left(r, \frac{f}{g}\right) \\
&\leq \sum_{p=1}^{n-1} N\left(r, \frac{1}{\frac{f}{g} - \eta_p}\right) + \sum_{m=1}^{n-3} N\left(r, \frac{1}{\frac{f}{g} - \gamma_m}\right) + S\left(r, \frac{f}{g}\right) \\
&\leq S_\rho(r, f) + S\left(r, \frac{f}{g}\right) \\
\Rightarrow \quad T\left(r, \frac{f}{g}\right) &= S_\rho(r, f) \\
\Rightarrow \quad T(r, \psi) = T\left(r, \frac{f}{g}\right) &= S_\rho(r, f)
\end{aligned}$$

Now consider,

$$\begin{aligned}
T(r, f) &= T\left(r, \frac{f}{g} g\right) \leq T\left(r, \frac{f}{g}\right) + T(r, g) \\
\Rightarrow \quad T(r, f) &\leq T(r, g) + S_\rho(r, f) \\
\Rightarrow \quad T(r, f) &= T(r, g) + S_\rho(r, f) \qquad (10)
\end{aligned}$$

From (9), we have

$$\begin{aligned}
F &= g^{n-1} \prod_{p=1}^{n-1} \left(\frac{f}{g} - \eta_p\right) + p(z) g^{n-3} \prod_{m=1}^{n-3} \left(\frac{f}{g} - \gamma_m\right) \\
\Rightarrow \quad g^{n-1} &= \frac{F}{\prod_{p=1}^{n-1} \left(\frac{f}{g} - \eta_p\right)} - \frac{p(z) g^{n-3} \prod_{m=1}^{n-3} \left(\frac{f}{g} - \gamma_m\right)}{\prod_{p=1}^{n-1} \left(\frac{f}{g} - \eta_p\right)} \qquad (11)
\end{aligned}$$

provided ψ is not identically equal to η_p and γ_m, for $p = 1, 2, \cdots, n-1$ and $m = 1, 2, \cdots, n-3$. From (10) and (11), we obtain

$$\begin{aligned}
(n-1)T(r, f) &= (n-1)T(r, g) + S_\rho(r, f) \\
&\leq T(r, F) + T\left(r, \prod_{p=1}^{n-1} \left(\frac{f}{g} - \eta_p\right)^{-1}\right) + T(r, p(z)) \\
&\quad + (n-3)T(r, g) + T\left(r, \prod_{m=1}^{n-3} \left(\frac{f}{g} - \gamma_m\right)\right) \\
&\quad + T\left(r, \prod_{p=1}^{n-1} \left(\frac{f}{g} - \eta_p\right)^{-1}\right) + S_\rho(r, f)
\end{aligned}$$

$$(12) \qquad \Rightarrow \quad (n-1)T(r,f) \leq (n-3)T(r,g) + S_\rho(r,f) + S(r,f)$$

Since $\rho(f) = \rho(g)$ which implies $T(r,f) = T(r,g)$. Equation (12) reduces to

$$\begin{aligned} (n-1)T(r,f) &\leq (n-3)T(r,f) + S_\rho(r,f) + S(r,f) \\ \Rightarrow \quad 2T(r,f) &\leq S_\rho(r,f) \\ \Rightarrow \quad T(r,f) &= S_\rho(r,f) \end{aligned}$$

which is a contradiction. Therefore, we must have $\psi = \eta_p$ for some $p = 1,2,\cdots,n-1$ and $\psi = \gamma_m$ for some $m = 1,2,\cdots,n-3$, which implies $f = \eta_p g$ and $f = \gamma_m g$ which again implies $f^n + p(z)f^{n-2} = g^n + p(z)g^{n-2}$ and $L(z,f) = L(z,g)$. Since $f = \eta_p g$ and by the linearity of the difference polynomial L, we get $L(z,f) = \eta_p L(z,g)$, since $\eta_p \neq 1$, we get a contradiction to $L(z,f) = L(z,g)$. Hence (3) possesses atmost one admissible transcendental entire solution of finite order such that all coefficients of $L(z,f)$ are small functions of $f(z)$. ■

Proof. of Theorem 2. Suppose $f_1(z)$ and $f_2(z)$ are two distinct finite order transcendental meromorphic solutions of (3) such that $N(r,f_i) = S(r,f_i)$ $(i = 1,2)$. From (3), we obtain

$$(13) \qquad \begin{aligned} G(z) &= \frac{f_1^n - f_2^n + p(z)\left(f_1^{n-2} - f_2^{n-2}\right)}{f_1 - f_2} \\ &= \frac{L(z,f_2) - L(z,f_1)}{f_1 - f_2} = \frac{L(z,f_1) - L(z,f_2)}{f_2 - f_1} \\ \Rightarrow \quad G(z) &= \prod_{p=1}^{n-1}(f_1 - \eta_p f_2) + p(z)\prod_{m=1}^{n-3}(f_1 - \gamma_m f_2) \end{aligned}$$

where,

$$\begin{aligned} G(z) = {} & (f_1 - \eta_1 f_2)(f_1 - \eta_2 f_2)\cdots(f_1 - \eta_{n-1} f_2) \\ & + p(z)(f_1 - \gamma_1 f_2)(f_1 - \gamma_2 f_2)\cdots(f_1 - \gamma_{n-3} f_2). \end{aligned}$$

Here $\eta_p \neq 1 (p = 1,2,\cdots,n-1)$ are the distinct n^{th} roots of the unity and $\gamma_m \neq 1 (m = 1,2,\cdots,n-3)$ are the distinct $(n-2)^{th}$ roots of the unity. Using (13) and Lemma 1, we obtain

$$\begin{aligned} m(r,G) &= m\left(r, \frac{L(z,f_1) - L(z,f_2)}{f_2 - f_1}\right) = m\left(r, \frac{L(z,f_1 - f_2)}{f_2 - f_1}\right) \\ &= S(r,f_1) + S(r,f_2) \end{aligned}$$

Since by hypothesis $N(r,f_i) = S(r,f_i)(i = 1,2)$, it follows that $N(r,G) = S(r,f_1) + S(r,f_2)$. Hence, we get

$$(14) \qquad T(r,G) = m(r,G) + N(r,G) = S(r,f_1) + S(r,f_2)$$

Now, we will discuss the following two cases for $G(z)$.

Case 1. If $G(z) \equiv 0$. From (13), we get

$$L(z, f_1) - L(z, f_2) = 0 \quad \Rightarrow \quad L(z, f_1) = L(z, f_2)$$

$$\begin{aligned} \Rightarrow\ & a_0 f_1(z) + a_1(z) f_1(z + c_1) + \cdots + a_k(z) f_1(z + c_k) \\ & = a_0 f_2(z) + a_1(z) f_2(z + c_1) + \cdots + a_k(z) f_2(z + c_k). \end{aligned} \tag{15}$$

Let $h = \frac{f_1}{f_2}$, then substituting $f_1 = h f_2$, equation (15) can be written as

$$\begin{aligned} & a_0(z) f_2(z)(h(z) - 1) + a_1(z) f_2(z + c_1)(h(z + c_1) - 1) \\ & \quad + \cdots + a_k(z) f_2(z + c_k)(h(z + c_k) - 1) = 0 \end{aligned}$$

$$\begin{aligned} & \Rightarrow h(z) \equiv 1 \text{ and } h(z + c_i) \equiv 1, \text{ where } c_i = 1, 2, \cdots, k. \\ & \Rightarrow f_1(z) \equiv f_2(z) \text{ and } f_1(z + c_i) \equiv f_2(z + c_i), \text{ where } c_i = 1, 2, \cdots, k. \end{aligned}$$

that is, $f_1 = f_2$. Thus (3) has an unique solution.

Case 2. If $G(z) \not\equiv 0$. Consider,

$$\begin{aligned} G(z) &= (f_1 - \eta_1 f_2)(f_1 - \eta_2 f_2) \cdots (f_1 - \eta_{n-1} f_2) \\ & \quad + p(z)(f_1 - \gamma_1 f_2)(f_1 - \gamma_2 f_2) \cdots (f_1 - \gamma_{n-3} f_2) \\ \Rightarrow G(z) &= f_2^{n-1} Q_1\left(\frac{f_1}{f_2}\right) + p(z) f_2^{n-3} Q_2\left(\frac{f_1}{f_2}\right) \end{aligned} \tag{16}$$

where $Q_1\left(\frac{f_1}{f_2}\right)$ is a polynomial in $\frac{f_1}{f_2}$ of degree $n-1$ and $Q_2\left(\frac{f_1}{f_2}\right)$ is a polynomial in $\frac{f_1}{f_2}$ of degree $n-3$ with constant coefficients. (16) can be written as

$$\begin{aligned} G(z) &= f_2^{n-1}\left[Q_1\left(\frac{f_1}{f_2}\right) + \frac{p(z)}{f_2^2} Q_2\left(\frac{f_1}{f_2}\right)\right] \\ \Rightarrow \frac{G(z)}{f_2^{n-1}} &= \left[Q_1\left(\frac{f_1}{f_2}\right) + \frac{p(z)}{f_2^2} Q_2\left(\frac{f_1}{f_2}\right)\right] \end{aligned}$$

$$\Rightarrow \quad T\left(r, Q_1\left(\frac{f_1}{f_2}\right) + \frac{p(z)}{f_2^2} Q_2\left(\frac{f_1}{f_2}\right)\right) = T\left(r, \frac{G(z)}{f_2^{n-1}}\right). \tag{17}$$

Using (14), (17) deduces to

$$(2n - 4) T\left(r, \frac{f_1}{f_2}\right) = (n - 3) T(r, f_2) + S(r, f_1) + S(r, f_2). \tag{18}$$

Similarly, we can write

$$(2n-4)T\left(r,\frac{f_2}{f_1}\right)=(n-3)T\left(r,f_1\right)+S\left(r,f_1\right)+S\left(r,f_2\right). \tag{19}$$

From (18) and (19), we obtain

$$T\left(r,f_1\right)+S\left(r,f_1\right)=T\left(r,f_2\right)+S\left(r,f_2\right). \tag{20}$$

Since $\rho(f_1)=\rho(f_2)$, (20) reduces to

$$S\left(r,f_1\right)=S\left(r,f_2\right). \tag{21}$$

Using (21), (18) reduces to

$$\begin{aligned} 2(n-2)T\left(r,\frac{f_2}{f_1}\right) &= (n-3)T\left(r,f_2\right)+S\left(r,f_2\right) \\ \Rightarrow \quad S\left(r,\frac{f_2}{f_1}\right) &= S\left(r,f_2\right) \end{aligned}$$

From (14) and First Fundamental Theorem ([14], Theorem 1.2), we can write

$$\begin{aligned} N\left(r,\frac{1}{G}\right) &= S(r,f_2) \\ \Rightarrow \sum_{p=1}^{n-1} N\left(r,\frac{1}{f_1-\eta_p f_2}\right) &= S(r,f_2) \text{ and } \sum_{m=1}^{n-3} N\left(r,\frac{1}{f_1-\gamma_m f_2}\right)=S(r,f_2) \\ \Rightarrow N\left(r,\frac{1}{f_1-\eta_p f_2}\right) &= S(r,f_2) \text{ and } N\left(r,\frac{1}{f_1-\gamma_m f_2}\right)=S(r,f_2) \end{aligned}$$

holds for all $p=1,2,\cdots,n-1$ and $m=1,2,\cdots,n-3$. Since $\frac{1}{\frac{f_1}{f_2}-\eta_p}=f_2\frac{1}{f_1-\eta_p f_2}$ and $\frac{1}{\frac{f_1}{f_2}-\gamma_m}=f_2\frac{1}{f_1-\gamma_m f_2}$, we obtain

$$N\left(r,\frac{1}{\frac{f_1}{f_2}-\eta_p}\right)=S(r,f_2) \text{ and } N\left(r,\frac{1}{\frac{f_1}{f_2}-\gamma_m}\right)=S(r,f_2) \tag{22}$$

holds for all $p = 1, 2, \cdots, n-1$ and $m = 1, 2, \cdots, n-3$. From (22) and the Second main theorem ([14], Theorem 1.8), we obtain

$$\begin{aligned}(n-3)T\left(r, \frac{f_1}{f_2}\right) &\leq \sum_{p=1}^{n-1} N\left(r, \frac{1}{\frac{f_1}{f_2}-\eta_p}\right) + S\left(r, \frac{f_1}{f_2}\right)\\ &\leq \sum_{p=1}^{n-1} N\left(r, \frac{1}{\frac{f_1}{f_2}-\eta_p}\right) + \sum_{m=1}^{n-3} N\left(r, \frac{1}{\frac{f_1}{f_2}-\gamma_m}\right) + S\left(r, \frac{f_1}{f_2}\right)\\ &\leq S\left(r, f_2\right) + S\left(r, \frac{f_1}{f_2}\right) = S\left(r, \frac{f_1}{f_2}\right)\\ \Rightarrow \quad (n-3)T\left(r, \frac{f_1}{f_2}\right) &\leq S\left(r, \frac{f_1}{f_2}\right)\end{aligned}$$

which is a contradiction to $n \geq 4$.

Hence (3) has $f(z)$ as its unique transcendental meromorphic solution with finite order such that $N(r, f) = S(r, f)$. ■

Proof. of Theorem 3. First we consider two cases for $q(z)$ and $r(z)$.

Case 1. If $q(z)$ is a constant or $r(z) \equiv 0$. Then (4) can be reduced to $f(z)^n + p(z)L(z, f) = Q(z)$, where $Q(z)$ is a polynomial.

$$\begin{aligned}\Rightarrow \quad f(z)^n &= Q(z) - p(z)\frac{L(z,f)}{f(z)}.f(z)\\ \Rightarrow \quad nT(r, f(z)) &\leq T(r, Q(z)) + T(r, p(z)) + T\left(r, \frac{L(z,f)}{f(z)}\right)\\ &\quad + T(r, f(z)) + S(r, f) \leq T(r, f(z)) + S(r, f)\\ \Rightarrow \quad (n-1)T(r, f(z)) &\leq S(r, f), \text{ which is a contradiction to } n > 1.\end{aligned}$$

Hence, if $q(z)$ is a constant or $r(z) \equiv 0$ then transcendental entire function $f(z)$ of finite order cannot be solution of (4).

Case 2. If $q(z)$ is a non-constant polynomial and $r(z) \not\equiv 0$. Assume that transcendental entire function $f(z)$ of finite order is a solution of (4). Differentiating (4), we get

$$nf(z)^{n-1}f' + p(z)L'(z, f) + p'(z)L(z, f) = \left[r(z)q'(z) + r'(z)\right] e^{q(z)}. \tag{23}$$

From (23) and (4), we get

$$\frac{nf(z)^{n-1}f'(z) + p(z)L'(z, f) + p'(z)L(z, f)}{f(z)^n + p(z)L(z, f)} = \frac{\left[r(z)q'(z) + r'(z)\right] e^{q(z)}}{r(z)e^{q(z)}}$$

$$\Rightarrow nf(z)^{n-1}f'(z) + p(z)L'(z,f) + p'(z)L(z,f)$$
$$= \left[q'(z) + \frac{r'(z)}{r(z)}\right] f(z)^n + \left[q'(z) + \frac{r'(z)}{r(z)}\right] p(z)L(z,f)$$

$$\Rightarrow f(z)^{n-1}\left[nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right) f(z)\right] \tag{24}$$
$$= \left(q'(z) + \frac{r'(z)}{r(z)}\right) p(z)L(z,f) - p(z)L'(z,f) - p'(z)L(z,f).$$

If $nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right) f(z) \equiv 0$, then integrating and simplifying, we get $f(z)^n = Br(z)e^{q(z)}$ implies $f(z) = g(z)e^{\frac{q(z)}{n}}$ where $g(z)^n = Br(z)$, B is a non-zero constant. Thus, (4) can be written as

$$(B-1)r(z)e^{q(z)} + p(z)L(z,f) \equiv 0. \tag{25}$$

Notice that if $B = 1$, then $L(z,f) \equiv 0$, which contradicts the hypothesis. Thus, $B \not\equiv 1$, substituting $f(z) = g(z)e^{\frac{q(z)}{n}}$ in $L(z,f)$ and considering $h(z) = e^{\frac{q(z)}{n}}$, then $L(z,f)$ can be expressed as

$$L(z,f) = a_0 g(z)h(z) + \sum_{s=1}^{k} a_s(z)g(z+c_s)h(z+c_s).$$

Consider,

$$T(r, L(z,f)) = m(r, L(z,f)) + N(r, L(z,f)) = m(r, L(z,f))$$
$$= m\left(r, \frac{L(z,f)}{h(z)}h(z)\right) \le m\left(r, \frac{L(z,f)}{h(z)}\right) + m(r, h(z)) + S(r,h).$$

Using Lemma 1, we deduce

$$T(r, L(z,f)) \le T(r, h(z)) + S(r, h(z)). \tag{26}$$

From (25), we get

$$T\left(r, (1-B)r(z)e^{q(z)}\right) = T(r, (1-B)r(z)h(z)^n) = T(r, p(z)L(z,f)) \tag{27}$$
$$\le T(r, p(z)) + T(r, L(z,f)) + S(r,h).$$

Since polynomial $p(z)$ is small function with respect to transcendental entire function $h(z)$, we have $T(r, p(z)) = S(r,h)$. Using (26), (27) reduces to

$$(n-1)T(r, h(z)) \le S(r,h)$$

which is contradiction to $n>1$. Hence $nf^{'}(z)-(q^{'}+\frac{r^{'}(z)}{r(z)})f(z)\not\equiv 0$.

Now, we consider the following two subcases for n.

Subcase 1. If $n>2$, (24) can be deduced to

$$f(z)^{n-2}\left[nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right] = \left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)p(z)\frac{L(z,f)}{f(z)}-p(z)\frac{L^{'}(z,f)}{f(z)}-\ p^{'}(z)\frac{L(z,f)}{f(z)} \tag{28}$$

and

$$f^{n-3}(z)\left[f(z)\left(nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right)\right] = \left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)p(z)\frac{L(z,f)}{f(z)}-p(z)\frac{L^{'}(z,f)}{f(z)}-p^{'}(z)\frac{L(z,f)}{f(z)}. \tag{29}$$

Applying Lemma 3 and Remark 1 to (28) and (29), we get

$$m\left(r,nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right)=S(r,f) \tag{30}$$

and

$$m\left(r,f(z)\left(nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right)\right)=S(r,f). \tag{31}$$

Since $f(z)$ is an entire function and from (30) and (31), we get

$$T\left(r,nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right)=S(r,f) \tag{32}$$

and

$$T\left(r,f(z)\left(nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right)\right)=S(r,f). \tag{33}$$

Now consider,

$$T(r,f(z))\ =T\left(r,\frac{f(z)\left(nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)\right)}{nf^{'}(z)-\left(q^{'}(z)+\frac{r^{'}(z)}{r(z)}\right)f(z)}\right)$$

$$(34) \Rightarrow T(r, f(z)) \leq T\left(r, f(z)\left(nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right)\right) + T\left(r, nf'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right) + S(r, f).$$

From (32) and (33), (34) reduces to

$$\begin{aligned} & T(r, f(z)) \leq S(r, f) \\ \Rightarrow\quad & T(r, f(z)) = S(r, f), \text{ a contradiction.} \end{aligned}$$

Subcase 2. If $n = 2$, (4) and (24) can be reduced to

$$f(z)^2 + p(z)L(z, f) = r(z)e^{q(z)} \tag{35}$$

$$f(z)\left[2f'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right] = \left(q'(z) + \frac{r'(z)}{r(z)}\right)p(z)L(z, f) - p(z)L'(z, f) - p'(z)L(z, f). \tag{36}$$

Let $G(z) = 2f'(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f(z)$. Then, (36) reduces to

$$G(z) = \left(q'(z) + \frac{r'(z)}{r(z)}\right)p(z)\frac{L(z, f)}{f(z)} - p(z)\frac{L'(z, f)}{f(z)} - p'(z)\frac{L(z, f)}{f(z)}. \tag{37}$$

Since $f(z)$ is an entire function and from Lemma 1 and the Lemma on the logarithmic derivative ([14], page no. 16), (37) deduces to

$$T(r, G(z)) = m(r, G(z)) + S(r, f) = S(r, f). \tag{38}$$

Differentiating $G(z)$, we get

$$2f''(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)' f(z) - \left(q'(z) + \frac{r'(z)}{r(z)}\right)f'(z) = \frac{G'(z)}{G(z)}G(z)$$

$$\Rightarrow 2f''(z) - \left(q'(z) + \frac{r'(z)}{r(z)} + 2\frac{G'(z)}{G(z)}\right)f'(z) - \left(q''(z) - \frac{G'(z)}{G(z)}q'(z) + \left(\frac{r'(z)}{r(z)}\right)' - \frac{G'(z)}{G(z)}\frac{r'(z)}{r(z)}\right)f(z) = 0$$

$$(39) \quad \Rightarrow 2\left(\left(\frac{f'(z)}{f(z)}\right)' + \left(\frac{f'(z)}{f(z)}\right)^2\right) - \left(q'(z) + \frac{r'(z)}{r(z)} + 2\frac{G'(z)}{G(z)}\right)\frac{f'(z)}{f(z)} - \left(q''(z) - q'(z)\frac{G'(z)}{G(z)} + \left(\frac{r'(z)}{r(z)}\right)' - \frac{G'(z)}{G(z)} \cdot \frac{r'(z)}{r(z)}\right) = 0.$$

Suppose that z_0 is a zero of $f(z)$ with multiplicity p. If z_0 is a zero of $r(z)$ as well, then the contribution of z_0 to $N\left(r, \frac{1}{f}\right)$ is $S(r, f)$. Assume that z_0 is not a zero of $r(z)$, we now discuss the following two Subcases:

Subcase A. Suppose z_0 is a zero of $G(z)$ with multiplicity k. From (39), we obtain that $p = 1 + k \leq 2k$, by (38) implies that the contribution of z_0 to $N\left(r, \frac{1}{f}\right)$ is $S(r, f)$.

Subcase B. Suppose z_0 is not a zero of $G(z)$. By (39), we obtain $p^2 - p = 0$, then such a zero of $f(z)$ must be simple and we notice that $q' + \frac{r'}{r} + 2\frac{G'}{G}$ must vanish at z_0. Thus, by (38) implies that the contribution of z_0 to $N\left(r, \frac{1}{f}\right)$ is $S(r, f)$. Hence, $N\left(r, \frac{1}{f}\right) = S(r, f)$. Thus, by Hadamard's factorization theorem, $f(z)$ can be expressed as $f(z) = B(z)e^{d(z)}$, where $B(z)$ is an entire function satisfying $N\left(r, \frac{1}{B(z)}\right) = S(r, f)$ and $d(z)$ is non-constant polynomial. Substituting $f(z)$ in (35), we get

$$B(z)^2 e^{2d(z)} + p(z)a_0(z)B(z)e^{d(z)} + p(z)\sum_{j=1}^{k} a_j(z)B\left(z + c_j\right)e^{d(z+c_j)} = r(z)e^{q(z)}$$

$$\Rightarrow -\frac{B(z)e^{d(z)}}{p(z)a_0(z)} + \frac{r(z)e^{q(z)}}{p(z)a_0(z)B(z)e^{d(z)}} - \frac{\sum_{j=1}^{k} a_j(z)B\left(z + c_j\right)e^{d(z+c_j)}}{a_0(z)B(z)e^{d(z)}} = 1$$

$$\Rightarrow f_1 + f_2 + f_3 = 1$$

where

$$f_1 = -\frac{B(z)e^{d(z)}}{p(z)a_0(z)}, \qquad f_2 = \frac{r(z)e^{q(z)}}{p(z)a_0(z)B(z)e^{d(z)}},$$

$$f_3 = -\frac{\sum_{j=1}^{k} a_j(z)B\left(z + c_j\right)e^{d(z+c_j)}}{a_0(z)B(z)e^{d(z)}}.$$

Notice that $f_1 = -\frac{B(z)e^{d(z)}}{p(z)a_0(z)}$ is not a constant and we deduce that

$$\sum_{l=1}^{3} N\left(r, \frac{1}{f_l}\right) + 2\sum_{l=1}^{3} \overline{N}(r, f_l) \leq S(r, f) < (\lambda + o(1))T(r).$$

Thus, by Lemma 4, we get either $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$. If $f_2(z) \equiv 1$, then by (35), we deduce $T(r, f) \leq S(r, f)$, which is a contradiction. If $f_3(z) \equiv 1$, then we get $L(z, f) \equiv 0$, by hypothesis, we again get a contradiction. Hence, transcendental entire function of finite order cannot be a solution of (4). ■

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RENUKADEVI S. DYAVANAL
DEPARTMENT OF MATHEMATICS
KARNATAK UNIVERSITY
DHARWAD-580003, INDIA
e-mail: renukadyavanal@gmail.com

MADHURA M. MATHAI
DEPARTMENT OF MATHEMATICS
KARNATAK UNIVERSITY
DHARWAD-580003, INDIA
e-mail: madhuramathai@gmail.com

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