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ON SECOND ORDER NONLOCAL BOUNDARY VALUE PROBLEM AT RESONANCE

Abstract. This work is devoted to the existence of solutions for a system of nonlocal resonant boundary value problem

$$x'' = f(t,x), \quad x'(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s),$$

where $f : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous and $g : [0,1] \to \mathbb{R}^k$ is a function of bounded variation.

Key words: nonlocal boundary conditions, resonant problem, nonlinear problem, Neumann problem.

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1. Introduction

Nonlocal boundary value problems (BVPs) arise in different areas of applied mathematics and physics. Such problems, for instance, have applications in chemical engineering, thermo-elasticity, underground water flow and population dynamics (see [2, 8, 18] and the references therein).

Nowadays, the problem of the existence of solutions for various types of nonlocal BVPs is the subject of many papers. Let us notice that BVPs with Riemann-Stieltjes integral boundary conditions include as special cases multi-point and integral BVPs. For such problems and comments on their importance, we refer the reader, for example, to [1, 3, 4, 6, 11, 19, 20].

In the paper the following system of ordinary differential equations

$$x'' = f(t,x), \quad x'(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s), \tag{1}$$

where $f = (f_1, \dots, f_k) : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous, $g = (g_1, \dots, g_k) : [0,1] \to \mathbb{R}^k$ has bounded variation, is studied.

As far as we are aware, the BVP (1) has not been studied in this generality so far. The motivation to deal with this problem is the fact that, in the case when $g \equiv 0$, it is a generalization of the Neumann BVP

$$x'' = f(t, x), \quad x'(0) = 0, \quad x'(1) = 0.$$

Until now, under suitable monotonicity conditions or nonresonance conditions, some existence or uniqueness theorems or methods for Neumann BVPs have been presented (compare [5, 10, 13, 14, 15] and the references therein).

In this paper we have shown that, under suitable assumptions on f, the BVP (1), has at least one solution. Here, we consider only the resonant case. First, we perturb the boundary condition so as to obtain an invertible problem. Then, we show that we can choose a convergent subsequence of the solutions to this problem and that its limit is a solution to the problem (1). To accomplish this, we assume that there is a uniform limit of the function f (cf. assumption (iv)). Similar assumption one can find in [7] and [9].

The method described above has been applied earlier for other nonlocal second order BVP. In [12], we have shown that the following BVP

$$x'' = f(t, x), \quad x'(0) = 0, \quad x'(1) = \int_0^1 x'(s)dg(s), \tag{2}$$

which is also a generalization of the Neumann problem, has a solution.

Solutions to the problems (1) and (2) are stationary solutions for a heat equation, corresponding to a heated bar, with a controller at 1 (comp., for instance, [17, 16]). In the problem (2), the heated bar adds or removes heat depending on the speed of the changes of the temperature detected by sensors put at any points of the bar (it depends on the function g) while, in the problem (1), the bar adds or removes heat depending on the temperature detected by sensors.

2. Settings

First, observe that the problem (1) can be written down as a system of BVPs

$$\begin{cases} x_i''(t) = f_i(t, x(t)), \\ x_i'(0) = 0, \\ x_i'(1) = \int_0^1 x_i(s)dg_i(s), \end{cases}$$

where $t \in [0, 1]$, $i = 1, \ldots, k$ and the integrals $\int_0^1 x_i'(s)dg_i(s)$ are meant in the sense of Riemann-Stieltjes.

Now, let us note that if $\int_0^1 dg_i(t) = 0$, $i = 1, \ldots, k$, then the homogeneous linear problem, i.e.,

$$x'' = 0, \quad x'(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s),$$

has nontrivial solutions - constant functions. Hence problem (1) is resonant. This means that the problem under consideration is not invertible and therefore we will use the perturbation method. Let us consider the following BVP

$$x'' = f(t,x), \quad t \in [0,1], \tag{3}$$

$$x'(0) = 0, \tag{4}$$

$$x'(1) = \int_0^1 x(s)dg(s) + \alpha_n x(0), \quad \alpha_n \in (0,1), \quad \alpha_n \to 0. \tag{5}$$

Notice that the problem (3), (4), (5) is always nonresonant.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^k$, while the scalar product in $\mathbb{R}^k$ corresponding to the Euclidean norm be denoted by $\langle \cdot , \cdot \rangle$. Moreover, let us consider the Banach space $C^1([0,1],\mathbb{R}^k)$ of all continuous functions $x : [0,1] \to \mathbb{R}^k$ which have continuous first derivatives x' with the norm

$$\|x\| = \max\left\{ \sup_{t\in[0,1]} |x(t)|\, , \sup_{t\in[0,1]} \left|x'(t)\right| \right\}. \tag{6}$$

The following compactness criterion in $C^1\left([0,1],\mathbb{R}^k\right)$ will be needed:

Lemma 1. *For a set $Z \subset C^1\left([0,1],\mathbb{R}^k\right)$ to be relatively compact, it is necessary and sufficient that:*

(a) there exists $M > 0$ such that for any $x \in Z$ and $t \in [0,1]$ we have $|x(t)| \le M$ and $|x'(t)| \le M$;

(b) the families $Z := \{x \mid x \in Z\}$ and $Z' := \{x' \mid x \in Z\}$ are equicontinuous.

Throughout this paper, we assume that:

(*i*) $f = (f_1, \ldots, f_k) : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function.

(*ii*) $g = (g_1, \ldots, g_k) : [0,1] \to \mathbb{R}^k$ has bounded variation on the interval $[0,1]$.

(*iii*) $\int_0^1 dg_i(t) = 0$, $i = 1, \ldots, k$.

(*iv*) For every t there exists a uniform finite limit

$$h(t,\xi) := \lim_{\lambda\to\infty} f(t,\lambda\,\xi)$$

with respect to $\xi \in \mathbb{R}^k$, $|\xi| = 1$ such that h is bounded on $[0,1] \times S^{k-1}$, with S^{k-1} the unit sphere in $\mathbb{R}^k$.

(v) Set

$$h_0(\xi) := \int_0^1 h(u,\xi)du - \int_0^1 \left(\int_0^u (u-s)h(s,\xi)ds \right) dg(u).$$

For every $\xi \in \mathbb{R}^k$, $|\xi| = 1$, we have $\langle \xi, h_0(\xi) \rangle < 0$.

3. Existence of solutions to the perturbed problem

Now, let us consider the equation (3) and integrate it from 0 to t. By (4), we get

$$x'(t) = \int_0^t f(s,x(s))ds \tag{7}$$

and

$$x(t) = x(0) + \int_0^t (t-s)f(s,x(s))ds \tag{8}$$

By (iii), (5), (7) and (8), we obtain

$$\int_0^1 f(s,x(s))ds = \int_0^1 \left(\int_0^u (u-s)f(s,x(s))ds \right) dg(u) + \alpha_n x(0),$$

so

$$x(0) = \frac{1}{\alpha_n} \left[\int_0^1 f(s,x(s))ds - \int_0^1 \left(\int_0^u (u-s)f(s,x(s))ds \right) dg(u) \right].$$

Let $n \in \mathbb{N}$ be fixed.

A function $x : [0,1] \to \mathbb{R}^k$ is called a solution to the problem (3), (4), (5) if the following holds:

- $x \in C^2([0,1], \mathbb{R}^k)$;
- $x''(t) = f(t,x(t))$ for every $t \in [0,1]$;
- $x'(0) = 0$, $x'(1) = \int_0^1 x(s)dg(s) + \alpha_n x(0)$.

Lemma 2. *Let the assumptions* $(i) - (iii)$ *be satisfied. A function* $x \in C^1([0,1], \mathbb{R}^k)$ *is a solution of the problem* (3), (4), (5) *if and only if* x *satisfies the following integral equation*

$$\begin{aligned} x(t) = & \int_0^t (t-s)f(s,x(s))ds \\ & + \frac{1}{\alpha_n} \left[\int_0^1 f(s,x(s))ds - \int_0^1 \left(\int_0^u (u-s)f(s,x(s))ds \right) dg(u) \right]. \end{aligned}$$

Now, using Schauder's fixed point Theorem we shall show that, for every fixed $n \in \mathbb{N}$, the BVP (3), (4), (5) has a solution. For this purpose, let us consider an operator $A_n : C^1([0,1], \mathbb{R}^k) \to C^1([0,1], \mathbb{R}^k)$ given by

$$\begin{aligned}(A_n x)(t) &= \int_0^t (t-s) f(s, x(s)) ds \\ &+ \frac{1}{\alpha_n}\left[\int_0^1 f(s, x(s)) ds - \int_0^1 \left(\int_0^u (u-s) f(s, x(s)) ds\right) dg(u)\right],\end{aligned}$$

where $n \in \mathbb{N}$ is fixed. Then

$$(A_n x)'(t) = \int_0^t f(s, x(s)) ds. \tag{9}$$

It is easy to observe that the operator A_n is well-defined.

By the assumption (iv), in particular, the function f is bounded. Set

$$M := \sup_{t \in [0,1], x \in \mathbb{R}^k} |f(t, x)|. \tag{10}$$

Then, by (9) and (10), we have

$$\sup_{t \in [0,1]} |(A_n x)'(t)| \le M. \tag{11}$$

Moreover, we get

$$\sup_{t \in [0,1]} |(A_n x)(t)| \le M + \frac{1}{\alpha_n} \left(M + M \operatorname{Var}(g)\right), \tag{12}$$

where $\operatorname{Var}(g)$ means the variation of g on the interval $[0,1]$.

From (ii), $L := \operatorname{Var}(g) < \infty$. Put $M_n := M + \frac{1}{\alpha_n}(M + M\,L)$, then $\|A_n x\| \le M_n$ for every $n \in \mathbb{N}$. Moreover, $(A_n x)''(t)$ and $(A_n x)'(t)$, $t \in [0,1]$, are bounded, hence the families $(A_n x)'$ and $(A_n x)$ are equicontinuous. Now, by Lemma 1, one can easily show that the operator A_n is completely continuous.

Set $B_n := \left\{x \in C^1\left([0,1], \mathbb{R}^k\right) \,\middle|\, \|x\| \le M_n\right\}$. Now, let us notice that, by Schauder's fixed point Theorem, the operator

$$A_n : B_n \to B_n,$$

has a fixed point in B_n for every n. Hence, the following lemma holds

Lemma 3. *Let the assumptions* $(i) - (iv)$ *be satisfied. Then, for each* $n \in \mathbb{N}$*, the problem* (3), (4), (5) *has at least one solution.*

4. The convergent subsequence

For each $n \in \mathbb{N}$ let φ_n be a solution to the problem (3), (4), (5).

First, we shall show, that the sequence (φ_n) is bounded in $C^1\left([0,1], \mathbb{R}^k\right)$. Assume, on the contrary, that the sequence (φ_n) is unbounded. Then, passing to a subsequence if necessary, we have $\|\varphi_n\| \to \infty$. By (11), we get that $\sup_{t\in[0,1]} |(\varphi_n)'(t)| \le M$, for every n. Hence,

$$\sup_{t\in[0,1]} |\varphi_n(t)| \to \infty,$$

when $n \to \infty$.

Now, let us consider a sequence $(\frac{\varphi_n}{\|\varphi_n\|}) \subset C^1\left([0,1], \mathbb{R}^k\right)$ and notice that the norm of the sequence equals 1. Hence, the sequence is bounded. Moreover, the family $(\frac{\varphi_n}{\|\varphi_n\|})$ (and simultaneously $\left(\frac{\varphi_n'}{\|\varphi_n\|}\right)$) is equicontinuous, since $\frac{\varphi_n'(t)}{\|\varphi_n\|}$ $\left(\text{or } \frac{\varphi_n''(t)}{\|\varphi_n\|}\right)$ is bounded (observe that $\varphi_n''(t) = f(t, \varphi_n(t))$ and thus $\varphi_n''(t)$ is continuous and, by (10), bounded). By Lemma 1, there exists a convergent subsequence of $(\frac{\varphi_n}{\|\varphi_n\|})$. Let us also denote this subsequence as $(\frac{\varphi_n}{\|\varphi_n\|})$.

Lemma 4. *The sequence*

$$\left(\frac{\varphi_n(t)}{\|\varphi_n\|}\right) \tag{13}$$

converges uniformly to a $\xi \in \mathbb{R}^k$ on $[0,1]$. Moreover, $|\xi| = 1$.

Proof. Notice that $\frac{\varphi_n(t)}{\|\varphi_n\|}$ is given by

$$\begin{aligned} \frac{\varphi_n(t)}{\|\varphi_n\|} &= \frac{\int_0^t (t-s) f(s, \varphi_n(s)) ds}{\|\varphi_n\|} \\ &\quad + \frac{\int_0^1 f(s, \varphi_n(s)) ds - \int_0^1 \left(\int_0^u (u-s) f(s, \varphi_n(s)) ds\right) dg(u)}{\alpha_n \|\varphi_n\|}. \end{aligned} \tag{14}$$

By (10), we obtain that

$$\frac{\int_0^t (t-s) f(s, \varphi_n(s)) ds}{\|\varphi_n\|} \tag{15}$$

converges uniformly to $0 \in \mathbb{R}^k$ on $[0,1]$. Consequently, by (14) and (15), we can easily observe that the limit (13) does not depend on t.

Since the norm of the sequence $(\frac{\varphi_n}{\|\varphi_n\|})$ in the space $C^1\left([0,1], \mathbb{R}^k\right)$ equals 1, $\frac{\varphi_n'(t)}{\|\varphi_n\|} \to 0$ and the limit of the sequence (13) does not depend on t, without using the formula (14), we get that $\frac{\varphi_n(t)}{\|\varphi_n\|}$ converges to $\xi \in \mathbb{R}^k$ such that $|\xi| = 1$. Moreover, this convergence is uniform. ■

Lemma 5. *For each $t \in [0,1]$, $f(t, \|\varphi_n\| \frac{\varphi_n(t)}{\|\varphi_n\|})$ tends to $h(t,\xi)$.*

Proof. From Lemma 4, $(y_n(t)) := \left(\frac{\varphi_n(t)}{\|\varphi_n\|}\right)$ converges uniformly to ξ. Notice that if $(y_n(t))$ tends to ξ, then also $\left(\frac{\varphi_n(t)}{|\varphi_n(t)|}\right)$ tends uniformly to ξ.

Given $\varepsilon > 0$, for each $t \in [0,1]$, there is $R > 0$ such that

$$|f(t, \lambda\xi) - h(t,\xi)| < \frac{\varepsilon}{2},$$

when $\lambda \geq R$ and $\xi \in \mathbb{R}^k$, $|\xi| = 1$.

One can easy observe that : for every t, $h(t,\xi_n) \to h(t,\xi)$, when $\xi_n \to \xi$, with $|\xi_n|, |\xi| = 1$. Consequently, for each t there exists n_1 such that

$$\left| h\left(t, \frac{\varphi_n(t)}{|\varphi_n(t)|}\right) - h(t,\xi)\right| < \frac{\varepsilon}{2},$$

for all $n \geq n_1$.

Let us consider a sequence $(c_n) \subset \mathbb{R}$ such that $c_n \to \infty$. Now, since $(y_n(t))$ is uniformly convergent, one can choose n_2 such that for all $n \geq n_2$ and all $t \in [0,1]$,

$$|y_n(t)| \geq \frac{1}{2} \quad \text{and} \quad c_n \geq 2R.$$

We shall show that for each $t \in [0,1]$, $f(t, c_n y_n(t))$ converges to $h(t,\xi)$. Indeed, for $n \geq \max\{n_1, n_2\}$, we obtain

$$\begin{aligned} |f(t, c_n y_n(t)) - h(t,\xi)| &= \left| f\left(t, c_n \frac{\varphi_n(t)}{\|\varphi_n(t)\|}\right) - h(t,\xi)\right| \\ &= \left| f\left(t, c_n |y_n(t)| \frac{\varphi_n(t)}{|\varphi_n(t)|}\right) - h(t,\xi)\right| \\ &\leq \left| f\left(t, c_n |y_n(t)| \frac{\varphi_n(t)}{|\varphi_n(t)|}\right) - h\left(t, \frac{\varphi_n(t)}{|\varphi_n(t)|}\right)\right| \\ &\quad + \left| h\left(t, \frac{\varphi_n(t)}{|\varphi_n(t)|}\right) - h(t,\xi)\right| < \varepsilon, \end{aligned}$$

what ends the proof. ■

Lemma 6. *Let ξ be the limit of the sequence* (13)*. Then*

$$\xi = \gamma \left[\int_0^1 h(s,\xi) ds - \int_0^1 \left(\int_0^u (u-s) h(s,\xi) ds \right) dg(u) \right],$$

where $\gamma \in (0,\infty)$.

Proof. Let us observe that, by (14), we obtain

$$
\begin{aligned}
(16)\qquad \xi &= \lim_{n\to\infty} \frac{\varphi_n(t)}{\|\varphi_n\|} = \frac{\int_0^t (t-s) f(s,\varphi_n(s))ds}{\|\varphi_n\|} \\
&\quad + \frac{\int_0^1 f(s,\varphi_n(s))ds - \int_0^1 \left(\int_0^u (u-s) f(s,\varphi_n(s))ds\right) dg(u)}{\alpha_n \|\varphi_n\|} \\
&= \lim_{n\to\infty} \left(\frac{\int_0^1 f(s, \|\varphi_n\| \frac{\varphi_n(s)}{\|\varphi_n\|})ds}{\alpha_n \|\varphi_n\|} \right. \\
&\quad \left. - \frac{\int_0^1 \left(\int_0^u (u-s) f(s, \|\varphi_n\| \frac{\varphi_n(s)}{\|\varphi_n\|})ds\right) dg(u)}{\alpha_n \|\varphi_n\|} \right).
\end{aligned}
$$

From (10) and Lemma 5, Lebesgue dominated convergence theorem implies that

$$
\lim_{n\to\infty} \int_0^1 f(s, \|\varphi_n\| \frac{\varphi_n(s)}{\|\varphi_n\|})ds = \int_0^1 h(s,\xi)ds
$$

and

$$
\begin{aligned}
&- \lim_{n\to\infty} \int_0^1 \left(\int_0^u (u-s) f(s, \|\varphi_n\| \frac{\varphi_n(s)}{\|\varphi_n\|})ds \right) dg(u) \\
&\qquad = - \int_0^1 \left(\int_0^u (u-s) h(s,\xi) ds \right) dg(u).
\end{aligned}
$$

Moreover, by (v), the sum of the limits is different from zero. Hence, since Lemma 4 holds, there exists $\gamma \in (0,\infty)$ such that $\gamma := \lim_{n\to\infty} 1/(\alpha_n \|\varphi_n\|)$.

Now, by assumption (iv) and Lemma 5, we obtain

$$
\begin{aligned}
(17)\qquad \xi &= \lim_{n\to\infty} \frac{\varphi_n(t)}{\|\varphi_n\|} \\
&= \gamma \left[\int_0^1 h(s,\xi)ds - \int_0^1 \left(\int_0^u (u-s) h(s,\xi)ds \right) dg(u) \right].
\end{aligned}
$$

■

Lemma 7. *Let the assumptions* $(i)-(v)$ *hold. Then the sequence* (φ_n) *is bounded in* $C^1\left([0,1],\mathbb{R}^k\right)$.

Proof. Assume that the sequence (φ_n) in unbounded. Then, by Lemma 4, Lemma 6 and (v), we reach a contradiction. Indeed, we get

$$
\begin{aligned}
(18)\qquad 1 = \langle \xi,\xi\rangle &= \gamma \left\langle \xi\, , \int_0^1 h(s,\xi)ds - \int_0^1 \left(\int_0^u (u-s) h(s,\xi)ds \right) dg(u) \right\rangle \\
&= \gamma \langle \xi, h_0(\xi)\rangle < 0.
\end{aligned}
$$

Hence, the sequence (φ_n) is bounded. ■

By Lemma 1, it is easy to note that the following lemma holds

Lemma 8. *Let the assumptions* $(i)-(v)$ *hold. Then the set* $Z = \{\varphi_n \mid n \in \mathbb{N}\}$ *is relatively compact in* $C^1\left([0,1], \mathbb{R}^k\right)$.

5. The existence result

Now, let us consider the problem (1). By a solution to the problem (1) we mean a function $x : [0,1] \to \mathbb{R}^k$ such that:
- $x \in C^2([0,1], \mathbb{R}^k)$;
- $x''(t) = f(t, x(t))$ for every $t \in [0,1]$;
- $x'(0) = 0$, $x'(1) = \int_0^1 x(s)dg(s)$.

By the above Lemmas, we get the existence result for problem (1). Indeed, Lemma 8 implies that (φ_n) has a convergent subsequence (φ_{n_l}), $\varphi_{n_l} \to \varphi$. We know that φ_{n_l} (φ'_{n_l}) converges uniformly to φ (φ') on $[0,1]$. One can see that $f(t, \varphi_{n_l})$ is uniformly convergent to $f(t, \varphi)$. Since

$$\varphi''_{n_l}(t) = f(t, \varphi_{n_l}(t)),$$

the sequence $\varphi''_{n_l}(t)$ is also uniformly convergent. Moreover, $\varphi''_{n_l}(t)$ converges uniformly to $\varphi''(t)$.

Note that we have actually shown that function $\varphi \in C^1\left([0,1], \mathbb{R}^k\right)$ is a solution of the equation of problem (1) (in fact, $\varphi \in C^2([0,1]\mathbb{R}^k)$, since f is continuous). By (4) and (5), it is easy to see that φ satisfies boundary conditions of problem (1).

We have proved the following theorem

Theorem 1. *Under assumptions* $(i)-(v)$ *the problem* (1) *has at least one solution.*

Example 1. Let $k = 2$, $g(t) = (\frac{1}{2}t^2 - \frac{1}{2}t, \frac{1}{2}t^2 - \frac{1}{2}t)$ and

$$\begin{aligned} f_1(t, x_1, x_2) &= \frac{-x_1^p - c_1(t)x_2^q + a_1(t)}{|x|^p + b_1(t)}, \\ f_2(t, x_1, x_2) &= \frac{c_2(t)x_1^q - x_2^p + a_2(t)}{|x|^p + b_2(t)}, \end{aligned}$$

where $p, q \in \mathbb{N}$, p is odd and $q < p$, a_i, b_i, c_i are bounded and continuous functions on $[0,1]$ and, additionally, b_i are positive, $i = 1, 2$. For every $\xi = (\xi_1, \xi_2)$, $|\xi| = 1$, we get

$$h(t, \xi) = \lim_{\lambda \to \infty} f(t, \lambda\, \xi) = \left(\frac{-\xi_1^p}{|\xi|^p}, \frac{-\xi_2^p}{|\xi|^p} \right)$$

and

$$h_0(\xi) = \left(-\frac{23\, \xi_1^p}{24}, -\frac{23\, \xi_2^p}{24} \right).$$

Then

$$\langle \xi, h_0(\xi) \rangle = -\frac{23}{24} \left(\xi_1^{p+1} + \xi_2^{p+1} \right) < 0.$$

Hence, the problem (1) has at least one nontrivial solution.

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