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MORE ON WEAK δs-CONTINUITY IN TOPOLOGICAL SPACES

Abstract. The notion of weakly δs-continuous functions is introduced by Ekici in [6]. In this paper, we further investigate some more properties of weakly δs-continuous functions. This type of continuity is a generalization of super continuity [16].

Key words: continuous, δ-semiopen, δ-semi-continuous, rarely δs-continuous, super-continuous.

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1. Introduction and preliminaries

Levine [12] defined semiopen sets which are weaker than open sets in topological spaces. After Levine's semiopen sets, mathematicians gave in several papers different and interesting new open sets as well as generalized open sets. In 1968, Veličko [24] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces. In 1997, Park et al. [19] have introduced the notion of δ-semiopen sets which are stronger than semiopen sets but weaker than δ-open sets and investigated the relationships between several types of open sets. In 1979, Popa [20] introduced the useful notion of rare continuity as a generalization of weak continuity [11]. The class of rarely continuous functions has been further investigated by Long and Herrington [13] and Jafari [7] and [8]. The concept of rare δs-continuity in topological spaces as a generalization of super continuity is introduced by Caldas et al. [3].

The notion of weakly δs-continuous functions is introduced by Ekici in [6]. The purpose of the present paper is to further investigate some more properties of weakly δs-continuous functions. This type of functions is weaker than both super continuous functions and δ-semi-continuous functions and stronger than rare δs-continuous functions.

Throughout this paper, (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of a space X, then $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively.

A subset A of X is called regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). Recall that a subset A of X is called semi-open [12] if $A \subset Cl(Int(A))$. The complement of a semi-open sets is called semi-closed. A subset A of X is called preopen [15] if $A \subset Int(Cl(A))$. A rare or codense set is a set A such that $Int(A) = \emptyset$, equivalently, if the complement $X \setminus A$ is dense. A point $x \in X$ is called a δ-cluster [24] of A if $S \cap U \neq \emptyset$ for each regular open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$. A subset A is called δ-closed if $Cl_\delta(A) = A$. The complement of a δ-closed set is called δ-open. The δ-interior of a subset A of a space (X, τ), denoted by $Int_\delta(A)$, is the union of all regular open sets of (X, τ) contained in A.

A subset A of a topological space X is said to be δ-semiopen sets [19] if there exists a δ-open set U of X such that $U \subset A \subset Cl(U)$, equivalently if $A \subset Cl(Int_\delta(A))$. The complement of a δ-semiopen set is called a δ-semiclosed set. A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \emptyset$ for every δ-semiopen set U of X containing x. The set of all δ-semicluster points of A is called the δ-semiclosure of A, denoted by $sCl_\delta(A)$ and the δ-semiinterior of A, denoted by $sInt_\delta(A)$, is defined as the union of all δ-semiopen sets contained in A. We denote the collection of all δ-semiopen (resp. δ-semiclosed, δ-open, regular open and open) sets by $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{SC}(X)$, $\delta O(X)$, $RO(X)$ and $O(X)$). We set $\delta SO(X, x) = \{U \mid x \in U \in \delta SO(X)\}$, $\delta O(X, x) = \{U \mid x \in U \in \delta O(X)\}$, $RO(X, x) = \{U \mid x \in U \in RO(X)\}$ and $O(X, x) = \{U \mid x \in U \in O(X)\}$.

Lemma 1. *The intersection (resp. union) of an arbitrary collection of δ-semiclosed (resp. δ-semiopen) sets in (X, τ) is δ-semiclosed (resp. δ-semiopen)*

Corollary 1. *Let A be a subset of a topological space (X, τ). Then the following properties hold:*
(1) $sCl_\delta(A) = \cap\{F \in \delta SC(X, \tau) : A \subset F\}$.
(2) $sCl_\delta(A)$ *is δ-semiclosed.*
(3) $sCl_\delta(sCl_\delta(A)) = sCl_\delta(A)$.

Lemma 2 ([1]). *For subsets A and A_i $(i \in I)$ of a space (X, τ), the following hold:*
(1) $A \subset sCl_\delta(A)$.
(2) *If $A \subset B$, then $sCl_\delta(A) \subset sCl_\delta(B)$.*
(3) $sCl_\delta(\cap\{A_i : i \in I\}) \subset \cap\{sCl_\delta(A_i) : i \in I\}$.
(4) $sCl_\delta(\cup\{A_i : i \in I\}) = \cup\{sCl_\delta(A_i) : i \in I\}$.
(5) *A is δ-semiclosed if and only $A = sCl_\delta(A)$.*

Lemma 3 ([19]). *For a subset A of a space (X, τ), the following hold:*
(1) *A is a δ-semiopen set if and only if $A = sInt_\delta(A)$.*
(2) *$X - sInt_\delta(A) = sCl_\delta(X - A)$ and $sInt_\delta(X - A) = X - sCl_\delta(A)$.*
(3) *$sInt_\delta(A))$ is a δ-semiopen set.*

Definition 1. *A function $f : X \to Y$ is said to be:*

(1) *Weakly continuous [11] if for each $x \in X$ and each open set V containing $f(x)$, there exists $U \in O(X, x)$ such that $f(U) \subset Cl(V)$.*

(2) *δ-semi-continuous [18] if for each $x \in X$ and each open set V containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset V$.*

(3) *Rarely quasi continuous [21] (resp. rarely δs-continuous [3]) if for each $x \in X$ and each $V \in O(Y, f(x))$, there exist a rare set R_V with $V \cap Cl(R_V) = \emptyset$ and $U \in SO(X, x)$ (resp. $U \in \delta SO(X, x)$) such that $f(U) \subset V \cup R_V$.*

(4) *Super-continuous [16] if the inverse image of every open set in Y is δ-open in X.*

(5) *semi-continuous [12] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists $U \in SO(X, x)$ such that $f(U) \subset V$.*

Definition 2. *A function $f : X \to Y$ is said to be:*

(1) *Weakly quasicontinuous [22] if for each $x \in X$ and for each open set U containing x and each open set G containing $f(x)$, there exists a nonempty open set V such that $V \subset U$ and $f(V) \subset Cl(G)$.*

(2) *Weakly-θ-continuous [5] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(Int(Cl(U))) \subset Cl(V)$.*

Definition 3. *A function $f : X \to Y$ is said to be I. δs-continuous [3] at $x \in X$ if for each set $V \in O(Y, f(x))$, there exists $U \in \delta SO(X, x)$ such that $Int[f(U)] \subset V$. If f has this property at each point $x \in X$, then we say that f is I. δs-continuous on X.*

Remark 1 ([3]). It should be noted that super-continuity implies I. δs-continuity and I. δs-continuity implies rare δs-continuity. But the converses are not true as shown by the following examples.

Example 1 ([3]). Let $X = Y = \{a, b, c\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}\}$. Then a function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$ and $f(c) = c$, is I. δs-continuous. Since f is not continuous, then it is not super continuous.

Example 2 ([3]). Let (Y, σ) be the same spaces as in the above Example. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is rare δs-continuous but it is not I. δs-continuous.

Remark 2. The following diagram [[18], Remark 4.1] holds:

$$\begin{array}{lll} \textit{super} - \textit{continuity} & \rightarrow & \textit{continuity} \\ \downarrow & & \downarrow \\ \delta - \textit{semi} - \textit{continuity} & \rightarrow & \textit{semi} - \textit{continuity} \end{array}$$

2. Weakly-δs-continuous and some properties

In [3], unaware of the paper of Ekici [6], the notion of weakly-δs-continuous under the name of almost weakly-δs-continuous was defined. In this paper, we used the name weakly-δs-continuous functions.

Definition 4. *A function $f : X \to Y$ is called weakly-δs-continuous [6] if for each $x \in X$ and each open set V containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset Cl(V)$.*

The following diagram holds:

$$\begin{array}{lllll} \textit{super C.} & \rightarrow & \textit{weak } \theta - C. & \rightarrow & \textit{weak C.} \\ \downarrow & & \downarrow & & \downarrow \\ \delta - \textit{semi} - C. & \rightarrow & \textit{weak } \delta s - C. & \rightarrow & \textit{weak quasi C.} \\ \downarrow & & \downarrow & & \downarrow \\ I.\ \delta s - C. & \rightarrow & \textit{rare } \delta s - C. & \rightarrow & \textit{rare quasi C.} \end{array}$$

It should be mentioned that in the above diagram C. means continuity.

Example 3. Let $X = \{a, b, c\}$, $\tau = \{X, \{a\}, \{b\}, \{a, b\}, \emptyset\}$ and $\sigma = \{X, \{a\}, \{b, c\}, \emptyset\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is δ-semi-continuous but it is not weakly continuous.

Example 4. Let X, τ and σ be the same as in Example 3. Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is I. δs-continuous and not weakly quasicontinuous.

Example 5. Let $X = \{a, b, c\}$ and $\tau = \sigma = \{X, \{a\}, \emptyset\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is continuous and hence weakly θ-continuous. But it is not I. δs-continuous.

Example 6. Let $X = \{a, b, c\}$, $\tau = \{X, \{a\}, \emptyset\}$ and $\sigma = \{X, \{a\}, \{c\}, \{a, c\}, \emptyset\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is weakly quasicontinuous(= weakly semi-continuous) [[9], Example 2] but it is not rarely δs-continuous.

Theorem 1. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is weakly δs-continuous,*
(2) *$sCl_\delta(f^{-1}(Int(Cl(V)))) \subset f^{-1}(Cl(V))$ for every subset $V \subset Y$,*
(3) *$sCl_\delta(f^{-1}(Int(F))) \subset f^{-1}(F)$ for every regular closed subset $F \subset Y$,*
(4) *$sCl_\delta(f^{-1}(U)) \subset f^{-1}(Cl(U))$ for every open subset $U \subset Y$,*
(5) *$f^{-1}(U) \subset sInt_\delta(f^{-1}(Cl(U)))$ for every open subset $U \subset Y$,*
(6) *$sCl_\delta(f^{-1}(U)) \subset f^{-1}(Cl(U))$ for each preopen subset $U \subset Y$,*
(7) *$f^{-1}(U) \subset sInt_\delta(f^{-1}(Cl(U)))$ for each preopen subset $U \subset Y$.*

Proof. (1) $\Rightarrow$ (2) : Let V be a subset of Y and $x \in X \backslash f^{-1}(Cl(V))$. Then $f(x) \in Y \backslash Cl(V)$. There exists an open set U containing $f(x)$ such that $U \cap V = \emptyset$. We have $Cl(U) \cap Int(Cl(V)) = \emptyset$. Since f is weakly δs-continuous, then there exists a δ-semiopen set W containing x such that $f(W) \subset Cl(U)$. Then $W \cap f^{-1}(Int(Cl(V))) = \emptyset$ and $x \in X \backslash sCl_\delta(f^{-1}(Int(Cl(V))))$. Hence, $sCl_\delta(f^{-1}(Int(Cl(V)))) \subset f^{-1}(Cl(V))$.

(2) $\Rightarrow$ (3) : Suppose F is any regular closed set in Y. Then $sCl_\delta(f^{-1}(Int(F))) = sCl_\delta(f^{-1}(Int(Cl(Int(F))))) \subset f^{-1}(Cl(Int(F))) = f^{-1}(F)$.

(3) $\Rightarrow$ (4) : Suppose U is an open subset of Y. Since $Cl(U)$ is regular closed in Y, then $sCl_\delta(f^{-1}(U)) \subset sCl_\delta(f^{-1}(Int(Cl(U)))) \subset f^{-1}(Cl(U))$.

(4) $\Rightarrow$ (5) : Suppose U is any open set of Y. Since $Y \backslash Cl(U)$ is open in Y, then $X \backslash sInt_\delta(f^{-1}(Cl(U))) = sCl_\delta(f^{-1}(Y \backslash Cl(U))) \subset f^{-1}(Cl(Y \backslash Cl(U))) \subset X \setminus f^{-1}(U)$. Hence, $f^{-1}(U) \subset sInt_\delta(f^{-1}(Cl(U)))$.

(5) $\Rightarrow$ (1) : Suppose $x \in X$ and U is any open subset of Y containing $f(x)$. Then $x \in f^{-1}(U) \subset sInt_\delta(f^{-1}(Cl(U)))$. Take $W = sInt_\delta(f^{-1}(Cl(U)))$. Thus $f(W) \subset Cl(U)$ and hence f is weakly δs-continuous at x in X.

(1) $\Rightarrow$ (6) : Suppose U is any preopen set of Y and $x \in X \backslash f^{-1}(Cl(U))$. There exists an open set O containing $f(x)$ such that $O \cap U = \emptyset$. We have $Cl(O \cap U) = \emptyset$. Since U is preopen, then $U \cap Cl(O) \subset Int(Cl(U)) \cap Cl(O) \subset Cl(Int(Cl(U)) \cap O) \subset Cl(Int(Cl(U) \cap O)) \subset Cl(Int(Cl(U \cap O))) \subset Cl(U \cap O) = \emptyset$. Since f is weakly δs-continuous and O is an open set containing $f(x)$, there exists a δ-semiopen set W in X containing x such that $f(W) \subset Cl(O)$. Then $f(W) \cap U = \emptyset$ and $W \cap f^{-1}(U) = \emptyset$. This implies that $x \in X \backslash sCl_\delta(f^{-1}(U))$ and then $sCl_\delta(f^{-1}(U)) \subset f^{-1}(Cl(U))$.

(6) $\Rightarrow$ (7) : Suppose U is any preopen set of Y. Since $Y \backslash Cl(U)$ is open in Y, then $X \setminus sInt_\delta(f^{-1}(Cl(U))) = sCl_\delta(f^{-1}(Y \setminus Cl(U))) \subset f^{-1}(Cl(Y \setminus Cl(U))) \subset X \setminus f^{-1}(U)$. This shows that $f^{-1}(U) \subset sInt_\delta(f^{-1}(Cl(U)))$.

(7) $\Rightarrow$ (1) : Suppose $x \in X$ and U is any open set of Y containing $f(x)$. We have $x \in f^{-1}(U) \subset sInt_\delta(f^{-1}(Cl(U)))$. Take $W = sInt_\delta(f^{-1}(Cl(U)))$. Then $f(W) \subset Cl(U)$. This means that f is weakly δs-continuous at x in X. ■

Theorem 2. *If $f : X \to Y$ is a weakly δs-continuous function and Y is Hausdorff, then f has δ-semiclosed point inverses.*

Proof. Let $y \in Y$ and $x \in \{x \in X : f(x) \neq y\}$. Since $f(x) \neq y$ and Y is Hausdorff, there exist disjoint open sets G_1, G_2 such that $f(x) \in G_1$ and $y \in G_2$. Since $G_1 \cap G_2 = \emptyset$, then $Cl(G_1) \cap G_2 = \emptyset$. we have $y \notin Cl(G_1)$. Since f is weakly δs-continuous, there exists a δ-semiopen set U containing x such that $f(U) \subset Cl(G_1)$. Assume that U is not contained in $\{x \in X : f(x) \neq y\}$. There exists a point $u \in U$ such that $f(u) = y$. Since $f(U) \subset Cl(G_1)$, we have $y = f(u) \in Cl(G_1)$. This is a contradiction. Hence, $U \subset \{x \in X : f(x) \neq y\}$ and $\{x \in X : f(x) \neq y\}$ is δ-semiopen in X. This shows that $\{x \in X : f(x) \neq y\}$ is δ-semiopen in X, equivalently $f^{-1}(y) = \{x \in X : f(x) = y\}$ is δ-semiclosed in X.

Recall that a point $x \in X$ is said to be in the θ-closure [17] of a subset A of X, denoted by $Cl_\theta(G)$, if $Cl(G) \cap A \neq \emptyset$ for each open set G of X containing x. A is called θ-closed if $A = Cl_\theta(A)$. The complement of a θ-closed set is called θ-open. ■

Theorem 3. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is weakly δs-continuous,*
(2) *$f(sCl_\delta(V)) \subset Cl_\theta(f(V))$ for each subset $V \subset X$,*
(3) *$sCl_\delta(f^{-1}(G)) \subset f^{-1}(Cl_\theta(G))$ for each subset $G \subset Y$,*
(4) *$sCl_\delta(f^{-1}(Int(Cl_\theta(G)))) \subset f^{-1}(Cl_\theta(G))$ for every subset $G \subset Y$.*

Proof. (1) $\Rightarrow$ (2) : Let $V \subset X$, $x \in sCl_\delta(V)$ and U be any open set of Y containing $f(x)$. There exists a δ-semiopen set W containing x such that $f(W) \subset Cl(U)$. Since $x \in sCl_\delta(V)$, then $W \cap V \neq \emptyset$. This implies that $\emptyset \neq f(W) \cap f(V) \subset Cl(U) \cap f(V)$ and $f(x) \in Cl_\theta(f(V))$. Hence, $f(sCl_\delta(V)) \subset Cl_\theta(f(V))$.

(2) $\Rightarrow$ (3) : Let $G \subset Y$. Then $f(sCl_\delta(f^{-1}(G))) \subset Cl_\theta(G)$ and hence $sCl_\delta(f^{-1}(G)) \subset f^{-1}(Cl_\theta(G))$.

(3) $\Rightarrow$ (4) : Let $G \subset Y$. Since $Cl_\theta(G)$ is closed in Y, then $sCl_\delta(f^{-1}(Int(Cl_\theta(G)))) \subset f^{-1}(Cl_\theta(Int(Cl_\theta(G)))) = f^{-1}(Cl(Int(Cl_\theta(G)))) \subset f^{-1}(Cl_\theta(G))$.

(4) $\Rightarrow$ (1) : Let U be any open set of Y. We have $U \subset Int(Cl(U)) = Int(Cl_\theta(U))$. Thus, $sCl_\delta(f^{-1}(U)) \subset sCl_\delta(f^{-1}(Int(Cl_\theta(U)))) \subset f^{-1}(Cl_\theta(U)) = f^{-1}(Cl(U))$. This implies from Theorem 1 that f is weakly δs-continuous. ■

Theorem 4. *If $f^{-1}(Cl_\theta(V))$ is δ-semiclosed in X for every subset $V \subset Y$, then f is weakly δs-continuous.*

Proof. Let $V \subset Y$. Since $f^{-1}(Cl_\theta(V))$ is δ-semiclosed in X, then $sCl_\delta(f^{-1}(V)) \subset sCl_\delta(f^{-1}(Cl_\theta(V))) = f^{-1}(Cl_\theta(V))$. This implies from Theorem 3 that f is weakly δs-continuous. ■

Theorem 5. *Let $f : X \to Y$ be a function. If f is weakly δs-continuous, then $f^{-1}(V)$ is δ-semiclosed in X for every θ-closed subset $V \subset Y$.*

Proof. It follows from Theorem 3. ■

Corollary 2. *Let $f : X \to Y$ be a function. If f is weakly δs-continuous, then $f^{-1}(V)$ is δ-semiopen in X for every θ-open subset $V \subset Y$.*

Definition 5. *A function $f : X \to Y$ is said to be*

(1) *(δ, s)-open if $f(A)$ is semiopen for every δ-semiopen subset $A \subset X$.*

(2) *neatly weak δs-continuous if for each $x \in X$ and each open set V of X containing $f(x)$, there exists a δ-semiopen set U containing x such that $Int(f(U)) \subset Cl(V)$.*

Theorem 6. *If a function $f : X \to Y$ is neatly weak δs-continuous and (δ, s)-open, then f is weakly δs-continuous.*

Proof. Let $x \in X$ and V be an open subset of Y containing $f(x)$. Since f is neatly weak δs-continuous, there exists a δ-semiopen set U of X containing x such that $Int(f(U)) \subset Cl(V)$. Since f is (δ, s)-open, then $f(U)$ is semiopen in Y. Then $f(U) \subset Cl(Int(f(U))) \subset Cl(V)$. Thus, f is weakly δs-continuous. ■

Theorem 7. *If $f : X \to Y$ is weakly δs-continuous and Y is Hausdorff, then for each $(x, y) \notin G(f)$, there exist a δ-semiopen set $V \subset X$ and an open set $U \subset Y$ containing x and y, respectively, such that $f(V) \cap Int(Cl(U)) = \emptyset$.*

Proof. Let $(x, y) \notin G(f)$. We have $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets U and V containing y and $f(x)$, respectively. We have $Int(Cl(U)) \cap Cl(V) = \emptyset$. Since f is weakly δs-continuous, there exists an δ-semiopen set G containing x such that $f(G) \subset Cl(V)$. Hence, $f(G) \cap Int(Cl(U)) = \emptyset$. ■

Definition 6. *A function $f : X \to Y$ is said to be faintly δs-continuous if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists a δ-semiopen set U containing x such that $f(U) \subset V$.*

Theorem 8. *Let $f : X \to Y$ be a function. The following are equivalent:*

(1) *f is faintly δs-continuous,*

(2) *$f^{-1}(V)$ is δ-semiopen in X for every θ-open subset $V \subset Y$,*

(3) *$f^{-1}(V)$ is δ-semiclosed in X for every θ-closed subset $V \subset Y$.*

Proof. Obvious. ■

Theorem 9. *Let $f : X \to Y$ be a function, where Y is regular. The following are equivalent:*
(1) *f is δ-semicontinuous,*
(2) *$f^{-1}(Cl_\theta(V))$ is δ-semiclosed in X for every subset $V \subset Y$,*
(3) *f is weakly δs-continuous,*
(4) *f is faintly δs-continuous.*

Proof. (1) $\Rightarrow$ (2) : Let $V \subset Y$. Since $Cl_\theta(V)$ is closed, then $f^{-1}(Cl_\theta(V))$ is δ-semiclosed in X.

(2) $\Rightarrow$ (3) : It follows from Theorem 4.

(3) $\Rightarrow$ (4) : Let V be a θ-closed subset of Y. By Theorem 3, we have $sCl_\delta(f^{-1}(V)) \subset f^{-1}(Cl_\theta(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is δ-semiclosed and hence f is faintly δs-continuous.

(4) $\Rightarrow$ (1) : Let V be an open subset of Y. Since Y is regular, V is θ-open in Y. Since f is faintly δs-continuous, then $f^{-1}(V)$ is δ-semiopen in X. Thus, f is δ-semicontinuous. ■

Definition 7. *A space (X, τ) is said to be δ-semi T_2 (see [1]) if for each pair of distinct points x and y in X, there exist $U \in \delta SO(X, x)$ and $V \in \delta SO(X, y)$ such that $U \cap V = \emptyset$.*

Theorem 10. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly δs-continuous injective function. If (Y, σ) is Urysohn, then (X, τ) is δ-semi T_2.*

Proof. Let x_1 and x_2 be any two distinct points of X. Since f is injective, $f(x_1) \neq f(x_2)$. Since (Y, σ) is Urysohn, there exist disjoint $V_1, V_2 \in \sigma$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $Cl(V_1) \cap Cl(V_2) = \emptyset$. Since f is weakly δs-continuous at x_i, then there exists δ-semiopen sets U_i for $i = 1, 2$ containing x_i such that $f(U_i) \subset Cl(V_i)$. This indicates that (X, τ) is δ-semi T_2. ■

Theorem 11. *If $f : X \to Y$ is weakly δs-continuous and $g : Y \to Z$ is continuous, then the composition $gof : X \to Z$ is weakly δs-continuous.*

Proof. Let $x \in X$ and A be an open set of Z containing $g(f(x))$. We have $g^{-1}(A)$ is an open set of Y containing $f(x)$. Then there exists a δ-semiopen set B containing x such that $f(B) \subset Cl(g^{-1}(A))$. Since g is continuous, then $(gof)(B) \subset g(Cl(g^{-1}(A))) \subset Cl(A)$. Thus, gof is weakly δs-continuous. ■

Definition 8. *We say that the product space $X = X_1 \times \ldots \times X_n$ has property $P_{\delta s}$ [3] if A_i is a δ-semiopen set in a topological space X_i, for $i = 1, 2, \ldots n$, then $A_1 \times \ldots \times A_n$ is also δ-semiopen in the product space $X = X_1 \times \ldots \times X_n$.*

Theorem 12. *If $f_i : X_i \to Y_i$ is weakly δs-continuous for each $i \in I = \{1, 2, 3, \ldots, n\}$ and $\prod X_i$ has property $P_{\delta s}$, then the function $f : \prod X_i \to \prod Y_i$ which is defined by $f((x_i)) = (f_i(x_i))$ is weakly δs-continuous.*

Proof. Let $x = (x_i) \in \prod X_i$ and V be an open set containing $f(x)$. There exists an open set $\prod U_i$ such that $f(x) \in \prod_{i=1}^{n} U_i \times \prod_{i \neq j} Y_j \subset V$, where U_i is open in Y_i. Since f_i is weakly δs-continuous, there exists δ-semiopen sets G_i in X_i containing x_i such that $f_i(G_i) \subset Cl(U_i)$ for each $i = 1, 2, ..., n$. Take $G = \prod_{i=1}^{n} G_i \times \prod_{i \neq j} X_j$. Then G is δ-semiopen in $\prod X_i$ containing x and $f(G) \subset \prod_{i=1}^{n} f_i(G_i) \times \prod_{i \neq j} Y_j \subset \prod_{i=1}^{n} Cl(U_i) \times \prod_{i \neq j} Y_j \subset Cl(V)$. This shows that f is weakly δs-continuous. ■

Theorem 13. *Let $f, g : X \to Y$ be weakly δs-continuous functions and Y be Urysohn. If $\delta SO(X)$ is closed under the finite intersections, then the set $\{x \in X : f(x) = g(x)\}$ is δ-semiclosed in X.*

Proof. Let $x \in X \backslash \{x \in X : f(x) = g(x)\}$. We have $f(x) \neq g(x)$. Since Y is Urysohn, then there exist open sets A and B of Y such that $f(x) \in A$, $g(x) \in B$ and $Cl(A) \cap Cl(B) = \emptyset$. Since f is weakly δs-continuous, there exists δ-semiopen set G in X containing x such that $f(G) \subset Cl(A)$. Since g is weakly δs-continuous, there exists a δ-semiopen set K of X containing x such that $g(K) \subset Cl(B)$. Take $W = G \cap K$. Then W is δ-semiopen containing x and $f(W) \cap g(W) \subset Cl(A) \cap Cl(B) = \emptyset$. This implies that $W \cap \{x \in X : f(x) = g(x)\} = \emptyset$ and hence $\{x \in X : f(x) = g(x)\}$ is δ-semiclosed in X. ■

Definition 9. *A subset U of a topological space X is called N-closed if there exists a finite number of points $x_1, x_2, \ldots, x_n$ in U such that $U \subset \cup_{i=1}^{n} Int(Cl(V(x_i)))$, where the family $\{V(x) \mid x \in U\}$ is a cover of U by open sets of X.*

Theorem 14. *Let $f : X \to Y$ be a function, where $\delta SO(X)$ is semiclosed under the finite intersections. If for each $(x, y) \notin G(f)$, there exist a δ-semiopen set $U \subset X$ and an open set $V \subset Y$ containing x and y, respectively, such that $f(U) \cap Int(Cl(V)) = \emptyset$, then inverse image of each N-closed set of Y is δ-semiclosed in X.*

Proof. Suppose that there exists a N-closed set $W \subset Y$ such that $f^{-1}(W)$ is not δ-semiclosed in X. We have a point $x \in sCl_\delta(f^{-1}(W)) \backslash f^{-1}(W)$. Since $x \notin f^{-1}(W)$, then $(x, y) \notin G(f)$ for each $y \in W$. There exist δ-semiopen sets $U_y(x) \subset X$ and an open set $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x)) \cap Int(Cl(V(y))) = \emptyset$. The family $\{V(y) : y \in W\}$ is a cover of W by open sets of Y. Since W is N-closed, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in W such that $W \subset \cup_{i=1}^{n} Int(Cl(V(y_i)))$.

Take $U = \cap_{i=1}^{n} U_{y_i}(x)$. We have $f(U) \cap W = \emptyset$. Since $x \in Cl_\delta(f^{-1}(W))$, then $f(U) \cap W \neq \emptyset$. This is a contradiction. ■

For a function $f : X \to Y$, the graph function $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$ for each $x \in X$.

Theorem 15. *If the graph function g of a function $f : X \to Y$ is weakly δs-continuous, then f is weakly δs-continuous.*

Proof. Let g be weakly δs-continuous and $x \in X$ and U be an open set of X containing $f(x)$. Then $X \times U$ is an open set containing $g(x)$. There exists a δ-semiopen set V containing x such that $g(V) \subset Cl(X \times U) = X \times Cl(U)$. This implies that $f(V) \subset Cl(U)$ and hence f is weakly δs-continuous. ■

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