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IMPROVED OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS WITH NON-POSITIVE NEUTRAL TERM

Abstract. In this paper, we present some oscillation criteria for second order nonlinear delay difference equation with non-positive neutral term of the form

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n f(x_{n-\sigma}) = 0, \quad n \geq n_0 > 0,$$

where $z_n = x_n - p_n x_{n-\tau}$, and α is a ratio of odd positive integers. Examples are provided to illustrate the results. The results obtained in this paper improve and complement to some of the existing results.

Key words: 39A11.

AMS Mathematics Subject Classification: Ooscillation, second order delay difference equation, non-positive neutral term.

1. Introduction

In this paper, we study the oscillatory behavior of the following second order difference equation with non-positive neutral term of the form

$$\Delta(a_n(\Delta z_n)^\alpha) + q_n f(x_{n-\sigma}) = 0, \quad n \geq n_0 > 0, \tag{1}$$

where $z_n = x_n - p_n x_{n-\tau}$, subject to the following hypotheses:

(H_1) $\{a_n\}$, $\{p_n\}$, and $\{q_n\}$ are sequences of real numbers with $0 \leq p_n \leq p < 1, q_n \geq 0$ and q_n is not identically zero for infinitely many values of n, and $\{a_n\}$ is a positive real sequence with $\sum_{s=n_0}^{n} \frac{1}{a_s^{\frac{1}{\alpha}}} \to \infty$ as $n \to \infty$;

(H_2) $f \in \mathcal{C}(R, R)$ such that $uf(u) > 0$ for all $u \neq 0$, and there exists a positive constant M such that $\frac{f(u)}{u^\alpha} \geq M$ for all $u \neq 0$;

(H_3) α is a ratio of odd positive integers, σ and τ are positive integers.

Let $\theta = \max\{\tau, \sigma\}$. By a solution of equation (1) we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying the equation (1) for all $n \geq n_0$. As usual a solution $\{x_n\}$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is said to be non-oscillatory. We consider only nontrivial solutions of equation (1), and we assume that equation (1) possesses such solutions.

From the review of literature, it is known that there are many results available on the oscillatory and asymptotic behavior of solutions of equation (1) when the neutral term is nonnegative, i.e., $p_n \leq 0$; see for example [1, 2, 3, 8, 14] and the references cited therein. However, there are few results available on the oscillatory behavior of solutions of equation (1) when the neutral term is non-positive; see, for example [5, 6, 7, 10, 11, 12, 13, 15, 16] and the references cited therein. In [1], we see that the oscillatory behavior of the equation

$$\Delta^2\Big(x_n - px_{n-\tau}\Big) + q_n x_{n-\sigma} = 0, \quad n \in \mathbb{N}(n_0), \tag{2}$$

is discussed and in [13], the authors studied the oscillatory and asymptotic behavior of the equation

$$\Delta\Big(a_n(\Delta(x_n - p_n x_{n-\tau}))\Big) + q_n x_{n-\sigma}^{\alpha} = 0, \quad n \in \mathbb{N}(n_0), \tag{3}$$

with $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} = \infty$. Also Thandapani et al. [11] studied the oscillation of the equation

$$\Delta\Big(a_n(\Delta(x_n - p_n x_{n-\tau}))^{\alpha}\Big) + q_n f(x_{n-\sigma}) = 0, \tag{4}$$

under the conditions

$$\frac{f(u)}{u^{\alpha}} \geq M, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty.$$

In these papers the authors obtained that every solution of equations (2), (3) and (4) is either oscillatory or tends to zero as $n \to \infty$.

This observation motivated us to study when all solutions of equation (1) are oscillatory. In Section 2, we present some lemmas which will be useful to prove our main results. In Section 3, we obtain some new sufficient conditions for the oscillation of all solutions of equation (1), and in Section 4, we provide some examples to illustrate the main results. Thus the results presented in this paper improve and complement to those established in [5, 6, 7, 10, 11, 12, 13, 15, 16].

2. Some preliminary lemmas

In this section, we provide two lemmas which are useful in proving the main results. Since α is a ratio of odd positive integers it is enough to prove all the results for positive solutions of equation (1) since the proof for the negative case in similar.

Lemma 1. *Assume that hypotheses $(H_1) - (H_3)$ hold. If $\{x_n\}$ is a positive solution of equation (1), then $\{z_n\}$ satisfies only one of the following two cases for all $n \geq N \geq n_0$:*
(C_1) $z_n > 0, \Delta z_n > 0$ *and* $\Delta(a_n(\Delta z_n)^\alpha) \leq 0$;
(C_2) $z_n < 0, \Delta z_n > 0$ *and* $\Delta(a_n(\Delta z_n)^\alpha) \leq 0$.

Proof. Let $\{x_n\}$ be a positive solution of equation (1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_1$. Now from equation (1), we have

$$\Delta(a_n(\Delta z_n)^\alpha) = -q_n f(x_{n-\sigma}) \leq 0 \tag{5}$$

for all $n \geq n_1$. Therefore $a_n(\Delta z_n)^\alpha$ is non-increasing and hence $(\Delta z_n)^\alpha$ is non-increasing and of one sign eventually. That is, there exists an integer $n_2 \geq n_1$ such that either $\Delta z_n > 0$ or $\Delta z_n < 0$ for all $n \geq n_2$. We shall show that $\Delta z_n > 0$ for all $n \geq n_2$. If not, then $\Delta z_n < 0$ for all $n \geq n_2$, and

$$a_n(\Delta z_n)^\alpha \leq a_{n_2}(\Delta z_{n_2})^\alpha < 0$$

for all $n \geq n_2$. Then

$$\Delta z_n \leq \frac{a_{n_2}^{\frac{1}{\alpha}} \Delta z_{n_2}}{a_n}$$

for all $n \geq n_2$. Summing the last inequality from n_2 to $n-1$, we obtain

$$z_n - z_{n_2} \leq a_{n_2}^{\frac{1}{\alpha}} \Delta z_{n_2} \sum_{s=n_2}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}.$$

Letting $n \to \infty$, and using (H_1), we see that $z_n \to -\infty$. Now we consider the two cases for $\{x_n\}$.

Case (i)**:** Suppose $\{x_n\}$ is an unbounded sequence. Then there exists a sequence $\{n_j\}$ such that $n_j \to \infty$ and $x_{n_j} \to \infty$, where $x_{n_j} = max\{x_s : n_0 \leq s \leq n_j\}$. Then

$$x_{n_j-\tau} = \max\{x_s : n_0 \leq s \leq n_j - \tau\} \leq \max\{x_s : n_0 \leq s \leq n_j\} = x_{n_j}.$$

Therefore for large values of n

$$z_{n_j} = x_{n_j} - p_{n_j} x_{n_j-\tau} \geq (1 - p_{n_j}) x_{n_j} > 0,$$

which is a contradiction for $z_n \to -\infty$ as $n \to \infty$.

Case *(ii)***:** Suppose $\{x_n\}$ is a bounded sequence then $\{z_n\}$ is also a bounded sequence. This contradicts $z_n \to -\infty$ as $n \to \infty$. Hence $\Delta z_n > 0$ for all $n \geq n_2$ and z_n satisfies only one of the two possible cases (C_1) and (C_2). ■

Lemma 2. *If $\{x_n\}$ is an eventually positive solution of equation* (1) *such that (C_1) of Lemma 1 holds, then*

$$x_n \geq z_n \geq R_n a_n \Delta z_n$$

where $R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}$.

Proof. The proof can be found in [11]. ■

3. Oscillation results

In this section, we obtain some sufficient conditions which ensure that all solutions of equation (1) are oscillatory. In what follows, we denote

$$\xi_n = a_n^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} a_s^{\frac{-1}{\alpha}},$$

where $n_1 \geq n_0$ is sufficiently large.

Theorem 1. *Assume that hypotheses (H_1)-(H_3) hold, and $\sigma > \tau$. If there exists a positive and nondecreasing sequence $\{\rho_n\}$ of real numbers, such that*

$$\sum_{n=n_1}^{\infty} \left[M\rho_{n+1} q_n - \frac{\Delta \rho_n a_{n-\sigma}}{\xi_{n-\sigma}^{\alpha}} \right] = \infty, \tag{6}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n-\sigma+\tau}^{n-1} \left(\frac{1}{a_s} \sum_{t=s}^{n-1} q_t \right)^{\frac{1}{\alpha}} > \frac{p}{M^{\frac{1}{\alpha}}} \tag{7}$$

for sufficiently large $n_1 \geq n_0$, then every solution of equation (1) *is oscillatory.*

Proof. Suppose $\{x_n\}$ is a positive solution of equation (1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_1$. Choose an integer $N \geq n_1$ such that the sequence $\{z_n\}$ satisfies only one of the two Cases (C_1) and (C_2) of Lemma 1. First we assume that $\{z_n\}$ satisfies Case (C_1) of Lemma 1. From the definition of z_n, we have

$$x_n = z_n + p_n x_{n-\tau} \geq z_n \tag{8}$$

for all $n \geq N$. Since $a_n(\Delta z_n)^\alpha$ is non-increasing, we have

$$a_n(\Delta z_n)^\alpha \leq a_{n-\sigma}(\Delta z_{n-\sigma})^\alpha. \tag{9}$$

Now

$$\begin{aligned} z_n &= z_N + \sum_{s=N}^{n-1} \Delta z_s \\ &\geq (a_n(\Delta z_n)^\alpha)^{\frac{1}{\alpha}} \sum_{s=N}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \\ &= \xi_n \Delta z_n. \end{aligned} \tag{10}$$

Define

$$w_n = \rho_n \frac{a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha} \tag{11}$$

for all $n \geq N$. Then $w_n > 0$ for all $n \geq N$, and

$$\begin{aligned} \Delta w_n &= \Delta \rho_n \frac{a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha} + \rho_{n+1} \frac{\Delta(a_n(\Delta z_n)^\alpha)}{z_{n-\sigma}^\alpha} \\ &\quad - \frac{\rho_{n+1} a_{n+1}(\Delta z_{n+1})^\alpha}{z_{n-\sigma}^\alpha z_{n-\sigma+1}^\alpha} \Delta(z_{n-\sigma}^\alpha) \\ &\leq \Delta \rho_n \frac{a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha} + \rho_{n+1} \frac{\Delta(a_n(\Delta z_n)^\alpha)}{z_{n-\sigma}^\alpha}. \end{aligned} \tag{12}$$

Using (5) and (H_1) in (12) we have

$$\Delta w_n \leq \Delta \rho_n \frac{a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha} - M \frac{\rho_{n+1} q_n x_{n-\sigma}^\alpha}{z_{n-\sigma}^\alpha}. \tag{13}$$

Applying (8) and (10) in (13), we have

$$\begin{aligned} \Delta w_n &\leq \Delta \rho_n \frac{a_{n-\sigma}(\Delta z_{n-\sigma})^\alpha}{z_{n-\sigma}^\alpha} - M \rho_{n+1} q_n \\ &\leq \frac{\Delta \rho_n a_{n-\sigma}}{\xi_{n-\sigma}^\alpha} - M \rho_{n+1} q_n, \quad n \geq N. \end{aligned}$$

Summing the last inequality from N to $n-1$, we obtain

$$w_n - w_N \leq \sum_{s=N}^{n-1} \left[\frac{\Delta\rho_s a_{s-\sigma}}{\xi_{s-\sigma}^{\alpha}} - M\rho_{s+1}q_s \right].$$

That is,

$$\sum_{s=N}^{n-1} \left(M\rho_{s+1}q_s - \frac{\Delta\rho_s a_{s-\sigma}}{\xi_{s-\sigma}^{\alpha}} \right) \leq w_N.$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (6). Next consider the Case (C_2) of Lemma 1. From the definition of z_n, we have

$$x_{n-k} > (-\frac{z_n}{p}), \quad n \geq N. \tag{14}$$

Using (H_2) and (14) in equation (1), we obtain

$$\Delta(a_n(\Delta z_n)^{\frac{1}{\alpha}}) - \frac{Mq_n}{p^{\alpha}} z_{n+\tau-\sigma}^{\alpha} \leq 0, \quad n \geq N.$$

Summing the last inequality from s to $n-1$ for $n > s+1$, we have

$$a_n(\Delta z_n)^{\alpha} - a_s(\Delta z_s)^{\alpha} - \frac{M}{p^{\alpha}} \sum_{t=s}^{n-1} q_t \; z_{t-\sigma+\tau}^{\alpha} \leq 0,$$

or

$$-\Delta z_s \leq \frac{M^{\frac{1}{\alpha}}}{p} \left(\frac{1}{a_s} \sum_{t=s}^{n-1} q_t z_{t-\sigma+\tau}^{\alpha} \right)^{\frac{1}{\alpha}}, \quad n \geq N.$$

Again summing the last inequality from $n-\sigma+\tau$ to $n-1$ for s and then using monotonicity of $\{z_n\}$ we have

$$z_{n-\sigma+\tau} - z_n \leq \frac{M^{\frac{1}{\alpha}}}{p} z_{n-\sigma+\tau} \sum_{s=n-\sigma+\tau}^{n-1} \left(\frac{1}{a_s} \sum_{t=s}^{n-1} q_t \right)^{\frac{1}{\alpha}}$$

or

$$\frac{p}{M^{\frac{1}{\alpha}}} \geq \sum_{s=n-\sigma+\tau}^{n-1} \left(\frac{1}{a_s} \sum_{t=s}^{n-1} q_t \right)^{\frac{1}{\alpha}}$$

which is a contradiction to (7). Now the proof is complete. ■

Next assume that $\rho_n \equiv 1$. then by Theorem 1, we have the following corollary.

Corollary 1. *Assume that condition (7) is satisfied. If* $\sum_{n=n_0}^{\infty} q_n = \infty$, *then every solution of equation (1) is oscillatory.*

Theorem 2. *Assume that hypotheses* (H_1), (H_2), *and condition (7) hold. If* $\alpha > 1$, *and there exists a positive and non-decreasing sequence* $\{\rho_n\}$ *of real numbers such that*

$$\sum_{n=n_1}^{\infty}\left(M\rho_{n+1}q_n - \frac{a_{n-\sigma(\Delta\rho_n)^{\alpha+1}}}{\rho_n^{\alpha+1}\rho_{n+1}^{\alpha}(\alpha+1)^{\alpha+1}}\right) = \infty \tag{15}$$

for sufficiently large $n_1 \geq n_0$, *then every solution of equation (1) is oscillatory.*

Proof. Suppose $\{x_n\}$ is a positive solution of equation (1). Then as in the proof of Theorem 1, $\{z_n\}$ satisfies (C_1) or (C_2) of Lemma 1.

First, assume that $\{z_n\}$ satisfies (C_1) of Lemma 1. Define a sequence $\{w_n\}$ as in Theorem 1. Then $w_n > 0$ for all $n \geq N \geq n_1 \geq n_0$, and

$$\Delta w_n = \Delta\rho_n \frac{a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha} + \rho_{n+1}\frac{\Delta(a_n(\Delta z_n)^\alpha)}{z_{n-\sigma}^\alpha} - \frac{\rho_{n+1}a_{n+1}\Delta(z_{n+1})^\alpha}{z_{n-\sigma}^\alpha z_{n-\sigma+1}^\alpha}\Delta z_{n-\sigma}^\alpha.$$

Using the definition of w_n and the monotonicity of $a_n(\Delta z_n)^\alpha$, we get

$$\begin{aligned}\Delta w_n &\leq \frac{\Delta\rho_n}{\rho_n}w_n - M\rho_{n+1}q_n - \rho_{n+1}\frac{a_n(\Delta z_n)^\alpha}{z_{n-\sigma}^\alpha z_{n-\sigma+1}^\alpha}\Delta z_{n-\sigma}^\alpha\\ &\leq \frac{\Delta\rho_n}{\rho_n}w_n - M\rho_{n+1}q_n - \rho_{n+1}\frac{w_n\Delta z_{n-\sigma}^\alpha}{z_{n-\sigma+1}^\alpha}\end{aligned} \tag{16}$$

for all $n \geq N$. By Mean value theorem, we have

$$\Delta z_{n-\sigma}^\alpha = \alpha t^{\alpha-1}\Delta z_{n-\sigma} \tag{17}$$

where $z_{n-\sigma} \leq t \leq z_{n-\sigma+1}$. Therefore

$$\begin{aligned}\Delta w_n &\leq \frac{\Delta\rho_n}{\rho_n}w_n - M\rho_{n+1}q_n - \alpha\rho_{n+1}w_n\frac{z_{n-\sigma}^{\alpha-1}\Delta z_{n-\sigma}}{z_{n-\sigma+1}^\alpha}\\ &\leq \frac{\Delta\rho_n}{\rho_n}w_n - M\rho_{n+1}q_n - \alpha\rho_{n+1}w_n\frac{\Delta z_{n-\sigma}}{z_{n-\sigma}}, \quad n \geq N.\end{aligned} \tag{18}$$

Now using (38) in the last inequality, we have

$$\begin{aligned}\Delta w_n &\leq \frac{\Delta\rho_n}{\rho_n}w_n - M\rho_{n+1}q_n - \alpha\rho_{n+1}w_n\frac{(a_n)^{\frac{1}{\sigma}}\Delta z_n}{a_{n-\sigma}^{\frac{1}{\alpha}}z_{n-\sigma}}\\ &\leq \frac{\Delta\rho_n}{\rho_n}w_n - M\rho_{n+1}q_n - \frac{\alpha\rho_{n+1}w_n^{\alpha+\frac{1}{\alpha}}}{a_{n-\sigma}^{\frac{1}{\alpha}}}, \quad n \geq N.\end{aligned}$$

Using the inequality $Au - Bu^{\alpha+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha_1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}$ with $u = w_n$, $A = \frac{\Delta\rho_n}{\rho_n}$, $B = \frac{\alpha\rho_{\alpha+1}}{a^\alpha_{n-\sigma}}$ in the last inequality, we obtain

$$\Delta w_n \leq -M\rho_{n+1}q_n + \frac{a_{n-\sigma}(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha+1}\rho_{n+1}^\alpha(\alpha+1)^{\alpha+1}}, n \geq N.$$

Summing the last inequality from N to $n-1$, we have

$$\sum_{s=N}^{n-1}\left[M\rho_{s+1}q_s - \frac{a_{s-\sigma}(\Delta\rho_s)^{\alpha+1}}{\rho_s^{\alpha+1}\rho_{s+1}^\alpha(\alpha+1)^{\alpha+1}}\right] \leq w_N - w_n \leq w_N.$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction to (15).

If Case (C_2) of Lemma 1 is satisfied by $\{z_n\}$, then as in the proof of Theorem 1 we obtain a contradiction to (7). This completes the proof. ■

Next we consider the case $\alpha = 1$.

Theorem 3. *Assume hypotheses (H_1), (H_2) hold, and $\alpha = 1$. If condition (3.2) and*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} q_s R_{s-\sigma} > \frac{1}{M}\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{19}$$

are satisfied, then every solution of equation (1) is oscillatory.

Proof. Suppose $\{x_n\}$ is a positive solution of equation (1). Then as in the proof of Theorem 1, $\{z_n\}$ satisfies Case (C_1) or (C_2) of Lemma 1. Let $\{z_n\}$ satisfies Case (C_1). By the definition of z_n, we have

$$x_n = z_n + p_n x_{n-\tau} \geq z_n \text{ for all } n \geq N. \tag{20}$$

Applying (19) and (H_2) in equation (1), we have

$$\Delta(a_n\Delta z_n) + Mq_n z_{n-\sigma} \leq 0, \quad n \geq N.$$

Using Lemma 2 in the last inequality, we obtain

$$\Delta(a_n\Delta z_n) + Mq_n R_{n-\sigma} a_{n-\sigma}\Delta z_{n-\sigma} \leq 0, \quad n \geq N.$$

Let $w_n = a_n\Delta z_n$ for $n \geq N$. Then $\{w_n\}$ is a positive solution of

$$\Delta w_n + Mq_n R_{n-\sigma} w_{n-\sigma} \leq 0, \quad n \geq N. \tag{21}$$

But by Theorem 7.6.1 of [4] and (19), the inequality (21) has no positive solution, which is a contradiction. Next assume Case (C_2) of Lemma 1 holds. Then as in the proof of Theorem 1 we again obtain a contradiction with (7). The proof is now completed. ■

3. Examples

In this section, we provide three examples in support of our results.

Example 1. Consider the following second order neutral difference equation

$$\Delta((\Delta(x_n - \frac{1}{2}x_{n-2}))^{\frac{1}{3}}) + 2x_{n-3}^{\frac{1}{3}} = 0, \quad n \geq 3. \tag{22}$$

It is easy to see that all conditions of Theorem 1 are satisfied and hence every solution of equation (22) is oscillatory. In fact $\{x_n\} = \{(-1)^{3n}\}$ is one such oscillatory solution of equation (22).

Example 2. Consider the following second order neutral difference equation

$$\Delta(n^3(\Delta(x_n - \frac{1}{2}x_{x-2}))^3) + (2n^2 + 2n + 1)x_{n-3}^3 = 0, \quad n \geq 3. \tag{23}$$

It is easy to see that all conditions of Theorem 2 are satisfied and hence every solution of equation (23) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (23).

Example 3. Consider the following second order neutral difference equation

$$\Delta^2(x_n - \frac{1}{2}x_{n-2}) + x_{n-3}(1 + x_{n-3}^2) = 0, \quad n \geq 3. \tag{24}$$

It is easy to see that all the conditions of Theorem 3 are satisfied and hence every solution of equation (24) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (24).

We conclude the paper with the following remark.

Remark 1. If results presented in [5-7, 10-13, 15, 16] are applied to examples 1-3, we obtain that the solutions of the equations (6)-(8) are either oscillatory or tend to zero as $n \to \infty$. But our Theorems 1 to 3 give stronger results in the sense that all solutions of equations (22) to (24) are oscillatory. Thus our results improve that of in [5-7, 10-13, 15, 16].

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