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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR $s-$CONVEX MAPPINGS VIA FRACTIONAL INTEGRALS OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

Abstract. In this paper, we obtain some Hermite-Hadamard type inequalities for $s-$convex function via fractional integrals with respect to another function which generalize the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. The results presented here provide extensions of those given in earlier works.

Key words:

AMS Mathematics Subject Classification: 26D07, 26D10, 26D15, 26A33.

1. Introduction

Definition 1. *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Definition 2 ([4])**.** *Let $s\in(0,1]$. A function $f:[0,\infty)\to[0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if*

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y)$$

for all $x,y\in[0,\infty)$ and $\lambda\in[0,1]$.

An $s-$convex function was introduced in Breckner's paper [4] and a number of properties and connections with $s-$convexity in the first sense were discussed in paper [13].

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[17, p.137], [10]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [1, 2, 10, 11, 17, 22, 23]).

In the following we present a brief synopsis of all necessary definitions and results that will be required. More details, one can consult [12, 15, 16, 18].

Definition 3. *Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \;\; x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \;\; x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

It is remarkable that Sarikaya et al.[20] first gave the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$.

Definition 4. *Let $f \in L_1[a, b]$. The Hadamard fractional integrals $\mathbf{J}_{a+}^{\alpha} f$ and $\mathbf{J}_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\mathbf{J}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)dt, \;\; x > a$$

and

$$\mathbf{J}_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}f(t)dt,\quad x<b$$

respectively.

Definition 5. *Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The left-sides $(I_{a^+;g}^{\alpha}f(x))$ and right-sides $(I_{b^-;g}^{\alpha}f(x))$ fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha<0$ are defined by*

$$I_{a^+;g}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)f(t)}{[g(x)-g(t)]^{1-\alpha}}dt,\quad x>a$$

and

$$I_{b^-;g}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)f(t)}{[g(t)-g(x)]^{1-\alpha}}dt,\quad x<b$$

respectively.

In [14], Jleli and Samet gave the following equality:

Lemma 1. *Let $\alpha>0$ and let $\Xi_{\alpha,g}:[0,1]\to\mathbb{R}$ be a function defined by*

$$\begin{aligned}\Xi_{\alpha,g}(t) &= [g(ta+(1-t)b)-g(a)]^{\alpha}-[g(tb+(1-t)a)-g(a)]^{\alpha}\\ &+ [g(b)-g(tb+(1-t)a)]^{\alpha}-[g(b)-g(ta+(1-t)b)]^{\alpha}.\end{aligned}$$

If $f\in C^1(I^{\circ})$, then

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}\left(I_{a^+;g}^{\alpha}F(b)+I_{b^-;g}^{\alpha}F(a)\right)\\ &\qquad=\frac{b-a}{4[g(b)-g(a)]^{\alpha}}\int_0^1\Xi_{\alpha,g}(t)f'(ta+(1-t)b)dt.\end{aligned}$$

For some recent results connected with fractional integral inequalities, see [3], [5]-[9], [19], [21], [24]-[27].

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for $s-$convex function involving fractional integrals with respect to another function. The results presented in this paper provide extensions of those given in earlier works.

2. Main results

Firstly, let us start with some notations given in [14]. Let $f:I^{\circ}\to\mathbb{R}$ be a function such that $a,b\in I^{\circ}$ and $0<a<b<\infty$. We suppose that

$f \in L^{\infty}(a,b)$ in such a way that $I^{\alpha}_{a^+;g}f(x)$ and $I^{\alpha}_{b^-;g}f(x)$ are well defined. We define the function

$$\begin{aligned} \widetilde{f}(x) &= f(a+b-x), \ \ x \in [a,b] \\ F(x) &= f(x)+\widetilde{f}(x), \ \ x \in [a,b]. \end{aligned}$$

Now we shall present the following notations:

$$\begin{aligned} H_1(\alpha,s;g) &= \int_0^1 \frac{[t^s+(1-t)^s]\, g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}}dt, \\ H_2(\alpha,s;g) &= \int_0^1 \frac{[t^s+(1-t)^s]\, g'\left((1-t)a+tb\right)}{\left[g\left((1-t)a+tb\right)-g(a)\right]^{1-\alpha}}dt. \end{aligned}$$

For $g(t)=t$, we have

$$H_1(\alpha,s;g)=H_2(\alpha,s;g)=(b-a)^{\alpha-1}\left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right]$$

where $\beta(x,y)$ is the Beta function.

For $\alpha>0$ and $s\in(0,1]$, we give the following operator

$$L_g^{\alpha,s}(x,y)=\int_a^{\frac{a+b}{2}}|x-u|^s\,|g(y)-g(u)|^{\alpha}\,du-\int_{\frac{a+b}{2}}^{b}|x-u|^s\,|g(y)-g(u)|^{\alpha}\,du$$

for $x,y\in[a,b]$. Particularly, for $g(t)=t$, we have

$$L_g^{\alpha,s}(b,b)=-L_g^{\alpha,s}(a,a)=(b-a)^{\alpha+s+1}\,\frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}$$

and

$$\begin{aligned} L_g^{\alpha,s}(a,b) &= -L_g^{\alpha,s}(b,a) \\ &= (b-a)^{\alpha+s+1}\left[\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,s+1\right)\right] \end{aligned}$$

where $\beta(z;x,y)$ is the incomplete Beta function.

Theorem 2. *Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and let $\alpha>0$. If f is a s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$,* *then the following Hermite-Hadamard type inequality for fractional integrals hold:*

$$
\begin{aligned}
(3)\qquad 2^{s-1} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2\,[g(b)-g(a)]^{\alpha}}\left[\frac{I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)}{2}\right]\\
&\le \left[\frac{f(a)+f(b)}{2}\right]\\
&\quad\times\frac{\alpha\,(b-a)}{[g(b)-g(a)]^{\alpha}}\left[\frac{H_1(\alpha,s;g)+H_2(\alpha,s;g)}{2}\right].
\end{aligned}
$$

Proof. Since f is an s-convex mapping in the second sense on $[a,b]$, we have

$$
(4)\qquad f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2^s}
$$

for $x,y\in[a,b]$. Now, for $t\in[0,1]$, let $x=ta+(1-t)b$ and $y=(1-t)a+tb$. Then we have

$$
(5)\qquad 2^s f\left(\frac{a+b}{2}\right)\le f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right).
$$

Multiplying both sides of (5) by

$$
\frac{b-a}{\Gamma(\alpha)}\frac{g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}}
$$

and integraing the resulting inequality with respect to t over $(0,1)$, we get

$$
\begin{aligned}
&\frac{2^s\,(b-a)}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\int_0^1\frac{g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}}dt\\
&\le\frac{b-a}{\Gamma(\alpha)}\int_0^1\frac{f\left(ta+(1-t)b\right)g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}}dt\\
&\quad+\frac{b-a}{\Gamma(\alpha)}\int_0^1\frac{f\left((1-t)a+tb\right)g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}}dt.
\end{aligned}
$$

Using the change of variable $\tau=(1-t)a+tb$, we have

$$
\frac{2^s}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\frac{[g(b)-g(a)]^{\alpha}}{\alpha}\le I^{\alpha}_{a^+;g}\widetilde{f}(b)+I^{\alpha}_{a^+;g}f(b),
$$

i.e.

$$\frac{2^s}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)[g(b)-g(a)]^{\alpha} \le I_{a+;g}^{\alpha}F(b). \tag{6}$$

Similarly, multiplying both sides of (5) by

$$\frac{b-a}{\Gamma(\alpha)} \frac{g'((1-t)a+tb)}{[g((1-t)a+tb)-g(a)]^{1-\alpha}}$$

and integrating the resulting inequality with respect to t over $(0,1)$, we obtain

$$\frac{2^s}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)[g(b)-g(a)]^{\alpha} \le I_{b-;g}^{\alpha}F(a). \tag{7}$$

Summing the inequalities (6) and (7), we get

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2[g(b)-g(a)]^{\alpha}}\left[\frac{I_{a+;g}^{\alpha}F(b)+I_{b-;g}^{\alpha}F(a)}{2}\right].$$

This completes the proof of first inequality in (3).

For the proof of the second inequality in (3), since f is s-convex in the second sence, we have

$$f(ta+(1-t)b) \le t^s f(a) + (1-t)^s f(b)$$

and

$$f((1-t)a+tb) \le (1-t)^s f(a) + t^s f(b).$$

By adding these inequalities, we have

$$f(ta+(1-t)b) + f((1-t)a+tb) \le [t^s+(1-t)^s][f(a)+f(b)]. \tag{8}$$

Multiplying both sides of (8) by

$$\frac{b-a}{\Gamma(\alpha)} \frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}$$

and integraing the resulting inequality with respect to t over $(0,1)$, we have

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)}\int_0^1 \frac{f(ta+(1-t)b)\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}dt \\
&\quad + \frac{b-a}{\Gamma(\alpha)}\int_0^1 \frac{f((1-t)a+tb)\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}dt \\
&\quad \le [f(a)+f(b)]\frac{b-a}{\Gamma(\alpha)}\int_0^1 \frac{[t^s+(1-t)^s]\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}dt.
\end{aligned}$$

Then, we get

$$I_{a^+;g}^{\alpha}\widetilde{f}(b)+I_{a^+;g}^{\alpha}f(b)\leq [f(a)+f(b)]\,\frac{b-a}{\Gamma(\alpha)}H_1(\alpha,s;g),$$

that is,

$$I_{a^+;g}^{\alpha}F(b)\leq [f(a)+f(b)]\,\frac{b-a}{\Gamma(\alpha)}H_1(\alpha,s;g). \tag{9}$$

Similarly, multiplying both sides of (8) by

$$\frac{b-a}{\Gamma(\alpha)}\frac{g'\left((1-t)a+tb\right)}{\left[g\left((1-t)a+tb\right)-g(a)\right]^{1-\alpha}}$$

and integraing the resulting inequality with respect to t over $(0,1)$, we get

$$I_{b^-;g}^{\alpha}F(a)\leq [f(a)+f(b)]\,\frac{b-a}{\Gamma(\alpha)}H_2(\alpha,s;g). \tag{10}$$

By adding the inequalities (9) and (10), we have

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\frac{I_{b^-;g}^{\alpha}F(a)+I_{a^+;g}^{\alpha}F(b)}{2}\right]\\&\qquad\leq\left[\frac{f(a)+f(b)}{2}\right]\frac{\alpha\,(b-a)}{[g(b)-g(a)]^{\alpha}}\left[\frac{H_1(\alpha,s;g)+H_2(\alpha,s;g)}{2}\right],\end{aligned}$$

which completes te proof. ■

Remark 1. If we put $s=1$ in (3), then we obtain Theorem 2.1 in [14].

Remark 2. If we choose $g(t)=t$, then we obtain the following inequality

$$\begin{aligned}2^{s-1}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)}{2}\right]\\&\leq\alpha\left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right]\frac{f(a)+f(b)}{2}\end{aligned}$$

which was proved by Set et al. in [25] .

Corollary 1. *Under assumption of Theorem 2 with* $g(t)=\ln t$, *we have the following inequality*

$$\begin{aligned}2^{s-1}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\frac{\mathbf{J}_{a^+}^{\alpha}F(b)+\mathbf{J}_{b^-}^{\alpha}F(a)}{2}\right]\\&\leq\left[\frac{f(a)+f(b)}{2}\right]\frac{\alpha\,(b-a)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\frac{H_1(\alpha,s;\ln)+H_2(\alpha,s;\ln)}{2}\right].\end{aligned}$$

Theorem 3. *Let g be as the above. If $f\in C^1(I^\circ)$ and $|f'|$is an s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right)\right| \leq \frac{I_g^{\alpha,s}(a,b)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

where

$$I_g^{\alpha,s}(a,b)=L_g^{\alpha,s}(b,b)+L_g^{\alpha,s}(a,b)-L_g^{\alpha,s}(b,a)-L_g^{\alpha,s}(a,a).$$

Proof. Taking madulus in Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right)\right| \leq \frac{b-a}{4\left[g(b)-g(a)\right]^{\alpha}}\int_0^1\left|\Xi_{\alpha,g}(t)\right|\left|f'(ta+(1-t)b)\right|dt.$$

Since $|f'|$ is an s-convex in the second sense on $[a,b]$, we get

$$\left|f'(ta+(1-t)b)\right|\leq t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|.$$

Hence,

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right)\right| \leq \frac{b-a}{4\left[g(b)-g(a)\right]^{\alpha}} \times\left[\left|f'(a)\right|\int_0^1 t^s\left|\Xi_{\alpha,g}(t)\right|dt+\left|f'(b)\right|\int_0^1(1-t)^s\left|\Xi_{\alpha,g}(t)\right|dt\right]. \tag{11}$$

Here, we have

$$\int_0^1 t^s\left|\Xi_{\alpha,g}(t)\right|dt=\frac{1}{(b-a)^{s+1}}\int_a^b(b-u)^s\left|\varphi(u)\right|dt$$

where

$$\begin{aligned}\varphi(u) &= \left[g(u)-g(a)\right]^{\alpha}-\left[g(a+b-u)-g(a)\right]^{\alpha}\\ &\quad+\left[g(b)-g(a+b-u)\right]^{\alpha}-\left[g(b)-g(u)\right]^{\alpha}.\end{aligned}$$

Since g is an increasing function, φ is a non-decreasing function on $[a, b]$. Additionaly,

$$\varphi(a) = -2\left[g(b) - g\left(a\right)\right]^{\alpha} < 0$$

and

$$\varphi\left(\frac{a+b}{2}\right) = 0.$$

Consequently, we get

$$\begin{cases} \varphi(u) \leq 0, & \text{if} \quad a \leq u \leq \frac{a+b}{2}, \\ \varphi(u) > 0, & \text{if} \quad \frac{a+b}{2} < u \leq b. \end{cases}$$

Therefore, we have

$$\int_a^b (b-u)^s \left|\varphi(u)\right| dt = I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned} I_1 &= \int_a^{\frac{a+b}{2}} (b-u)^s \left[g(b) - g(u)\right]^{\alpha} du \\ &\quad - \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(b) - g(u)\right]^{\alpha} du = L_g^{\alpha,s}(b,b), \end{aligned}$$

$$\begin{aligned} I_2 &= -\int_a^{\frac{a+b}{2}} (b-u)^s \left[g(u) - g(a)\right]^{\alpha} du \\ &\quad + \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(u) - g(a)\right]^{\alpha} du = -L_g^{\alpha,s}(b,a), \end{aligned}$$

$$\begin{aligned} I_3 &= \int_a^{\frac{a+b}{2}} (b-u)^s \left[g(a+b-u) - g(a)\right]^{\alpha} du \\ &\quad - \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(a+b-u) - g(a)\right]^{\alpha} du = -L_g^{\alpha,s}(a,a), \end{aligned}$$

and

$$I_4 = -\int_a^{\frac{a+b}{2}} (b-u)^s [g(b) - g(a+b-u)]^\alpha du + \int_{\frac{a+b}{2}}^b (b-u)^s [g(b) - g(a+b-u)]^\alpha du = L_g^{\alpha,s}(a,b).$$

Thus, from the previous equalities it follows that

$$\int_0^1 t^s |\Xi_{\alpha,g}(t)| dt = \frac{L_g^{\alpha,s}(b,b) + L_g^{\alpha,s}(a,b) - L_g^{\alpha,s}(b,a) - L_g^{\alpha,s}(a,a)}{(b-a)^{s+1}}. \tag{12}$$

Similarly, it is clear that

$$\int_0^1 (1-t)^s |\Xi_{\alpha,g}(t)| dt = \frac{L_g^{\alpha,s}(b,b) + L_g^{\alpha,s}(a,b) - L_g^{\alpha,s}(b,a) - L_g^{\alpha,s}(a,a)}{(b-a)^{s+1}}. \tag{13}$$

If we put equality (12) and (13) in (11), we obtain the desired result. ■

Remark 3. If we put $s = 1$ in (3), then we obtain Theorem 2.5 in [14].

Remark 4. If we choose $g(t) = t$, then we obtain the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \le \frac{b-a}{2} \left\{ \beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; \alpha+1, s+1\right) + \frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)} \right\} \left[|f'(a)| + |f'(b)| \right].$$

This inequality was given by Set et al. in [25, Theorem 4 (for $q = 1$)]

Corollary 2. *Under assumption of Theorem 2 with $g(t) = \ln t$, we have the following inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{4\left(\ln \frac{b}{a}\right)^\alpha} \left(\mathbf{J}_{a+}^\alpha F(b) + \mathbf{J}_{b-}^\alpha F(a) \right) \right| \le \frac{I_{\ln}^{\alpha,s}(a,b)}{4\left(\ln \frac{b}{a}\right)^\alpha} \left[|f'(a)| + |f'(b)| \right]$$

where

$$I_{\ln}^{\alpha,s}(a,b) = L_{\ln}^{\alpha,s}(b,b) + L_{\ln}^{\alpha,s}(a,b) - L_{\ln}^{\alpha,s}(b,a) - L_{\ln}^{\alpha,s}(a,a).$$

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