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TOTAL LIMITED PACKING IN GRAPHS

Abstract. We define a k-total limited packing number in a graph, which generalizes the concept of open packing number in graphs, and give several bounds on it. These bounds involve many well known parameters of graphs. Also, we establish a connection among the concepts of tuple domination, tuple total domination and total limited packing that implies some results.

Key words: k-total limited packing, k-tuple domination, k-tuple total domination.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order n and edge set $E = E(G)$. The minimum and maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For a vertex $v \in V$, $N(v)$ is the open neighborhood of v, which is the set of vertices adjacent to v and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. Let $[A, B]$ be the set of edges with end points in both A and B. We use [11] for terminology and notation which are not defined here.

A set $S \subseteq V$ is a *dominating set (total dominating set*) if each vertex in $V \backslash S$ (in V) is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (total dominating set). A subset $S \subseteq V$ is a 2-*packing* if for every pair of vertices $u, v \in S$, $d(u, v) > 2$. The 2-*packing number*, $\rho(G)$, is the maximum cardinality of a 2-packing in G. In [9] Henning and Slater studied the concept of *open packing* in graphs. A set $L \subseteq V$ is an *open packing set* if for every vertex $v \in V$, $|N(v) \cap L| \leq 1$. The *open packing number* $\rho_0(G)$ is the maximum cardinality of an open packing set.

In [3], Harary and Haynes introduced the concept of *tuple domination.* A set $D \subseteq V$ is a *k-tuple dominating set* in G if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The *k-tuple domination number*, denoted $\gamma_{\times k}(G)$, is the smallest number of vertices in a k-tuple dominating set. In fact the authors showed that every graph G with $\delta \geq k-1$ has a k-tuple dominating set and hence a k-tuple domination number.

A generalization of total domination titled *k-tuple total domination* was studied by Henning and Kazemi in [7] (this concept had been studied by Zhao et al. [12] as *total k-domination*). A subset $S \subseteq V$ is a *k-tuple total dominating set* in G if $|N(v) \cap S| \geq k$ for all $v \in V(G)$. The *k-tuple total domination number*, denoted $\gamma_{\times k,t}(G)$, is the smallest number of vertices in a k-tuple total dominating set. Of course, we can consider it as a total version of k-tuple domination, as well.

Gallant et al. [2] introduced the concept of *limited packing*, which generalizes 2-packing in graphs. They exhibited some real-world applications of it to network security, NIMBY, market saturation and codes. A set of vertices $B \subseteq V$ is called a *k limited packing set* in G provided that for all $v \in V(G)$, we have $|N[v] \cap B| \leq k$. The *k-limited packing number*, denoted $L_k(G)$, is the largest number of vertices in a k-limited packing set. It is easy to see that $L_1(G) = \rho(G)$. Of course, we can consider the concept of limited packing as the dual of tuple domination in a graph. For more information the reader can consult [10].

The above discussions give us a motivation to introduce the concept of *total limited packing* in graphs. Let G be a graph, and $k \in N$. A set of vertices $L \subseteq V(G)$ is called a *k-total limited packing* in G provided that for all $v \in V(G)$, we have $|N(v) \cap L| \leq k$. The *k-total limited packing number*, denoted $L_{k,t}(G)$, is the largest number of vertices in a k-total limited packing set. We can consider total limited packing first as a generalization of open packing, second as a dual of tuple total domination and third as a total version of limited packing. In fact, one can apply total limited packing to the subjects that we consider for limited packing as applications. Obviously, a k-total limited packing is also a $k+1$-total limited packing, therefore we have the following inequalities:

$$\rho_0(G) = L_{1,t}(G) \leq L_{2,t}(G) \leq \ldots \leq L_{\Delta,t}(G) = |V(G)|.$$

Also, a k-limited packing is a k-total limited packing and a k-total limited packing is a $k+1$-limited packing in G. Hence,

$$L_k(G) \leq L_{k,t}(G) \leq L_{k+1}(G).$$

In this paper, we investigate the concept of k-total limited packing and obtain some upper and lower bounds on $L_{k,t}(G)$. Also, we give a connection between this concept and k-tuple total domination that leads to some results in this area. These bounds involve $\rho_0(G), \gamma_t(G), \gamma_{\times k}(G), \gamma_{\times k,t}(G)$ and some other parameters. The reader can find comprehensive information about many domination parameters until 1998 in [4] and [5].

2. Bounds on $L_{k,t}(G)$ and $\gamma_{\times k,t}(G)$

At this point we give the values of the k-total limited packing number of some familiar graphs in the following observation. We assume that $k < \Delta$, otherwise $L_{k,t}(G) = |V(G)|$.

Observation 1. *Let $m, n, k \in N$ with $n \geq 3$. Then,*
(i) $L_{k,t}(K_n) = k$,
(ii) $L_{k,t}(K_{m,n}) = min\{k, m\} + min\{k, n\}$.

We apply these k-total limited packing numbers when we show that our bounds are sharp. If we impose constraints on the minimum degree δ we can obtain the following.

Theorem 1. *Let G be a graph of order n, k be a positive integer and $\delta \geq k$. Then $L_{k,t}(G) \leq \lfloor \frac{k}{\delta} n \rfloor$, and this bound is sharp.*

Proof. Let L be a maximum k-total limited packing set in G. We count the number, $|[L, V - L]|$, of edges with end points in both L and $V - L$. Since L is a k-total limited packing set, every vertex in $V - L$ has at most k neighbors in L. Therefore $|[L, V - L]| \leq k(n - |L|)$. On the other hand every vertex in L has at least $\delta - k$ neighbors in $V - L$. Hence $(\delta - k)|L| \leq |[L, V - L]|$. Together these inequalities imply $|L| \leq \frac{kn}{\delta}$ and hence, $L_{k,t}(G) \leq \lfloor \frac{k}{\delta} n \rfloor$. Now we show that this bound is sharp. Consider the complete graph K_n and let $k < n - 1$ and $n > 2$. Then, $L_{k,t}(K_n) = k = \lfloor k + \frac{k}{n-1} \rfloor = \lfloor \frac{k}{\delta} n \rfloor$. ■

By a recursive process one can bound the size of k-total limited packing in terms of k and $\rho_0(G)$ as follows.

Theorem 2. *Let G be a graph of order n, and $2 \leq k \leq \Delta$. Then $L_{k,t}(G) \geq L_{k-1,t}(G) + 1$. Moreover $L_{k,t}(G) \geq \rho_0(G) + k - 1$, and this bound is sharp.*

Proof. Let L be a maximum $(k-1)$-total limited packing set in G. Let $L = V$. If u is a vertex with $deg(u) = \Delta$, then $\Delta = |N(u)| = |N(u) \cap L| \leq k - 1$. This is a contradiction. Therefore, there is a vertex u which belongs to $V - L$. It is easy to check that $|N(v) \cap (L \cup \{u\}| \leq k$, for all $v \in V(G)$. This shows that $L \cup \{u\}$ is a k-total limited packing set in G. Hence,

$$L_{k,t}(G) \geq |L \cup \{u\}| = |L| + 1 = L_{k-1,t}(G) + 1.$$

Repeating these inequalities, we have

$$L_{k,t}(G) \geq L_{k-1,t}(G) + 1 \geq \ldots \geq L_{1,t}(G) + k - 1 = \rho_0(G) + k - 1.$$

For sharpness we consider the graph $K_{1,n}$, when $n > 2$ and $k \leq n$. By Observation 1, $L_{k,t}(K_{1,n}) = 1 + k = 2 + k - 1 = L_{1,t}(K_{1,n}) + k - 1 = \rho_0(K_{1,n}) + k - 1$. This completes the proof. ■

One can obtain a sharp lower bound on $\gamma_{\times k,t}(G)$ corresponding to the previous theorem for $L_{k,t}(G)$ as follows.

Theorem 3. *Let G be a graph with $\delta \geq k$. Then $\gamma_{\times k,t}(G) \geq \gamma_{\times (k-1),t}(G) + 1$. Moreover, $\gamma_{\times k,t}(G) \geq \gamma_t(G) + k - 1$, and this bound is sharp.*

Proof. Let S be a minimum k-tuple total dominating set in G, and $u \in S$. It is easy to check that $|N(v) \cap (S - \{u\})| \geq k - 1$, for all $v \in V(G)$. Therefore $S - \{u\}$ is a $(k-1)$-tuple total dominating set in G. Hence,

$$\gamma_{\times (k-1),t}(G) \leq |S - \{u\}| = |S| - 1 \leq \gamma_{\times k,t}(G) - 1.$$

Repeating these inequalities, we have

$$\gamma_{\times k,t}(G) \geq \gamma_{\times (k-1),t}(G) + 1 \geq \ldots \geq \gamma_{\times 1,t}(G) + k - 1 = \gamma_t(G) + k - 1.$$

For sharpness it is sufficient to consider the graph K_n, when $k \leq n - 1$. Then $\gamma_{\times k,t}(K_n) = k + 1 = 2 + k - 1 = \gamma_t(K_n) + k - 1$. ■

3. Relationships among $L_{k,t}(G), \gamma_{\times k}(G)$ and $\gamma_{\times k,t}(G)$

In this section we establish a link between the concepts of total limited packing and tuple total domination. By this connection we will be able to obtain some new sharp bounds. First, we need the following useful lemma.

Lemma 1. *Let G be a graph.Then the following statements hold.*

(i) Let $\delta \geq k$. If S is a k-tuple total dominating, then $V - S$ is a $(\Delta - k)$-total limited packing set in G.

(ii) Let $\delta \geq k + 1$. If L is a k-total limited packing, then $V - L$ is a $(\delta - k)$-tuple total dominating set in G.

Proof. (i) Let S be a k-tuple total dominating set in G. Since every vertex v in $V(G)$ has at least k neighbors in S, then it has at most $\Delta - k$ neighbors in $V - S$. Therefore, $V - S$ is a $(\Delta - k)$-total limited packing in G.

(ii) Let L be a k-total limited packing set in G. Every vertex v in $V(G)$ has at most k neighbors in L. Therefore, it has at least $\delta - k$ neighbors in $V - L$. Hence, $V - L$ is a $(\delta - k)$-tuple total dominating set in G. ■

We are now in a position to present an upper bound on $L_{k,t}(G)$ that involves the total domination number of G.

Theorem 4. *Let G be a graph of order n, k be a positive integer and $\delta \geq k+1$. Then $L_{k,t}(G) \leq n - \gamma_t(G) - \delta + k + 1$, and this bound is sharp.*

Proof. Let L be a maximum k-total limited packing in G. By Lemma 1 the set $V - L$ is a $(\delta - k)$-tuple total dominating set in G. Now Theorem 2 implies

$$n - |L| \geq \gamma_{\times(\delta-k),t}(G) \geq \gamma_t(G) + \delta - k - 1.$$

Hence, $L_{k,t}(G) = |L| \leq n - \gamma_t(G) - \delta + k + 1$. Now, we show that this bound is sharp. Consider the graph K_n, when $n \geq 3$ and $k \leq n-2$. Then $L_{k,t}(K_n) = k = n - 2 - (n-1) + k + 1 = n - \gamma_t(K_n) - \delta + k + 1$. ■

Theorem 4 is an improvement of the following theorem that is presented in [8].

Theorem 5. *Let G be a graph of order n with $\delta(G) \geq 2$. Then $\rho^o(G) \leq n - \gamma_t(G)$.*

We can directly involve tuple (total) domination and total limited packing numbers to obtain upper and lower bounds on each other, respectively. We obtain two upper bounds on $L_{k,t}(G)$ directly in terms of $\gamma_{\times k}(G)$ and $\gamma_{\times k,t}(G)$ in the two following theorems.

Theorem 6. *Let G be a graph of order n, and k, k' be positive integers such that $\delta \geq k' - 1$. Then $L_{k,t}(G) \leq \min\{\frac{k+1}{k'}, \frac{k}{k'-1}\}\gamma_{\times k'}(G)$.*

Proof. Let L be a maximum k-total limited packing and D be a minimum k'-tuple dominating set in G. For notational convenience, we define A_1 and A_2 by

$$\{(l,d) |\ l \in L,\ d \in D \ \text{ and } \ l \in N[d]\}$$

and

$$\{(l,d) |\ l \in L,\ d \in D \ \text{ and } \ l \in N(d)\},$$

respectively. Since D is a k'-tuple dominating set $|N[l] \cap D| \geq k'$ for every vertex $l \in L$. This shows that

$$k'|L| \leq |A_1| \ \text{ and } \ (k'-1)|L| \leq |A_2|. \tag{1}$$

On the other hand, since L is a k-total limited packing set in G, every vertex $d \in D$ has at most k neighbors in L. Therefore

$$|A_1| \leq (k+1)|D| \ \text{ and } \ |A_2| \leq k|D|. \tag{2}$$

Together inequalities (1) and (2) imply

$$k'L_{k,t}(G) \leq (k+1)\gamma_{\times k'}(G) \ \text{ and } \ (k'-1)L_{k,t}(G) \leq k\gamma_{\times k'}(G).$$

Hence, $L_{k,t}(G) \leq \min\{\frac{k+1}{k'}\gamma_{\times k'}(G), \frac{k}{k'-1}\gamma_{\times k'}(G)\}$ and this completes the proof. ∎

Also, by a similar fashion we can present the following theorem.

Theorem 7. *Let G be a graph of order n, and k, k' be positive integers such that $\delta \geq k'$. Then $L_{k,t}(G) \leq \frac{k}{k'}\gamma_{\times k',t}(G)$.*

Proof. Let L and S be a maximum k-limited packing and a minimum k'-tuple total dominating set in G, respectively. Also, let us consider A as the set of ordered pairs $\{(l, s)|\ l \in L,\ s \in S \text{ and } l \in N(s)\}$. Similar to the proof of Theorem 6 we deduce that $k'|L| \leq |A| \leq k|S|$. Hence, $k'L_{k,t}(G) \leq k\gamma_{\times k',t}(G)$. ∎

This theorem generalizes a result in [6] which states that for a graph G with $\delta \geq r$, $\gamma_{\times r,t}(G) \geq r\rho_o(G)$, when $k = 1$ and $k' = r$.

We conclude this section with some results on r-regular graphs. Let $r \geq k + 1$. According to the proof of Theorem 4

$$n - L_{k,t}(G) \geq \gamma_{\times(r-k),t}(G)$$

On the other hand, if S is a minimum $(r - k)$-tuple total dominating, then Lemma 1 implies $V - S$ is a k-total limited packing set in G. Therefore,

$$n - \gamma_{\times(r-k),t}(G) \leq L_{k,t}(G).$$

Together these two inequalities imply

$$L_{k,t}(G) + \gamma_{\times(r-k),t}(G) = n, \tag{3}$$

for all $r(\geq k + 1)$-regular graphs G of order n.

Let us now point to the concept of lower open packing. The *lower open packing number* of G, denoted $\rho_{oL}(G)$, is the minimum cardinality of a maximal open packing of G. Henning and Slater [9] showed that if G is a graph of order n and maximum degree Δ, then

$$\rho_{oL}(G) \geq \frac{n}{\Delta(\Delta - 1) + 1}. \tag{4}$$

Moreover, we need the following useful lemma.

Lemma 2 ([1]). *If G is a graph of order n with $\delta \geq 3$, then $\gamma_t(G) \leq n/2$.*

Considering Theorem 1 along with the above discussion we can bound the open packing and 2-total limited packing number of a cubic graph G as follows.

Theorem 8. *Let G be a connected cubic graph of order n. Then*
(i) $n/7 \leq \rho_o(G) \leq n/3$.
(ii) $n/2 \leq L_{2,t}(G) \leq 2n/3$.

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