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A GENERAL FIXED POINT THEOREM FOR A PAIR OF MULTI-VALUED MAPPINGS IN PARTIAL METRIC SPACES

Abstract. The purpose of this paper is to prove a general fixed point theorem for a pair of multi-valued mappings satisfying a new type of implicit relation in partial metric spaces, which generalizes Theorem 2.2 [4], Theorem 3.1 [3], Theorem 3.2 [7], Corollary 2.3 [4], Theorem 2.8 [16] and obtain other particular results.

Key words: fixed point, pair of multi-valued mappings, partial metric space, implicit relation.

AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

In 1994, Matthews [9] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks and proved the Banach contraction principle in such spaces.

Many authors studied the fixed points for mappings satisfying some contractive condition in complete partial metric spaces. In 2012, Aydi et al. [3] introduced the definition of partial Hausdorff metric and also proved the existence of the Banach contraction principle for multi-valued mappings in complete partial metric spaces.

Quite recently, in [4] is proved a common fixed point theorem for a pair of multi-valued mappings satisfying a contractive condition in partial metric spaces.

Other results for fixed points of multi-valued mappings in complete partial metric spaces are recently obtained in [1], [5], [6], [7], [8], [16] and in other paper.

A Nadler type theorem for multi-valued mappings in partial metric spaces is proved in [3]. Quite recently, a new generalized Nadler type theorem in complete partial metric spaces is proved in [7].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [11], [12] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi-metric spaces, ultra-metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two or three metric spaces, for single valued mappings, hybrid pairs of mappings and set-valued mappings. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces and G-metric spaces. With this method the proofs of some fixed points theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures. The study of fixed points for self mappings in complete partial metric spaces satisfying an implicit relation is initiated in [17].

The study of coincidence and fixed points for multi-valued mappings in metric spaces satisfying implicit relations is initiated in [13], [14], [15] and in other papers.

The purpose of this paper is to prove a general fixed point theorem for a pair of multi-valued mappings satisfying a new type of implicit relation in partial metric spaces, which generalizes Theorems 2.2 [4], Theorem 3.1 [3], Theorem 3.2 [7], Corollary 2.3 [4], Theorem 2.8 [16] and obtain other particular results.

2. Preliminaries

Definition 1 ([9]). *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

$(P_1) : p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$,

$(P_2) : p(x,x) \leq p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is called a partial metric space.

If $p(x,y) = 0$, then $x = y$, but the converse does not hold always.

Each partial metric p on X generates a T_0-topology τ_p which has as base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,y) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x, x_n)$.

If p is a partial metric on X, then the function $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X.

Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$(1) \qquad \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Definition 2 ([9]). *Let (X, p) be a partial metric space.*

a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

b) (X, p) is said to be complete if every Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Lemma 1 ([9]). *Let (X, p) be a partial metric space. Then:*

a) A sequence in X is a Cauchy sequence in (X, p) if and only if is a Cauchy sequence in (X, p^s);

b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

Let (X, p) be a partial metric space. We denote by $CB^p(X)$ the family of all nonempty closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p, where the closedness in take from (X, τ_p) and boundness if given as follows: A is a bounded subset in (X, p) if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$,, we have $a \in B_p(x_0, M)$, that is $p(x_0, a) < p(a, a) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, we define

$$\begin{aligned} p(x, A) &= \inf\{p(x, a) : a \in A\}, \\ \delta_p(A, B) &= \sup\{p(a, B) : a \in A\} \end{aligned}$$

and

$$\delta_p(B, A) = \sup\{p(b, A) : b \in B\}.$$

Then $p(x, A) = 0$ implies $p^s(x, A) = 0$, where $p^s(x, A) = \inf\{p^s(x, A) : a \in A\}$.

Lemma 2 ([2]). *Let (X, p) be a partial metric space and A a nonempty subset in (X, p). Then $x \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.*

The following properties of $\delta_p : CB^p(X) \times CB^p(X) \to [0, \infty)$ are established in [3].

Lemma 3 ([3]). *For $A, B, C \in CB^p(X)$, we have the following:*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$,

(ii) $\delta_p(A, A) \leq \delta_p(A, B)$,

(iii) $\delta_p(A, B) = 0$ *implies* $A \subset B$,

(iv) $\delta(A, B) \leq \delta(A, C) + \delta(C, B) - \inf\{p(c, c) : c \in C\}$.

Let (X,p) be a partial metric space. For $A, B \in CB^p(X)$ we define [3]

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

Lemma 4 ([3]). *For all $A, B, C \in CB^p(X)$, we have:*
(i) $H_p(A,A) \leq H_p(A,B)$,
(ii) $H_p(A,B) = H_p(B,A)$,
(iii) $H_p(A,B) \leq H_p(A,C) + H(C,B) - \inf\{p(c,c) : c \in C\}$.

Corollary 1 ([3]). *Let (X,p) be a partial metric space. For $A, B \in CB^p(X)$, $H_p(A,B) = 0$ implies $A = B$.*

Remark 1. The converse of Corollary 1 is not true in general.

The mapping H_p is called a partial Hausdorff metric induced by p [3].

Lemma 5 ([3]). *Let (X,p) be a partial metric space, $A, B \in CB^p(X)$ and $k > 1$. For any $a \in A$ there exists $b = b(a) \in B$ such that $p(a,b) \leq kH(A,B)$.*

Lemma 6 ([16]). *Let $x_n \to x$ as n tends to infinite in a partial metric space with $p(x,x) = 0$. Then $\lim_{n\to\infty} p(x_n, B) = p(x, B)$ for any $B \in CB^p(X)$.*

The following result generalizes Nadler theorem [10] for partial metric space.

Theorem 1 ([3] Theorem 3.1). *Let (X,p) be a complete partial metric space. If $T : (X,p) \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$ we have*

$$H_p(Tx,Ty) \leq hp(x,y),$$

where $h \in (0,1)$, then T has a fixed point.

Quite recently, a generalization of Theorem 1 is obtained.

Theorem 2 ([7] Theorem 3.2). *Theorem 2. Let (X,p) be a complete partial metric space and let $T : (X,p) \to CB^p(X)$ be a multi-valued mapping such that for all $x, y \in X$*

$$H(Tx,Ty) \leq \alpha p(x,y) + \beta[p(x,Tx) + p(y,Ty)] + \gamma[p(x,Ty) + p(y,Tx)],$$

where $\alpha, \beta, \gamma \geq 0$, $0 < \alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

In a recent paper [4] the following results are obtained.

Theorem 3 ([4] Theorem 2.2). *Let (X,p) be a complete partial metric space and $T,S:(X,p)\to CB^p(X)$ be two multi-valued mappings satisfying, for all $x,y\in X$, the following condition:*

$$(2)\quad H_p(Tx,Sy)\le \alpha\max\{p(x,y),p(x,Tx),p(y,Sy),\frac{p(x,Sy)+p(y,Tx)}{2}\},$$

$\alpha\in[0,1)$. *Then, T and S have a common fixed point. Moreover, if T or S is a single valued mapping, then the common fixed point is unique.*

Corollary 2 ([4] Corollary 2.3). *Let (X,p) be a complete partial metric space and $T,S:(X,p)\to CB^p(X)$ be two multi-valued mappings satisfying, for all $x,y\in X$, the following condition:*

$$\begin{aligned} H_p(Tx,Sy) \ \le\ & a_1p(x,y)+a_2p(x,Tx)+a_3p(y,Sy) \\ & + a_4[p(x,Sy)+p(y,Tx)], \end{aligned}$$

where $a_1,a_2,a_3,a_4\ge 0$ and $0<a_1+a_2+a_3+2a_4<1$. Then, T and S have a common fixed point. Moreover, if T or S is a single valued mapping, then the common fixed point is unique.

Theorem 4 ([16] Theorem 2.8). *Let (X,p) be a complete partial metric space and $S,T:(X,p)\to CB^p(X)$ be mappings satisfying (2). Then T and S have a common fixed point. Further, if we assume that $p(x,y)\le p(x,Sx)$ or $p(x,y)\le p(y,Tx)$ for all $x,y\in X$, then T and S have a unique common fixed point.*

3. Implicit relations

Definition 3. *Let $\mathfrak{F}_p$ be the set of all continuous functions $F(t_1,\dots,t_5):\mathbb{R}_+^5\to\mathbb{R}$ satisfying the following conditions:*

(F_1) : *F is increasing in variable t_1 and nonincreasing in variables t_3,t_4,t_5;*

(F_2) : *There exist $h_1,h_2\in(0,1)$ and $k>1$ such that for all $u,v\ge 0$, $t>0$ and $u\le kt$ then:*

(F_{2a}) : *$F(t,v,v,u,u+v)\le 0$ implies $u\le h_1v$,*

(F_{2b}) : *$F(t,v,u,v,u+v)\le 0$ implies $u\le h_2v$.*

Example 1. $F(t_1,\dots,t_5)=t_1-a\max\{t_2,t_3,t_4,\frac{t_5}{2}\}$, where $\alpha\in(0,1)$.

(F_1) : Obviously.

(F_2) : Let $1<k<\frac{1}{\alpha}$ be such that if $u,v\ge 0,t>0$ and $u\le kt$, $F(t,v,v,u,u+v)=t-\alpha\max\{u,v,\frac{u+v}{2}\}\le 0$. Then $u\le \alpha k\max\{u,v,\frac{u+v}{2}\}$. If $u>v$, then $u(1-\alpha k)\le 0$, a contradiction. Hence, $u\le v$, which implies $u\le h_1v$, where $0<h_1=\alpha k<1$.

Similarly, $u\le kt$ and $F(t,v,u,v,u+v)\le 0$ implies $u\le h_2v$.

Example 2. $F(t_1,\dots,t_5)=t_1-at_2-bt_3-ct_4-dt_5$, where $a,b,c,d\geq 0$ and $0<a+b+c+2d<1$.

(F_1) : Obviously.

(F_2) : Let $1<k<\frac{1}{a+b+c+2d}$ be such that if $u,v\geq 0$, $t>0$ and $u\leq kt$, $F(t,v,v,u,u+v)=t-[av+bv+cu+d(u+v)]\leq 0$. Then $u\leq kt$ implies $u\leq h_1v$, where $0<h_1=\frac{k(a+b+d)}{1-k(c+d)}<1$.

Similarly, $u\leq kt$ and $F(t,v,u,v,u+v)\leq 0$ implies $u\leq h_2v$, where $0<h_2=\frac{k(a+c+d)}{1-k(c+d)}<1$.

If $h=\max\{h_1,h_2\}$ and $u\leq kt$ then $u\leq hv$.

Example 3. $F(t_1,\dots,t_5)=t_1-a\max\{t_2,t_3,t_4,t_5\}$, where $\alpha\in\left[0,\frac{1}{2}\right)$.

(F_1) : Obviously.

(F_2) : Let $1<k<\frac{1}{2\alpha}$ be such that if $u,v\geq 0,t>0$ and $u\leq kt$, $F(t,v,v,u,u+v)=t-\alpha\max\{v,u,u+v\}\leq 0$. Then $u\leq kt$ implies $u\leq k\alpha(u+v)$, hence $u\leq hv$, where $0<h=\frac{\alpha k}{1-\alpha k}<1$.

Similarly, $u\leq kt$ and $F(t,v,u,v,u+v)\leq 0$ implies $u\leq hv$.

Example 4. $F(t_1,\dots,t_5)=t_1-\max\{ct_2,ct_3,ct_4,at_5\}$, where $0<a$, $c<\frac{1}{2}$.

(F_1) : Obviously.

(F_2) : Let $1<k<\frac{1}{2\max\{a,c\}}$ be such that if $u,v\geq 0,t>0$ and $u\leq kt$, $F(t,v,v,u,u+v)=t-\max\{cu,cv,a(u+v)\}\leq 0$. Since $u\leq kt$, then $u\leq k\max\{a,c\}(u+v)$ which implies $u\leq hv$, where $0<h=\frac{k\max\{a,c\}}{1-k\max\{a,c\}}<1$.

Similarly, $u\leq kt$ and $F(t,v,u,v,u+v)\leq 0$ implies $u\leq hv$.

Example 5. $F(t_1,\dots,t_5)=t_1^2-at_2t_3-bt_4^2-ct_5^2$, where $a,b,c\geq 0$ and $0<a+b+4c<1$.

(F_1) : Obviously.

(F_2) : Let $1<k<\frac{1}{\sqrt{a+b+4c}}$ be such that if $u,v\geq 0$, $t>0$ and $u\leq kt$, $F(t,v,v,u,u+v)=t^2-av^2-bu^2-c(u+v)^2\leq 0$. If $u>v$, since $u\leq kv$, then $u^2[1-k^2(a+b+4c)]\leq 0$, a contradiction. Hence $u\leq v$ which implies $u\leq hv$, where $0<h=k\sqrt{a+b+4c}<1$.

Similarly, $u\leq kt$ and $F(t,v,u,v,u+v)\leq 0$ implies $u\leq hv$.

Example 6. $F(t_1,\dots,t_5)=t_1-at_2-bt_3-c\max\{2t_4,t_5\}$, where $a,b,c\geq 0$ and $0<a+b+2c<1$.

(F_1) : Obviously.

(F_2) : Let $1<k<\frac{1}{a+b+2c}$ be such that if $u,v\geq 0,t>0$ and $u\leq kt$, $F(t,v,v,u,u+v)=t-av-bv-c\max\{2u,u+v\}\leq 0$. Since $u\leq kt$, if $u>v$, then $u[1-k(a+b+2c)]\leq 0$, a contradiction. Hence $u\leq v$ which implies $u\leq hv$, where $0<h=k(a+b+2c)<1$.

Similarly, $u\leq kt$ and $F(t,v,u,v,u+v)\leq 0$ implies $u\leq hv$.

4. Main results

Theorem 5. *Let (X,p) be a complete partial metric space and T,S : $(X,p) \to CB^p(X)$ be two multi-valued mappings satisfying, for all $x,y \in X$, the following condition*

$$(3)\qquad \begin{cases} F(H_p(Tx,Sy), p(x,y), p(x,Tx), \\ p(y,Sy), p(x,Sy) + p(y,Tx)) \leq 0, \end{cases}$$

where $F \in \mathfrak{F}_p$. Then, T and S have a common fixed point. Moreover, if T or S is single-valued, then the common fixed point is unique.

Proof. Let $x_0 \in X$ be and $x_1 \in Sx_0$. By Lemma 5, there exists $x_2 \in Tx_1$ such that

$$p(x_2, x_1) \leq kH_p(Tx_1, Sx_0).$$

Continuing in this manner, we can construct a sequence $\{x_n\}$ in X such that

$$(4)\qquad x_{2n+1} \in Sx_{2n} \quad \text{and} \quad x_{2n+2} \in Tx_{2n+1}$$

$$(5)\qquad p(x_{2n+1}, x_{2n}) \leq kH_p(Sx_{2n}, Tx_{2n-1}),$$

$$(6)\qquad p(x_{2n+2}, x_{2n+1}) \leq kH_p(Tx_{2n+1}, Sx_{2n}).$$

By (3) we get

$$\begin{cases} F(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, Tx_{2n-1}), \\ p(x_{2n}, Sx_{2n}), p(x_{2n-1}, Sx_{2n}) + p(x_{2n}, Tx_{2n-1})) \leq 0. \end{cases}$$

By (4) and (F_1) we have

$$\begin{cases} F(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})) \leq 0. \end{cases}$$

Since by (P_4),

$$p(x_{2n-1}, x_{2n+1}) \leq p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n}),$$

by (F_1) we obtain

$$(7)\qquad \begin{cases} F(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})) \leq 0. \end{cases}$$

By (5), (7) and (F_{2a}) it follows that

$$p(x_{2n}, x_{2n+1}) \leq h_1 p(x_{2n-1}, x_{2n}).$$

Similarly, by (3) we have

$$\begin{cases} F(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n+1}, x_{2n}), p(x_{2n+1}, Tx_{2n+1}), \\ p(x_{2n}, Sx_{2n}), p(x_{2n+1}, Sx_{2n}) + p(x_{2n}, Tx_{2n+1})) \leq 0. \end{cases}$$

By (F_1) we get

$$\begin{cases} F(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n+1}, x_{2n}), p(x_{2n+1}, x_{2n+2}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})) \leq 0. \end{cases}$$

Since by (P_4),

$$p(x_{2n}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) - p(x_{2n+1}, x_{2n+1}),$$

by (F_1) we obtain

$$\text{(8)} \qquad \begin{cases} F(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n+1}, x_{2n}), p(x_{2n+1}, x_{2n+2}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})) \leq 0. \end{cases}$$

Then by (6), (8) and (F_{2b}) we have

$$p(x_{2n+1}, x_{2n+2}) \leq h_2 p(x_{2n}, x_{2n+1}).$$

Let $h = \max\{h_1, h_2\}$. Then

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n) \leq \ldots \leq h^p p(x_0, x_1).$$

For every $k \in \mathbb{N}$ we have

$$\begin{aligned} p(x_n, x_{n+k}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{n+k-1}, x_{n+k}) \\ &\leq (h^n + h^{n+1} + \ldots + h^{n+k-1}) p(x_0, x_1) \\ &= \frac{h^n}{1-h} p(x_0, x_1) \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

By the definition of p^s we get

$$p^s(x_n, x_{n+k}) \leq 2p(x_n, x_{n+k}) \to 0 \ \text{ as } \ n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete then by Lemma 1 (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $v \in X$ with respect to the metric p^s, that is

$$\lim_{n\to\infty} p^s(x_n, v) = 0.$$

Moreover, by (2) we have

$$p(v,v) = \lim_{n\to\infty} p(x_n, v) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{9}$$

Also, we get

$$\begin{cases} F(H_p(Tx_{2n+1}, Sv), p(x_{2n+1}, v), p(x_{2n+1}, Tx_{2n+1}), \\ p(v, Sv), p(x_{2n+1}, Sv) + p(v, Tx_{2n+1})) \leq 0. \end{cases}$$

By (F_1) we get

$$\begin{cases} F(p(x_{2n+2}, Sv), p(x_{2n+1}, v), p(x_{2n+1}, x_{2n+2}), \\ p(v, Sv), p(x_{2n+1}, Sv) + p(v, x_{2n+2})) \leq 0. \end{cases}$$

By (9) and Lemma 6 letting n tends to infinity we obtain

$$F(p(v, Sv), 0, 0, p(v, Sv), p(v, Sv)) \leq 0.$$

Since $p(v, Sv) < kp(v, Sv)$, by (F_{2a}) we have $p(v, Sv) = 0$. Since Sv is closed then $v \in Sv$.

Similarly, by (F_{2b}) we obtain $v \in Tv$.

Now, we show that the common point v is unique if T is a single-valued mappings. Assume that $u \in X$ is another common fixed point of T and S.

By (3) we have

$$\begin{cases} F(H_p(Tu, Sv), p(u, v), p(u, Tu), \\ p(v, Sv), p(u, Sv) + p(v, Tu)) \leq 0. \end{cases}$$

Since $u = \{Tu\}$ and $v \in Sv$ we have

$$\begin{aligned} p(u,v) &\leq H_p(Tu, Sv), \\ p(v, Sv) &= p(v,v), \end{aligned}$$

which implies by (F_1) that

$$F(p(u,v), p(u,v), p(u,u), p(v,v), p(u,v) + p(u,v)) \leq 0.$$

By $p(u,u) \leq p(u,v)$, $p(v,v) \leq p(u,v)$ and by $(F1)$, we obtain

$$F(p(u,v), p(u,v), p(u,v), p(u,v), p(u,v) + p(u,v)) \leq 0.$$

Since $p(u,v) \leq kp(u,v)$ if $p(u,v) > 0$, by (F_{2a}) or (F_{2b}) we have

$$p(u,v) \leq hp(u,v) < p(u,v),$$

a contradiction. Hence, $p(u,v) = 0$ which implies $u = v$ and v is the unique common fixed point of T and S. If S is single-valued mapping instead T, the proof is similar. ■

Corollary 3. *Theorem 3.*

Proof. The proof it follows by Theorem 5 and Example 1. ■

Example 7. Let $X = \{0,1,2\}$ be endowed with the partial metric $p : X \times X \to \mathbb{R}_+$ defined by $p(0,0) = p(1,1) = 0$, $p(0,1) = p(1,0) = \frac{1}{4}$, $p(2,2) = \frac{1}{3}$, $p(0,2) = p(2,0) = \frac{2}{5}$, $p(1,2) = p(2,1) = \frac{13}{20}$.

We define $T : X \to X$ and $S : X \to CB^p(X)$ by

$$Tx = \begin{cases} 0, & x \in \{0,1\} \\ 1, & x = 2 \end{cases}$$

and

$$Sx = \begin{cases} \{0\}, & x \neq 2 \\ \{0,1\}, & x = 2 \end{cases}$$

as in Example 2.11 [4], and

$$M(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Sy), \frac{p(x,Sy)+p(y,Tx)}{2}\right\}.$$

Note that Sx is closed and bounded for all $x \in X$ under the given partial metric p.

We distinguish the following cases:

(i) If $x \in \{0,1\}$ then $H_p(\{T0\}, S1) = 0$ and $H_p(\{Tx\}, Sy) \leq \alpha M(\{Tx\}, Sy)$ for $\alpha \in [0,1)$.

(ii) If $x = 0$ and $y = 2$, then $H_p(\{T0\}, S2) = H_p(\{0\}, \{0,1\}) = \frac{1}{4}$ and

$$\begin{cases} M(0,2) & = \max\left\{p(0,2), p(0,T0), p(2,S2), \frac{p(0,S2)+p(2,T0)}{2}\right\} \\ & = \max\left\{\frac{2}{5}, 0, \frac{2}{5}, \frac{0+\frac{2}{5}}{2}\right\} = \frac{2}{5}. \end{cases}$$

Hence $Hp(\{T0\}, S2) = \frac{1}{4} \leq \alpha M(0,2)$ for $\alpha \in \left[\frac{5}{8}, 1\right)$.

(iii) Similarly, if $x = 2$ and $y = 0$, then $H_p(\{T2\}, S0) = H_p(\{1\}, 0) = \frac{1}{4} \leq \alpha M(2,0) = \frac{13}{20}\alpha$. Hence, $H_p(\{T2\}, S0) = \frac{1}{4} \leq \alpha M(2,0) = \frac{13}{20}\alpha$ for $\alpha \in \left[\frac{5}{13}, 1\right)$.

(*iv*) If $x = 2$, $y = 1$, then $H_p(\{T2\}, S1) = H_p(\{1\},\{0\}) = \frac{1}{4}$ and $M(2,1) = \frac{13}{20}$. Hence $H_p(\{T2\}, S1) = \frac{1}{4} \leq \alpha M(2,1) = \frac{13}{20}\alpha$. Hence $Hp(\{T2\}, S1) = \frac{1}{4} \leq \alpha M(2,1) = \frac{13}{20}$ for $\alpha \in \left[\frac{5}{13}, 1\right)$.

(*v*) Similarly, if $x = 1$, $y = 2$, we have $H_p(\{T1\}, S2) = \frac{1}{4} \leq \alpha M(1,2) = \frac{13}{20}\alpha$ for $\alpha \in \left[\frac{5}{13}, 1\right)$.

(*vi*) If $x = 2$, $y = 2$, similarly we obtain $H_p(\{T2\}, S2) = \frac{1}{4} \leq \alpha M(2,2) = \frac{13}{20}\alpha$ for $\alpha \in \left[\frac{5}{13}, 1\right)$.

Hence, $Hp(\{Tx\}, Sy) \leq \alpha M(x,y)$ for $\alpha \in \left[\frac{5}{8}, 1\right)$. Thus all the conditions of Corollary 3 are satisfied and $x = 0$ is the unique common fixed point.

Corollary 4. *Corollary 2.*

Proof. The proof it follows by Theorem 5 and Example 2 with $a_2 = a_3$. ■

If $T = S$, by Theorem 5 we obtain

Theorem 6. *(X,p) be a complete partial metric space and $T : (X,p) \to CB^p(X)$ be a multi-valued mapping satisfying, for all $x, y \in X$, the following condition*

$$F(H_p(Tx,Ty), p(x,y), p(x,Tx), p(y,Ty), p(x,Ty) + p(y,Tx)) \leq 0,$$

where $F \in \mathfrak{F}_p$. Then, T has a fixed point. Moreover, if T is single-valued mapping, then the common fixed point is unique.

Remark 2. (*a*) The proof of Theorem 1 it follows by Theorem 6 and Example 2 with $b = c = d = 0$.

(*b*) The proof of Theorem 2 it follows by Theorem 6 and Example 2 with $b = c = \beta$.

Definition 4. *A function $F(t_1, \ldots, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ satisfies the condition (F_1'): F is increasing in variable t_1 and nonincreasing in variables t_2, t_3, t_4, t_5.*

Theorem 7. *Let (X,p) be a complete partial metric space and $T, S : X \to CB^p(X)$ be two multivalued mappings satisfying (2) for all $x, y \in X$,where F satisfy (F_1') and F_2. Then, T and S have a common fixed point. Further if $p(x,y) \leq p(y,Sx)$ or $p(x,y) \leq p(y,Tx)$, then T and S have a unique common fixed point in X.*

Proof. As in Theorem 5, T and S have a common fixed point z. Suppose that z' is another fixed point of T and S. By hypothesis

$$p(z,z') \leq p(z',Sz) \leq H_p(Sz,Tz') \tag{10}$$

Then by (2) we have

$$\begin{cases} F(H_p(Tz, Sz'), p(z', z), p(z', Tz'), \\ p(z, Sz), p(z, Tz') + p(z', Sz)) \leq 0. \end{cases}$$

From (F_1') and (10) we obtain:

$$\begin{cases} F(H_p(Tz', Sz), H_p(Tz', Sz), H(Tz', Tz'), \\ Hp(Sz, Sz), H_p(Sz, Tz') + H_p(z', Sz)) \leq 0. \end{cases}$$

By (F_1') and Lemma 4 (i) we obtain

$$\begin{cases} F(H_p(Tz', Sz), H_p(Tz', Sz), H_p(Tz', Sz), \\ H_p(Tz', Sz), H_p(Tz', Sz) + Hp(Tz', Sz)) \leq 0. \end{cases}$$

Since $H_p(Tz', Sz) \leq kH_p(Tz', Sz)$ then by (F_2) we obtain $H_p(Tz', Sz) \leq h_1 H_p(Tz', Sz)$ which implies $H_p(Tz, Sz') = 0$ and by (10), $z = z'$. ■

Corollary 5. *Theorem 4.*

Proof. The proof it follows by Theorem 7 and Example 1. ■

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