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**ULAM-HYERS STABILITY THEOREM BY TRIPLED
FIXED POINT THEOREM**

ABSTRACT. This paper deals with tripled fixed point theorem, and the approach is based on Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics. We are also study Ulam-Hyers stability results for the tripled fixed points of a triple of contractive type single-valued and respectively multi-valued operators on complete metric spaces.

KEY WORDS: metric space, tripled fixed point, single-valued operator, vector-valued metric, Perov type contraction.

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1. Introduction

In 1922, Banach's contraction principle [2] has become a very popular and important tool in modern analysis, especially in nonlinear analysis including its applications to differential and integral equations, variational inequality theory, complementarity problems, equilibrium problems, minimization problems and many others. Banach contraction principle was extended for single-valued contraction on spaces endowed with vector-valued metrics by Perov in [22], while the case of multi-valued contractions is treated by A. Petruşel [24].

Beside this one of the most popular generalization of fixed point theorems is coupled fixed point theorem for continuous and discontinuous operators introduced in 1987 by D. Guo and V. Lakshmikantham [10] in connection with coupled quasi solutions of an initial value problem for ordinary differential equations. In 2006, Gnana-Bhaskar and Lakshmikantham [3] introduced the concept of mixed monotone property in partially ordered metric space. Afterward, Lakshmikantham and Ćirić in [21] extended these results by giving the definition of the g-monotone property. In 2011, Vasile Berinde and Marin Borcut [5] extended and generalized the results of [3] and introduced

the concept of a tripled fixed point and the mixed monotone property of a mapping. For more details on coupled and tripled fixed point results, we refer to [8, 12, 13, 15, 16, 17, 18, 19] and cited therein.

It should be noted that through the coupled fixed point technique we cannot solve a system with the following form:

$$\begin{cases} x^3 + 2yz - 6x + 3 = 0, \\ y^3 + 2xz - 6y + 3 = 0, \\ z^3 + 2yx - 6z + 3 = 0. \end{cases}$$

On the base of above example we can say that the importance of coupled fixed point and tripled fixed point theorem is different and depends on the given problem.

Let X be a nonempty set. A mapping $d : X \times X \rightarrow \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

- (a) $d(x, y) \geq 0$ for all $x, y \in X$,
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

If $x, y \in \mathbb{R}^m, x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$, then, by definition: $x \leq y$ if and only if $x_i \leq y_i$ for $i \in \{1, 2, \dots, m\}$.

A set endowed with a vector-valued metric d is called generalized metric space. The notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix.

Notice that we will make an identification between row and column vectors in \mathbb{R}^m .

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see [1], [26], [29]).

Theorem 1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent,*

- (i) *A is convergent towards zero,*
- (ii) *$A^n \rightarrow 0$ as $n \rightarrow \infty$,*
- (iii) *The eigenvalues of A are in the open unit disc, i.e $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with*

$$\det(A - \lambda I) = 0,$$

- (iv) *The matrix $(I - A)$ is nonsingular and*

$$(1) \quad (I - A)^{-1} = I + A + \dots + A^n + \dots$$

- (v) *The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements*

(vi) $A^n q \rightarrow 0$ and $qA^n \rightarrow 0$ as $n \rightarrow \infty$, for each $q \in \mathbb{R}^m$.

We recall now Perov’s fixed point theorem (see [22]).

Theorem 2. *Let (X, d) be a complete generalized metric space and the operator $f : X \rightarrow X$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that $d(f(x), f(y)) \leq Ad(x, y)$ for all $x, y \in X$. If A is a matrix convergent towards zero, then:*

- (i) $Fix(f) = \{x^*\}$ (Here $Fix(f)$ denotes the set of fixed points of f),
- (ii) the sequence of successive approximations $(x_n)_{n \in \mathbb{N}}, x_n = f^n(x_0)$ is convergent and has the limit x^* , for all $x_0 \in X$,
- (iii) one has the following estimation

$$(2) \quad d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1),$$

(iv) if $g : X \rightarrow X$ is an operator such that there exist $y^* \in Fix(g)$ and $\epsilon \in (\mathbb{R}_m^+)^*$ with $d(f(x), g(x)) \leq \epsilon$, for each $x \in X$, then

$$d(x^*, y^*) \leq (I - A)^{-1}\epsilon,$$

(v) if $g : X \rightarrow X$ is an operator and there exists $\epsilon \in (\mathbb{R}_+^m)^*$ such that $d(f(x), g(x)) \leq \epsilon$, for all $x \in X$, then for the sequence $y_n = g^n(x_0)$ we have the following estimation

$$(3) \quad d(y_n, x^*) \leq (I - A)^{-1}\epsilon + A^n(I - A)^{-1}d(x_0, x_1).$$

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial equations:

$$\begin{aligned} x &= T_1(x, y, z) \\ y &= T_2(x, y, z) \\ z &= T_3(x, y, z) \end{aligned}$$

where $T_1, T_2, T_3 : X \times X \times X \rightarrow X$ are three given operators.

By definition, a solution $(x, y, z) \in X \times X \times X$ of the above system is called a tripled fixed point for the triple (T_1, T_2, T_3) . In a similar way, the case of an operatorial inclusion (using the symbol \in instead of $=$) could be considered.

This paper deal with existence and uniqueness of tripled fixed point theorem the approach is based on Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics. We are also studying Ulam-Hyers stability results for the tripled fixed points of a triple of contractive type single-valued and respectively multi-valued operators on complete metric spaces. For related results to Perov’s fixed point theorem and for some generalizations and applications of it we refer to [7], [9], [25].

2. Existence, uniqueness and stability results for tripled fixed points

For the proof of our main theorem we need the following notions and results.

Definition 1. Let (X, d) be a generalized metric space and $f : X \rightarrow X$ be an operator. Then, the fixed point equation

$$(4) \quad x = f(x)$$

is said to be generalized Ulam-Hyers stable if there exists an increasing function, $\psi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$, continuous at 0 with $\psi(0) = 0$, such that, for any $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for $i \in \{1, \dots, m\}$ and any solution $y^* \in X$ of the in inequality

$$(5) \quad d(y, f(y)) \leq \epsilon$$

there exists a solution x^* of (4) such that

$$(6) \quad d(x^*, y^*) \leq \psi(\epsilon).$$

In particular, if $\psi(t) = Ct$, $t \in \mathbb{R}_+^m$, (where $C \in M_{mm}(\mathbb{R}_+)$), then the fixed point equation (4) is called Ulam-Hyers stable.

Our first abstract result is a direct consequence of Perov's fixed point theorem.

Theorem 3. Let (X, d) be a generalized metric space and let $f : X \rightarrow X$ be an operator with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that A converges to zero and

$$d(f(x), f(y)) \leq Ad(x, y), \quad \text{for all } x, y \in X.$$

Then the fixed point equation

$$x = f(x), \quad x \in X$$

is Ulam-Hyers stable.

Proof. From Perov's fixed point theorem we get that $Fix(f) = \{x^*\}$. Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for each $i \in \{1, \dots, m\}$ and let y^* be a solution of the in equation

$$d(y, f(y)) \leq \epsilon.$$

Then we successively have that

$$\begin{aligned} d(x^*, y^*) &= d(f(x^*), y^*) \\ &\leq d(f(x^*), f(y^*)) + d(f(y^*), y^*) \\ &\leq Ad(x^*, y^*) + \epsilon. \end{aligned}$$

Thus, using Theorem 2, we get that

$$d(x^*, y^*) \leq (I - A)^{-1}\epsilon.$$

■

Definition 2. Let (X, d) be a metric space and let $T_1, T_2, T_3 : X \times X \times X \rightarrow X$ be three operators. Then the system of operatorial equations

$$(7) \quad \left. \begin{aligned} x &= T_1(x, y, z) \\ y &= T_2(x, y, z) \\ z &= T_3(x, y, z) \end{aligned} \right\}$$

is said to be Ulam-Hyers stable if there exist $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 > 0$ such that for each $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ and each triple $(u^*, v^*, w^*) \in X \times X \times X$ such that

$$(8) \quad \left. \begin{aligned} d(u^*, T_1(u^*, v^*, w^*)) &\leq \epsilon_1 \\ d(v^*, T_2(u^*, v^*, w^*)) &\leq \epsilon_2 \\ d(w^*, T_3(u^*, v^*, w^*)) &\leq \epsilon_3 \end{aligned} \right\}$$

there exists a solution $(x^*, y^*, z^*) \in X \times X \times X$ of (8) such that

$$(9) \quad \left. \begin{aligned} d(u^*, x^*) &\leq c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 \\ d(v^*, y^*) &\leq c_4\epsilon_1 + c_5\epsilon_2 + c_6\epsilon_3 \\ d(w^*, z^*) &\leq c_7\epsilon_1 + c_8\epsilon_2 + c_9\epsilon_3 \end{aligned} \right\}.$$

For examples and other considerations regarding Ulam-Hyers stability and generalized Ulam-Hyers stability of the operatorial equations and inclusions see I.A. Rus [27], Bota-Petruşel [6], Petru-Petruşel-Yao [23].

Our first main result is the following existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the tripled fixed point of single-valued operators (T_1, T_2, T_3) . The conclusions (i)-(ii) are originally proved by R. Precup [25], but for the sake of completeness we recall here the whole proof.

Theorem 4. Let (X, d) be a complete metric space, $T_1, T_2, T_3 : X \times X \times X \rightarrow X$ be three operators such that

$$(10) \quad \left. \begin{aligned} d(T_1(x, y, z), T_1(u, v, w)) &\leq k_1d(x, u) + k_2d(y, v) + k_3d(z, w) \\ d(T_2(x, y, z), T_2(u, v, w)) &\leq k_4d(x, u) + k_5d(y, v) + k_6d(z, w) \\ d(T_3(x, y, z), T_3(u, v, w)) &\leq k_7d(x, u) + k_8d(y, v) + k_9d(z, w) \end{aligned} \right\}$$

for all $(x, y, z), (u, v, w) \in X \times X \times X$. We suppose that

$$A = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{pmatrix}$$

converges to zero. Then,

(i) there exists a unique element $(x^*, y^*, z^*) \in X \times X \times X$ such that

$$(11) \quad \left. \begin{aligned} x^* &= T_1(x^*, y^*, z^*) \\ y^* &= T_2(x^*, y^*, z^*) \\ z^* &= T_3(x^*, y^*, z^*) \end{aligned} \right\}$$

(ii) the sequence $(T_1^n(x, y, z), T_2^n(x, y, z), T_3^n(x, y, z)), n \in N$ converges to (x^*, y^*, z^*) as $n \rightarrow \infty$, where

$$(12) \quad \left. \begin{aligned} T_1^{n+1}(x, y, z) &= T_1^n(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)) \\ T_2^{n+1}(x, y, z) &= T_2^n(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)) \\ T_3^{n+1}(x, y, z) &= T_3^n(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)) \end{aligned} \right\}$$

for all $n \in N$,

(iii) we have the following estimation:

$$(13) \quad \begin{pmatrix} d(T_1^n(x_0, y_0, z_0), x^*) \\ d(T_2^n(x_0, y_0, z_0), y^*) \\ d(T_3^n(x_0, y_0, z_0), z^*) \end{pmatrix} \leq A^n(I - A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0, y_0, z_0)) \\ d(y_0, T_2(x_0, y_0, z_0)) \\ d(z_0, T_3(x_0, y_0, z_0)) \end{pmatrix}$$

(iv) let $F_1, F_2, F_3 : X \times X \times X \rightarrow X$ be three operators such that, there exist $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ with

$$\begin{aligned} d(T_1(x, y, z), F_1(x, y, z)) &\leq \epsilon_1 \\ d(T_2(x, y, z), F_2(x, y, z)) &\leq \epsilon_2 \\ d(T_3(x, y, z), F_3(x, y, z)) &\leq \epsilon_3 \end{aligned}$$

for all $(x, y, z) \in X \times X \times X$. If $(a^*, b^*, c^*) \in X \times X \times X$ is such that

$$(14) \quad \left. \begin{aligned} a^* &= F_1(a^*, b^*, c^*) \\ b^* &= F_2(a^*, b^*, c^*) \\ c^* &= F_3(a^*, b^*, c^*) \end{aligned} \right\}$$

then

$$(15) \quad \begin{pmatrix} d(a^*, x^*) \\ d(b^*, y^*) \\ d(c^*, z^*) \end{pmatrix} \leq (I - A)^{-1} \epsilon$$

where

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

(v) let $F_1, F_2, F_3 : X \times X \times X \rightarrow X$ be three operators such that, there exist $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ with

$$(16) \quad \left. \begin{aligned} d(T_1(x, y, z), F_1(x, y, z)) &\leq \epsilon_1 \\ d(T_2(x, y, z), F_2(x, y, z)) &\leq \epsilon_2 \\ d(T_3(x, y, z), F_3(x, y, z)) &\leq \epsilon_3 \end{aligned} \right\}$$

for all $(x, y, z) \in X \times X \times X$. If we consider the sequence

$$(F_1^n(x, y, z), F_2^n(x, y, z), F_3^n(x, y, z)), \quad n \in N,$$

given by

$$(17) \quad \left. \begin{aligned} F_1^{n+1}(x, y, z) &= F_1(F_1^n(x, y, z), F_2^n(x, y, z), F_3^n(x, y, z)) \\ F_2^{n+1}(x, y, z) &= F_2(F_1^n(x, y, z), F_2^n(x, y, z), F_3^n(x, y, z)) \\ F_3^{n+1}(x, y, z) &= F_3(F_1^n(x, y, z), F_2^n(x, y, z), F_3^n(x, y, z)) \end{aligned} \right\}$$

for all $n \in N$ and

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix},$$

then

$$\begin{pmatrix} d(F_1^n(x_0, y_0, z_0), x^*) \\ d(F_2^n(x_0, y_0, z_0), y^*) \\ d(F_3^n(x_0, y_0, z_0), z^*) \end{pmatrix} \leq A^n (I - A)^{-1} \begin{pmatrix} d(x_0, F_1(x_0, y_0, z_0)) \\ d(y_0, F_2(x_0, y_0, z_0)) \\ d(z_0, F_3(x_0, y_0, z_0)) \end{pmatrix}$$

(vi) the system of operatorial equations

$$(18) \quad \left. \begin{aligned} x &= T_1(x, y, z) \\ y &= T_2(x, y, z) \\ z &= T_3(x, y, z) \end{aligned} \right\}$$

is Ulam-Hyers stable.

Proof. For (i)-(ii) let us define $T : X \times X \times X \rightarrow X \times X \times X$ by

$$T(x, y, z) = \begin{pmatrix} T_1(x, y, z) \\ T_2(x, y, z) \\ T_3(x, y, z) \end{pmatrix} = (T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)).$$

Denote $Z = X \times X \times X$ and consider $\tilde{d} : Z \times Z \rightarrow \mathbb{R}_+^3$,

$$\tilde{d}((x, y, z), (u, v, w)) = \begin{pmatrix} d(x, u) \\ d(y, v) \\ d(z, w) \end{pmatrix}.$$

Then we have

$$\begin{aligned} (19) \quad \tilde{d}(T(x, y, z), T(u, v, w)) &= \tilde{d} \left(\begin{pmatrix} T_1(x, y, z) \\ T_2(x, y, z) \\ T_3(x, y, z) \end{pmatrix}, \begin{pmatrix} T_1(u, v, w) \\ T_2(u, v, w) \\ T_3(u, v, w) \end{pmatrix} \right) \\ &= \begin{pmatrix} d(T_1(x, y, z), T_1(u, v, w)) \\ d(T_2(x, y, z), T_2(u, v, w)) \\ d(T_3(x, y, z), T_3(u, v, w)) \end{pmatrix} \\ &\leq \begin{pmatrix} k_1 d(x, u) + k_2 d(y, v) + k_3 d(z, w) \\ k_4 d(x, u) + k_5 d(y, v) + k_6 d(z, w) \\ k_7 d(x, u) + k_8 d(y, v) + k_9 d(z, w) \end{pmatrix} \\ &= \begin{pmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{pmatrix} \begin{pmatrix} d(x, u) \\ d(y, v) \\ d(z, w) \end{pmatrix} \\ &= A\tilde{d}((x, y, z), (u, v, w)). \end{aligned}$$

If we denote $(x, y, z) = \alpha$, $(u, v, w) = \beta$, we get that

$$\tilde{d}(T(\alpha), T(\beta)) \leq A\tilde{d}(\alpha, \beta).$$

Applying Perov's fixed point Theorem 1 (i), we get that there exists a unique element $(x^*, y^*, z^*) \in X \times X \times X$ such that

$$(x^*, y^*, z^*) = T(x^*, y^*, z^*)$$

and is equivalent with

$$\left. \begin{aligned} x^* &= T_1(x^*, y^*, z^*) \\ y^* &= T_2(x^*, y^*, z^*) \\ z^* &= T_3(x^*, y^*, z^*) \end{aligned} \right\}.$$

Moreover, for each $\alpha \in X \times X \times X$, we have that $T(\alpha) \rightarrow \alpha^*$ as $n \rightarrow \infty$, where

$$T^0(\alpha) = \alpha, \quad T^1(\alpha) = T(x, y, z) = (T_1(x, y, z), T_2(x, y, z), T_3(x, y, z))$$

$$\begin{aligned} T^2(\alpha) &= T(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)) \\ &= (T_1^2(x, y, z), T_2^2(x, y, z), T_3^2(x, y, z)) \end{aligned}$$

and generally

$$(20) \quad \left. \begin{aligned} T_1^{n+1}(\alpha) &= T_1^n(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)) \\ T_2^{n+1}(\alpha) &= T_2^n(T_1^2(x, y, z), T_2^2(x, y, z), T_3^2(x, y, z)) \\ T_3^{n+1}(\alpha) &= T_3^n(T_1^2(x, y, z), T_2^2(x, y, z), T_3^2(x, y, z)) \end{aligned} \right\}.$$

We obtain that $T(\alpha) = (T_1(\alpha), T_2(\alpha), T_3(\alpha)) \rightarrow \alpha^* = (x^*, y^*, z^*)$ as $n \rightarrow \infty$, for all $\alpha = (x, y, z) \in X \times X \times X$. So, for all $(x, y, z) \in X \times X \times X$, we have that

$$(21) \quad \left. \begin{aligned} T_1(x, y, z) &\rightarrow x^* \text{ as } n \rightarrow \infty \\ T_2(x, y, z) &\rightarrow y^* \text{ as } n \rightarrow \infty \\ T_3(x, y, z) &\rightarrow z^* \text{ as } n \rightarrow \infty \end{aligned} \right\}.$$

(iii) By Perov's theorem (iii) we successively have

$$\begin{aligned} &\left(\begin{aligned} d(T_1^n(x_0, y_0, z_0), x^*) \\ d(T_2^n(x_0, y_0, z_0), y^*) \\ d(T_3^n(x_0, y_0, z_0), z^*) \end{aligned} \right) = \tilde{d}((T^n(x_0, y_0, z_0)), (x^*, y^*, z^*)) \\ &\leq A^n(I - A)^{-1} \tilde{d}((x_0, y_0, z_0), (T_1(x_0, y_0, z_0), T_2(x_0, y_0, z_0), T_3(x_0, y_0, z_0))) \\ &\leq A^n(I - A)^{-1} \left(\begin{aligned} d(x_0, T_1(x_0, y_0, z_0)) \\ d(y_0, T_2(x_0, y_0, z_0)) \\ d(z_0, T_3(x_0, y_0, z_0)) \end{aligned} \right). \end{aligned}$$

(iv) If we consider $F : X \times X \times X \rightarrow X \times X \times X$ such that

$$(22) \quad F(x, y, z) = \begin{pmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{pmatrix}$$

and

$$(23) \quad \begin{aligned} \tilde{d}(T(x, y, z), F(x, y, z)) &= \tilde{d} \left(\begin{pmatrix} T_1(x, y, z) \\ T_2(x, y, z) \\ T_3(x, y, z) \end{pmatrix}, \begin{pmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{pmatrix} \right) \\ &= \begin{pmatrix} d(T_1(x, y, z), F_1(x, y, z)) \\ d(T_2(x, y, z), F_2(x, y, z)) \\ d(T_3(x, y, z), F_3(x, y, z)) \end{pmatrix} \leq \epsilon \end{aligned}$$

then, applying Perov's fixed point theorem 2 (iv) we get

$$(24) \quad \tilde{d}((x^*, y^*, z^*), (a^*, b^*, c^*)) \leq (I - A)^{-1}\epsilon.$$

(v) By (23) we get that

$$\tilde{d}(T(x, y, z), F((x, y, z))) \leq \epsilon.$$

Notice that $F^n(x, y, z) = F(F^{n-1}(x, y, z))$, for all $(x, y, z) \in X \times X \times X$.

Using the assertion (iii) of this theorem, we can successively write:

$$\begin{aligned} \tilde{d}(F^n(x_0, y_0, z_0), (x^*, y^*, z^*)) &\leq \tilde{d}(F^n(x_0, y_0, z_0), T^n(x_0, y_0, z_0)) \\ &\quad + \tilde{d}(T^n(x_0, y_0, z_0), (x^*, y^*, z^*)) \\ &\leq \tilde{d}(F^n(x_0, y_0, z_0), T^n(x_0, y_0, z_0)) \\ &\quad + A^n(I - A)^{-1}\tilde{d}(T(x_0, y_0, z_0), (x_0, y_0, z_0)). \end{aligned}$$

On the other hand, we have

$$(25) \quad \begin{aligned} &\tilde{d}(F^n(x_0, y_0, z_0), T^n(x_0, y_0, z_0)) \\ &= \tilde{d}(F(F^{n-1}(x_0, y_0, z_0)), T(T^{n-1}(x_0, y_0, z_0))) \\ &\leq \tilde{d}(F(F^{n-1}(x_0, y_0, z_0)), T(F^{n-1}(x_0, y_0, z_0))) \\ &\quad + \tilde{d}(T(F^{n-1}(x_0, y_0, z_0)), T(T^{n-1}(x_0, y_0, z_0))) \\ &\leq \epsilon + A\tilde{d}((F^{n-1}(x_0, y_0, z_0), T^{n-1}(x_0, y_0, z_0))) \\ &\leq \epsilon + A[\epsilon + A\tilde{d}((F^{n-2}(x_0, y_0, z_0), T^{n-2}(x_0, y_0, z_0)))] \\ &\leq \dots \leq \epsilon\epsilon(I + A + A^2 + \dots + A^n + \dots) \\ &= \epsilon(I - A)^{-1}. \end{aligned}$$

Thus, we finally get the conclusion

$$\begin{aligned} \tilde{d}(F^n(x_0, y_0, z_0), (x^*, y^*, z^*)) &\leq \epsilon(I - A)^{-1} \\ &\quad + A^n(I - A)^{-1}\tilde{d}(T(x_0, y_0, z_0), (x_0, y_0, z_0)). \end{aligned}$$

(vi) By (i) and (ii) there exists a unique element $(x^*, y^*, z^*) \in X \times X \times X$ such that (x^*, y^*, z^*) is a solution for (18) and the sequence

$$(T_1^n(x, y, z), T_2^n(x, y, z), T_3^n(x, y, z)) \rightarrow (x^*, y^*, z^*) \quad \text{as } n \rightarrow \infty.$$

Let $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ and $(u^*, v^*, w^*) \in X \times X \times X$ such that

$$(26) \quad \left. \begin{aligned} d(u^*, T_1(u^*, v^*, w^*)) &\leq \epsilon_1 \\ d(v^*, T_2(u^*, v^*, w^*)) &\leq \epsilon_2 \\ d(w^*, T_3(u^*, v^*, w^*)) &\leq \epsilon_3 \end{aligned} \right\}.$$

Then we have

$$\begin{aligned} &\tilde{d}((u^*, v^*, w^*), (x^*, y^*, z^*)) \\ &\leq \tilde{d}((u^*, v^*, w^*), (T_1(u^*, v^*, w^*), T_2(u^*, v^*, w^*), T_3(u^*, v^*, w^*))) \\ &\quad + \tilde{d}((T_1(u^*, v^*, w^*), T_2(u^*, v^*, w^*), T_3(u^*, v^*, w^*)), (x^*, y^*, z^*)) \\ &= \tilde{d}((u^*, v^*, w^*), (T_1(u^*, v^*, w^*), T_2(u^*, v^*, w^*), T_3(u^*, v^*, w^*))) \\ &\quad + \tilde{d}\left(\left(\begin{array}{c} T_1(u^*, v^*, w^*) \\ T_2(u^*, v^*, w^*) \\ T_3(u^*, v^*, w^*) \end{array}\right), \left(\begin{array}{c} T_1(x^*, y^*, z^*) \\ T_2(x^*, y^*, z^*) \\ T_3(x^*, y^*, z^*) \end{array}\right)\right) \\ &= \left(\begin{array}{c} d(u^*, T_1(u^*, v^*, w^*)) \\ d(v^*, T_2(u^*, v^*, w^*)) \\ d(w^*, T_3(u^*, v^*, w^*)) \end{array}\right) + \left(\begin{array}{c} d(T_1(u^*, v^*, w^*), T_1(x^*, y^*, z^*)) \\ d(T_2(u^*, v^*, w^*), T_2(x^*, y^*, z^*)) \\ d(T_3(u^*, v^*, w^*), T_3(x^*, y^*, z^*)) \end{array}\right) \\ &\leq \left(\begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{array}\right) + \tilde{d}(T(u^*, v^*, w^*), T(x^*, y^*, z^*)) \\ &\leq \epsilon + A\tilde{d}((u^*, v^*, w^*), (x^*, y^*, z^*)). \end{aligned}$$

Since $(I - A)$ is invertible and $(I - A)^{-1}$ has positive elements, we immediately obtain

$$\tilde{d}((u^*, v^*, w^*), (x^*, y^*, z^*)) \leq (I - A)^{-1}\epsilon$$

or equivalently

$$\left(\begin{array}{c} d(u^*, x^*) \\ d(y^*, v^*) \\ d(z^*, w^*) \end{array}\right) \leq (I - A)^{-1}\epsilon.$$

If we denote

$$(I - A)^{-1} = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix},$$

then we obtain

$$(27) \quad \left. \begin{aligned} d(u^*, x^*) &\leq c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 \\ d(y^*, v^*) &\leq c_4\epsilon_1 + c_5\epsilon_2 + c_6\epsilon_3 \\ d(z^*, w^*) &\leq c_7\epsilon_1 + c_8\epsilon_2 + c_9\epsilon_3 \end{aligned} \right\}$$

proving that the operatorial system (18) is Ulam-Hyers stable. \blacksquare

Remark 1. Notice that, if (X, d) is a metric space and $T : X \times X \times X \rightarrow X$ is an operator and we define

$$T_1(x, y, z) = T(x, y, z), \quad T_2(x, y, z) = T(y, x, y) \quad \text{and} \quad T_3(x, y, z) = T(z, y, x)$$

then the above approach leads to some well-known tripled fixed point theorems, see [5]. Moreover, in a forthcoming paper, the same approach will be applied for the case of tripled fixed points for mixed monotone operators, see, for example, [11], [20], [28].

We will consider now the case of multi-valued operators. We need first some notations. Let (X, d) be a generalized metric space with $d : X \times X \rightarrow \mathbb{R}_+^m$ given by

$$d(x, y) = \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_m(x, y) \end{pmatrix}.$$

Then, for $x \in X$ and $A \subseteq X$ we denote:

$$D_d(x, A) = \begin{pmatrix} D_{d_1}(x, A) \\ \vdots \\ D_{d_m}(x, A) \end{pmatrix} = \begin{pmatrix} \inf_{a \in A} d_1(x, a) \\ \vdots \\ \inf_{a \in A} d_m(x, a) \end{pmatrix}.$$

$$P(X) = \{Y \subseteq X \mid Y \text{ is nonempty}\}$$

$$P_{cl}(X) = \{Y \subseteq P(X) \mid Y \text{ closed}\}.$$

We also denote

$$D((x, y, z), A \times B \times C) = \begin{pmatrix} D_d(x, A) \\ D_d(y, B) \\ D_d(z, C) \end{pmatrix}.$$

Our second main result is an existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the tripled fixed point of a triple of multi-valued operators (T_1, T_2, T_3) . For the proof of our main result, we give the following theorem.

Theorem 5. *Let (X, d) be a complete generalized metric space and let $T : X \rightarrow P_{cl}(X)$ be a multi-valued A -contraction, i.e. there exists $A \in M_{mm}(\mathbb{R}_+)$ which converges towards zero as $n \rightarrow \infty$ and for each $x, y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u, v) \leq Ad(x, y)$. Then T is a MWP-operator, i.e. $Fix(T) \neq \phi$, and for each $(x, y) \in Graph(T)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for T starting from (x, y) which converges to a fixed point x^* of T . Moreover $d(x, x^*) \leq (I - A)^{-1}d(x, y)$, for all $(x, y) \in Graph(T)$.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then by the A -contraction condition, there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq Ad(x_0, x_1)$. Now, for $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that

$$d(x_2, x_3) \leq Ad(x_1, x_2) \leq A^2d(x_0, x_1).$$

In this way, by an iterative construction, we get a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$\left. \begin{aligned} x_0 &\in X \\ x_{n+1} &\in T(x_n) \\ d(x_n, x_{n+1}) &\leq A^n d(x_0, x_1) \end{aligned} \right\}.$$

for all $n \in \mathbb{N}$.

Thus, by the above relation, we get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p}) \\ &\leq Ad(x_0, x_1) + A^2d(x_0, x_1) + \dots + A^{p-1}d(x_0, x_1) \\ &= A(I + A + \dots + A^{n+p-1})d(x_0, x_1) \end{aligned}$$

Letting $n \rightarrow \infty$ we get that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy. Hence there exists $x^* \in X$ such that $x^* = \lim_{n \rightarrow \infty} x_n$.

We prove that $x^* \in T(x^*)$. Indeed, for $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that

$$d(x_n, u_n) \leq Ad(x_{n-1}, x^*),$$

for all $n \in \mathbb{N}$.

On the other side

$$d(x^*, u_n) \leq d(x^*, x_n) + d(x_n, u_n) \leq d(x^*, x_n) + Ad(x_{n-1}, x^*) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence $\lim_{n \rightarrow \infty} u_n = x^*$. But $u_n \in T(x^*)$, for $n \in \mathbb{N}$ and because $T(x^*)$ is closed, we have that $x^* \in T(x^*)$.

Moreover we can write

$$d(x_n, x_{n+p}) \leq A(I + A + \dots + A^{p-1} + \dots)d(x_0, x_1) = A(I - A)^{-1}d(x_0, x_1).$$

Letting $p \rightarrow \infty$ we get that

$$d(x_n, x^*) \leq A(I - A)^{-1}d(x_0, x_1).$$

for all $n \geq 1$. Thus

$$\begin{aligned} d(x_0, x^*) &\leq d(x_0, x_1) + d(x_1, x^*) \leq d(x_0, x_1) + A(I - A)^{-1}d(x_0, x_1) \\ &= (I + A(I - A)^{-1})d(x_0, x_1) = (I + A + A^2 + \dots)d(x_0, x_1) \\ &= (I - A)^{-1}d(x_0, x_1). \end{aligned}$$

■

Definition 3. Let (X, d) generalized metric space and $F : X \rightarrow P(X)$. The fixed point inclusion

$$(28) \quad x \in F(x), x \in X$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ increasing, continuous at 0 with $\psi(0) = 0$ such that for each $\epsilon = (\epsilon_1, \dots, \epsilon_m) > 0$ and for each ϵ -solution y^* of (28), i.e.

$$D_d(y^*, F(y^*)) \leq \epsilon$$

there exists a solution x^* of the fixed point inclusion (28) such that

$$d(y^*, x^*) \leq \psi(\epsilon).$$

In particular, if $\psi(t) = Ct$, for each $t \in \mathbb{R}_+^m$ (where $C \in M_{mm}(\mathbb{R}_+)$), then (28) is said to be Ulam-Hyers stable.

Definition 4. A subset U of a generalized metric space (X, d) is called proximal if for each $x \in X$ there exists $u \in U$ such that $d(x, u) = D_d(x, U)$.

Theorem 6. Let (X, d) be a complete generalized metric space and let $T : X \rightarrow P_{cl}(X)$ be a multi-valued A -contraction with proximal values. Then, the fixed point inclusion (28) is Ulam-Hyers stable.

Proof. Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_1 > 0$, for each $i \in 1, 2, \dots, m$ and let $y^* \in X$ an ϵ -solution of (28), i.e.,

$$D_d(y^*, T(y^*)) \leq \epsilon.$$

By the second conclusion of Theorem 5 we have that for any $(x, y) \in Graph(T)$

$$(29) \quad d(x, x^*(x, y)) \leq (I - A)^{-1}d(x, y),$$

where $x^*(x, y)$ denotes the fixed point of T which is obtained by Theorem 5 by successive approximations starting from (x, y) .

Since $T(y^*)$ is proximal there exists $u \in T(y^*)$ such that

$$d(y^*, u^*) = D_d(y, T(y^*)).$$

Hence, by 29

$$d(y^*, x^*(y^*, u^*)) \leq (I - A)^{-1}d(y^*, u^*) \leq (I - A)^{-1}\epsilon.$$

■

Theorem 7. *Let (X, d) be a complete generalized metric space and let $T : X \rightarrow P_{cl}(X)$ be a multi-valued A -contraction such that there exists $x^* \in X$ with $T(x^*) = \{x^*\}$. Then the fixed point inclusion (28) is Ulam-Hyers stable.*

Proof. Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$, for each $i \in 1, 2, \dots, m$ and let $y^* \in X$ an ϵ -solution of (28), i.e.,

$$D_d(y^*, T(y^*)) \leq \epsilon.$$

By the A -contraction condition, for $x = y^*, y = x^*$ and $u \in T(y^*)$ we get that

$$d(u^*, x^*) \leq Ad(y^*, x^*).$$

Then, for any $u \in T(y^*)$ we have

$$d(y^*, x^*) \leq d(y^*, u) + d(u^*, x^*) \leq d(y^*, u) + A.d(y^*, x^*).$$

Hence

$$d(y^*, x^*) \leq (I - A)^{-1}d(y^*, u),$$

for any $u \in T(y^*)$. Thus

$$d(y^*, x^*) \leq (I - A)^{-1}D_d(y^*, T(y^*)) \leq (I - A)^{-1}\epsilon.$$

■

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial inclusions:

$$(30) \quad \left. \begin{aligned} x &\in T_1(x, y, z) \\ y &\in T_2(x, y, z) \\ z &\in T_3(x, y, z) \end{aligned} \right\}$$

where $T_1, T_2, T_3 : X \times X \times X \rightarrow P(X)$ are three given multi-valued operators. By definition, a solution $(x, y, z) \in X \times X \times X$ of the above system is called a tripled fixed point for (T_1, T_2, T_3) .

Definition 5. Let (X, d) be a metric space and let $T_1, T_2, T_3 : X \times X \times X \rightarrow P(X)$ be three multi-valued operators. Then the operatorial inclusions system (30) is said to be Ulam-Hyers stable if there exist $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 > 0$ such that for each $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ and for each triple $(u^*, v^*, w^*) \in X \times X \times X$ which satisfies the relations

$$(31) \quad \left. \begin{aligned} d(u^*, \alpha) &\leq \epsilon_1 \text{ for all } \alpha \in T_1(u^*, v^*, w^*) \\ d(v^*, \beta) &\leq \epsilon_2 \text{ for all } \beta \in T_2(u^*, v^*, w^*) \\ d(w^*, \gamma) &\leq \epsilon_3 \text{ for all } \gamma \in T_3(u^*, v^*, w^*) \end{aligned} \right\}$$

there exists a solution $(x^*, y^*, z^*) \in X \times X \times X$ of (30) such that

$$(32) \quad \left. \begin{aligned} d(u^*, x^*) &\leq c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 \\ d(v^*, y^*) &\leq c_4\epsilon_3 + c_5\epsilon_4 + c_6\epsilon_3 \\ d(w^*, z^*) &\leq c_7\epsilon_3 + c_8\epsilon_4 + c_9\epsilon_3 \end{aligned} \right\}$$

Definition 6. Let (X, d) be a metric space. By definition, we say that $S : X \times X \times X \rightarrow P(X)$ has proximal values with respect to the first variable if for any $x, y, z \in X$ there exists $u \in S(x, y, z)$ such that

$$d(x, u) = D_d(x, S(x, y, z)).$$

Definition 7. Let (X, d) be a metric space. By definition we say that $S : X \times X \times X \rightarrow P(X)$ has proximal values with respect to the second variable if for any $x, y, z \in X$ there exists $v \in S(x, y, z)$ such that

$$d(y, v) = D_d(y, S(x, y, z)).$$

Definition 8. Let (X, d) be a metric space. By definition we say that $S : X \times X \times X \rightarrow P(X)$ has proximal values with respect to the third variable if for any $x, y, z \in X$ there exists $w \in S(x, y, z)$ such that

$$d(z, w) = D_d(z, S(x, y, z)).$$

Now we are in the position to give our next main results.

Theorem 8. Let (X, d) be a complete metric space and let $T_1, T_2, T_3 : X \times X \times X \rightarrow P_{cl}(X)$ be three multi-valued operators. Suppose that T_1 has proximal values with respect to the first variable, T_2 with respect to the second variable and T_3 with respect to the third variable. For each

$(x, y, z), (u, v, w) \in X \times X \times X$ and each $\alpha_1 \in T_1(x, y, z)$, $\alpha_2 \in T_2(x, y, z)$, $\alpha_3 \in T_3(x, y, z)$ there exist $\beta_1 \in T_1(u, v, w)$, $\beta_2 \in T_2(u, v, w)$, $\beta_3 \in T_3(u, v, w)$ satisfying

$$\left. \begin{aligned} d(\alpha_1, \beta_1) &\leq k_1d(x, u) + k_2d(y, v) + k_3d(z, w) \\ d(\alpha_2, \beta_2) &\leq k_4d(x, u) + k_5d(y, v) + k_6d(z, w) \\ d(\alpha_3, \beta_3) &\leq k_7d(x, u) + k_8d(y, v) + k_9d(z, w) \end{aligned} \right\}$$

We suppose that

$$A = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{pmatrix}$$

converges to zero. Then,

- (i) there exists $(x^*, y^*, z^*) \in X \times X \times X$ a solution for (30).
- (ii) the system of operatorial inclusions (30) is Ulam-Hyers stable.

Proof. (i)-(ii) Let us define $T : X \times X \times X \rightarrow P_{cl}(X) \times P_{cl}(X) \times P_{cl}(X)$ by

$$T(x, y, z) = T_1(x, y, z) \times T_2(x, y, z) \times T_3(x, y, z).$$

Denote $\Gamma = X \times X \times X$ and consider $\tilde{d} : \Gamma \times \Gamma \rightarrow \mathbb{R}_+^3$,

$$\tilde{d}((x, y, z), (u, v, w)) = \begin{pmatrix} d(x, u) \\ d(y, v) \\ d(z, w) \end{pmatrix}.$$

Then, from the hypotheses of the theorem, we get that for each $s = (x, y, z)$, $t = (u, v, w) \in X \times X \times X$ and each $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in T(x, y, z)$, there exists $\beta = (\beta_1, \beta_2, \beta_3) \in T(u, v, w)$ satisfying the relation

$$\tilde{d}(\alpha, \beta) \leq A\tilde{d}(s, t),$$

which proves that T is a multi-valued A -contraction. Since $T_1(x, y, z) \subset X$ is proximal with respect to the first variable we have that, for any $x, y, z \in X$ there exists $u \in T_1(x, y, z)$ such that

$$d(x, u) = D_d(x, T_1(x, y, z)).$$

Since $T_2(x, y, z) \subset X$ is proximal with respect to the second variable we get that, for any $x, y, z \in X$ there exists $v \in T_2(x, y, z)$ such that

$$d(y, v) = D_d(y, T_2(x, y, z)).$$

Since $T_3(x, y, z) \subset X$ is proximal with respect to the third variable we get that, for any $x, y, z \in X$ there exists $w \in T_3(x, y, z)$ such that

$$d(z, w) = D_d(z, T_3(x, y, z)).$$

Then the set $T(x, y, z) = T_1(x, y, z) \times T_2(x, y, z) \times T_3(x, y, z)$ is proximal, since for any $x, y, z \in X$ there exists $(u, v, w) \in T(x, y, z)$ such that

$$\tilde{d}((x, y, z), (u, v, w)) = D_{\tilde{d}}((x, y, z), T(x, y, z)).$$

The conclusions follow now from Theorem 5 and Theorem 6. ■

Theorem 9. *Let (X, d) be a complete metric space and let $T_1, T_2, T_3 : X \times X \times X \rightarrow P_{cl}(X)$ be three multi-valued operators. Suppose there exist $x_1, x_2, x_3 \in X$ such that*

$$(33) \quad T_1(x^*, y^*, z^*) = \{x^*\}, \quad T_2(x^*, y^*, z^*) = \{y^*\}, \quad T_3(x^*, y^*, z^*) = \{z^*\}.$$

For each $(x, y, z), (u, v, w) \in X \times X \times X$ and each $\alpha_1 \in T_1(x, y, z)$, $\alpha_2 \in T_2(x, y, z)$, $\alpha_3 \in T_3(x, y, z)$ there exist $\beta_1 \in T_1(u, v, w)$, $\beta_2 \in T_2(u, v, w)$, $\beta_3 \in T_3(u, v, w)$ satisfying

$$\left. \begin{aligned} d(\alpha_1, \beta_1) &\leq k_1 d(x, u) + k_2 d(y, v) + k_3 d(z, w) \\ d(\alpha_2, \beta_2) &\leq k_4 d(x, u) + k_5 d(y, v) + k_6 d(z, w) \\ d(\alpha_3, \beta_3) &\leq k_7 d(x, u) + k_8 d(y, v) + k_9 d(z, w) \end{aligned} \right\}.$$

We suppose that

$$A = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{pmatrix}.$$

converges to zero. Then:

- (i) *there exists $(x^*, y^*, z^*) \in X \times X \times X$ a solution for (30).*
- (ii) *the operatorial system (30) is Ulam-Hyers stable.*

Proof. For the prove of (i)-(ii) let us define $T : X \times X \times X \rightarrow P_{cl}(X) \times P_{cl}(X) \times P_{cl}(X)$ by

$$T(x, y, z) = T_1(x, y, z) \times T_2(x, y, z) \times T_3(x, y, z).$$

Then from the hypotheses of the theorem we get that

$$T(x^*, y^*, z^*) = T_1(x^*, y^*, z^*) \times T_2(x^*, y^*, z^*) \times T_3(x^*, y^*, z^*) = (x^*, y^*, z^*).$$

So, T has at least one strict fixed point. We denote $\Gamma = X \times X \times X$ and consider $\tilde{d} : \Gamma \times \Gamma \rightarrow \mathbb{R}_+^3$

$$\tilde{d}((x, y, z), (u, v, w)) = \begin{pmatrix} d(x, u) \\ d(y, v) \\ d(z, w) \end{pmatrix}.$$

Then from the hypotheses of the theorem, we have that for each $s = (x, y, z)$, $t = (u, v, w) \in X \times X \times X$ and each $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in T(x, y, z)$, there exists $\beta = (\beta_1, \beta_2, \beta_3) \in T(u, v, w)$ satisfying the relation

$$\tilde{d}(\alpha, \beta) \leq A\tilde{d}(s, t),$$

which proves that T is a multi-valued A -contraction. The conclusions follow now from Theorem 5 and Theorem 6. \blacksquare

Remark 2. Notice again that, if (X, d) is a metric space and $T : X \times X \times X \rightarrow P(X)$ is a multi-valued operator and we define

$$T_1(x, y, z) = T(x, y, z), \quad T_2(x, y, z) = T(y, x, y), \quad \text{and} \quad T_3(x, y, z) = T(z, y, x)$$

then the above approach leads to some tripled fixed point theorems in the classical sense.

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