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S.S. Dragomir

NEW INEQUALITIES OF CBS-TYPE FOR POWER SERIES OF COMPLEX NUMBERS

Abstract. Let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. In this paper we show amongst other that, if $\alpha, z \in \mathbb{C}$ are such that $|\alpha|, |\alpha| |z|^2 < R$, then

$$\left|f(\alpha) f\left(\alpha z^2\right) - f^2(\alpha z)\right| \leq f_A(|\alpha|) f_A\left(|\alpha| |z|^2\right) - |f_A(|\alpha| z)|^2 .$$

where $f_A(z) = \sum_{n=0}^{\infty} |\alpha_n| z^n$.

Applications for some fundamental functions defined by power series are also provided.

Key words: power series, CBS-type inequalities.

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1. Introduction

If we consider an analytic function $f(z)$ defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and apply the well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2, \tag{1}$$

holding for the complex numbers a_j, b_j, $j \in \{1,\ldots,n\}$, then we can deduce that

$$|f(z)|^2 = \left|\sum_{n=0}^{\infty} a_n z^n\right|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1-|z|^2} \sum_{n=0}^{\infty} |a_n|^2 \tag{2}$$

for any $z \in D(0,R) \cap D(0,1)$, where R is the radius of convergence of f.

The above inequality gives some information about the magnitude of the function f provided that numerical series $\sum_{n=0}^{\infty}|a_n|^2$ is convergent and z is not too close to the boundary of the open disk $D(0,1)$.

If we restrict ourselves more and assume that the coefficients in the representation $f(z)=\sum_{n=0}^{\infty}a_n z^n$ are nonnegative, and the assumption incorporates various examples of complex functions that will be indicated in the sequel, on utilizing the weighted version of the CBS-inequality, namely

$$\left|\sum_{j=1}^{n} w_j a_j b_j\right|^2 \le \sum_{j=1}^{n} w_j |a_j|^2 \sum_{j=1}^{n} w_j |b_j|^2, \tag{3}$$

where $w_j\ge 0$, while $a_j,b_j\in\mathbb{C}$, $j\in\{1,\ldots,n\}$, we can state that

$$\begin{aligned} |f(zw)|^2 &= \left|\sum_{n=0}^{\infty} a_n z^n w^n\right|^2 \le \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} \\ &= f\left(|z|^2\right) f\left(|w|^2\right) \end{aligned} \tag{4}$$

for any $z,w\in\mathbb{C}$ with $|z|^2,|w|^2\in D(0,R)$.

In an effort to provide a refinement for the celebrated Cauchy-Bunyakovsky -Schwarz inequality for complex numbers (1) de Bruijn established in 1960, [2] (see also [8, p. 89] or [3, p. 48]) the following result:

Lemma 1 (de Bruijn, 1960)**.** *If* $\mathbf{b}=(b_1,\ldots,b_n)$ *is an* n*-tuple of real numbers and* $z=(z_1,\ldots,z_n)$ *an* n*-tuple of complex numbers, then*

$$\left|\sum_{k=1}^{n} b_k z_k\right|^2 \le \frac{1}{2}\sum_{k=1}^{n} b_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right]. \tag{5}$$

Equality holds in (5) if and only if for $k\in\{1,\ldots,n\}$, $b_k=Re(\lambda z_k)$, *where* λ *is a complex number such that the quantity* $\lambda^2\sum_{k=1}^{n} z_k^2$ *is a nonnegative real number.*

On utilizing this result, Cerone & Dragomir established in [1] some inequalities for power series with nonnegative coefficients as follows:

Theorem 1 (Cerone & Dragomir, 2007 [1])**.** *Let* $f(z):=\sum_{n=0}^{\infty}a_n z^n$ *be an analytic function defined by a power series with nonnegative coefficients* a_n, $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)\subset\mathbb{C}$, $R>0$. *If* a *is a real number and* z *a complex number such that* $a^2,|z|^2\in D(0,R)$, *then:*

$$|f(az)|^2 \le \frac{1}{2} f\left(a^2\right)\left[f\left(|z|^2\right) + \left|f\left(z^2\right)\right|\right]. \tag{6}$$

For other similar results and applications for special functions see the research papers [1], [4]-[6] and the survey [7].

2. The results

Denote by:

$$D(0,R)=\begin{cases}\{z\in\mathbb{C}:|z|<R\}, & \text{if } \ R<\infty\\ \mathbb{C}, & \text{if } \ R=\infty,\end{cases}$$

and consider the functions:

$$\lambda\mapsto f(\lambda):D(0,R)\to\mathbb{C},\quad f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$$

and

$$\lambda\mapsto f_A(\lambda):D(0,R)\to\mathbb{C},\quad f_A(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\,\lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}
f(\lambda)&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1);\\
g(\lambda)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};\\
h(\lambda)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};\\
l(\lambda)&=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);
\end{aligned}\tag{7}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned}
f_A(\lambda)&=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1);\\
g_A(\lambda)&=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};\\
h_A(\lambda)&=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};\\
l_A(\lambda)&=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).
\end{aligned}\tag{8}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n, \quad \lambda \in \mathbb{C}, \tag{9}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \quad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \quad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \quad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \quad \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function.*

The following result holds:

Theorem 2. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $\alpha, z \in \mathbb{C}$ are such that $|\alpha|, |\alpha||z|^2 < R$, then*

$$\left|f(\alpha) f\left(\alpha z^2\right) - f^2(\alpha z)\right| \le f_A(|\alpha|) f_A\left(|\alpha||z|^2\right) - |f_A(|\alpha| z)|^2. \tag{10}$$

Proof. Let $n \ge 1$. Observe that

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \alpha^j \alpha^k \left(z^j - z^k\right)^2 &= \sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \alpha^j \alpha^k \left(z^{2j} - 2z^j z^k + z^{2k}\right) \\
&= \sum_{j=0}^{n} a_j \alpha^j z^{2j} \sum_{k=0}^{n} a_k \alpha^k + \sum_{j=0}^{n} a_j \alpha^j \sum_{k=0}^{n} a_k \alpha^k z^{2k} \\
&\quad - 2\sum_{j=0}^{n} a_j \alpha^j z^j \sum_{k=0}^{n} a_k \alpha^k z^k \\
&= 2\left[\sum_{j=0}^{n} a_j \alpha^j \sum_{j=0}^{n} a_j \alpha^j z^{2j} - \left(\sum_{j=0}^{n} a_j \alpha^j z^j\right)^2\right],
\end{aligned}$$

which gives us the useful identity

$$\sum_{j=0}^{n} a_j \alpha^j \sum_{j=0}^{n} a_j \alpha^j z^{2j} - \left(\sum_{j=0}^{n} a_j \alpha^j z^j \right)^2 = \frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \alpha^j \alpha^k \left(z^j - z^k \right)^2 \tag{11}$$

for any $\alpha, z \in \mathbb{C}$ and $n \geq 1$.

Taking the modulus in (11) and utilizing the generalized triangle inequality we have

$$\begin{aligned} &\left| \sum_{j=0}^{n} a_j \alpha^j \sum_{j=0}^{n} a_j \alpha^j z^{2j} - \left(\sum_{j=0}^{n} a_j \alpha^j z^j \right)^2 \right| \\ &\leq \frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \left| z^j - z^k \right|^2 \\ &= \frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \left[|z|^{2j} - 2\mathrm{Re} \left(z^j \bar{z}^k \right) + |z|^{2k} \right] \\ &= \frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \, |z|^{2j} \\ &\quad + \frac{1}{2} \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \, |z|^{2k} \\ &\quad - \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \, \mathrm{Re} \left(z^j \bar{z}^k \right). \end{aligned} \tag{12}$$

Observe that

$$\begin{aligned} \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \, |z|^{2j} &= \sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \, |z|^{2k} \\ &= \sum_{j=0}^{n} |a_j| \, |\alpha|^j \sum_{j=0}^{n} |a_j| \, |\alpha|^j \, |z|^{2j} \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\sum_{j=0}^{n} \sum_{k=0}^{n} |a_j| \, |a_k| \, |\alpha|^j \, |\alpha|^k \, \mathrm{Re} \left(z^j \bar{z}^k \right) \\ &\qquad = \mathrm{Re} \left(\sum_{j=0}^{n} |a_j| \, |\alpha|^j \, z^j \sum_{k=0}^{n} |a_k| \, |\alpha|^k \, \bar{z}^k \right) \end{aligned} \tag{14}$$

$$= \operatorname{Re}\left(\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\overline{\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j}\right) = \left|\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\right|^2.$$

Making use of (12)-(14) we get

$$\left|\sum_{j=0}^{n}a_j\alpha^j\sum_{j=0}^{n}a_j\alpha^j z^{2j} - \left(\sum_{j=0}^{n}a_j\alpha^j z^j\right)^2\right| \tag{15}$$
$$\leq \sum_{j=0}^{n}|a_j|\,|\alpha|^j\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,|z|^{2j} - \left|\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\right|^2$$

for any $\alpha, z \in \mathbb{C}$ and $n \geq 1$.

Since all series whose partial sums involved in the inequality (15) are convergent, then by letting $n \to \infty$ in (15) we deduce the desired result (10). ∎

Remark 1. If $f(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ is a function defined by power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$, then

$$\left|f(\alpha)\,f\left(\alpha z^2\right) - f^2(\alpha z)\right| \leq f(|\alpha|)\,f\left(|\alpha|\,|z|^2\right) - |f(|\alpha|\,z)|^2 \tag{16}$$

for $\alpha, z \in \mathbb{C}$ with $|\alpha|,\ |\alpha|\,|z|^2 < R$.

Corollary 1. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $x, y \in \mathbb{C}$ are such that $|x|^2,\ |y|^2 < R$, then*

$$\left|f\left(x^2\right)f\left(y^2\right) - f^2(xy)\right| \leq f_A\left(|x|^2\right)f_A\left(|y|^2\right) - |f_A(\bar{x}y)|^2 \tag{17}$$

and

$$\left|f\left(x^2\right)f\left(\sigma^2(x)\,y^2\right) - f^2(\sigma(x)\,xy)\right| \leq f_A\left(|x|^2\right)f_A\left(|y|^2\right) - |f_A(xy)|^2 \tag{18}$$

where $\sigma(x) := \frac{x}{\bar{x}}$ is the "sign" of the complex number $x \neq 0$.

Proof. If we take in (10) $\alpha = x^2$ and $z = \frac{y}{x}$, then we have

$$\left|f\left(x^2\right)f\left(y^2\right) - f^2(xy)\right| \leq f_A\left(|x|^2\right)f_A\left(|y|^2\right) - \left|f_A\left(|x|^2\frac{y}{x}\right)\right|^2, \tag{19}$$

which is equivalent to (17).

If we take in (10) $\alpha = x^2$ and $z = \frac{y}{\bar{x}}$, then we have

$$\left|f\left(x^2\right)f\left(x^2\left(\frac{y}{\bar{x}}\right)^2\right) - f^2\left(x^2\frac{y}{\bar{x}}\right)\right| \leq f_A\left(|x|^2\right)f_A\left(|y|^2\right) - |f_A(xy)|^2,$$

which is equivalent to (18). ∎

Remark 2. If $a \in \mathbb{R}$ and $y \in \mathbb{C}$ are such that a^2, $|y|^2 < R$, then

$$\left|f\left(a^2\right) f\left(y^2\right) - f^2\left(ay\right)\right| \le f_A\left(a^2\right) f_A\left(|y|^2\right) - \left|f_A\left(ay\right)\right|^2. \tag{20}$$

In particular, if the power series is with nonnegative coefficients, then

$$\left|f\left(a^2\right) f\left(y^2\right) - f^2\left(ay\right)\right| \le f\left(a^2\right) f\left(|y|^2\right) - \left|f\left(ay\right)\right|^2 \tag{21}$$

for $a \in \mathbb{R}$ and $y \in \mathbb{C}$ such that a^2, $|y|^2 < R$.

We also remark that, since

$$\left|f\left(ay\right)\right|^2 - f\left(a^2\right) f\left(y^2\right) \le \left|f\left(a^2\right) f\left(y^2\right) - f^2\left(ay\right)\right|,$$

then by (21) we get

$$\left|f\left(ay\right)\right|^2 - f\left(a^2\right) f\left(y^2\right) \le f\left(a^2\right) f\left(|y|^2\right) - \left|f\left(ay\right)\right|^2,$$

which is equivalent to Cerone-Dragomir's result

$$\left|f\left(ay\right)\right|^2 \le \frac{1}{2} f\left(a^2\right)\left[f\left(|y|^2\right) + \left|f\left(y^2\right)\right|\right],$$

where $a \in \mathbb{R}$ and $y \in \mathbb{C}$ such that a^2, $|y|^2 < R$.

The following result also holds:

Theorem 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $\alpha, z \in \mathbb{C}$ are such that $|\alpha|$, $|\alpha|\,|z|^2 < R$, then*

$$\begin{aligned} &\left|f\left(\alpha |z|^2\right) f(\alpha) - f(\alpha z) f(\alpha \bar{z})\right|^2 \\ &\quad \le f_A\left(|\alpha|\,|z|^2\right)\left[f_A\left(|\alpha|\,|z|^2\right)|f(\alpha)|^2 + |f(\alpha z)|^2 f_A(|\alpha|)\right. \\ &\qquad \left. - 2\,|f(\alpha)|^2 \operatorname{Re}\left(f_A(|\alpha|\, z)\,\overline{\left(f(\alpha z)/f(\alpha)\right)}\right)\right]. \end{aligned} \tag{22}$$

Proof. Let $n \ge 1$. Observe that

$$\sum_{j=0}^{n} a_j \alpha^j \left(z^j - \frac{f(\alpha z)}{f(\alpha)}\right) \bar{z}^j = \sum_{j=0}^{n} a_j \alpha^j |z|^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \bar{z}^j \tag{23}$$

for any $\alpha, z \in \mathbb{C}$.

Taking the modulus in (23) and utilizing the generalized triangle inequality we get

$$\left|\sum_{j=0}^{n} a_j \alpha^j |z|^{2j} - \frac{f(\alpha z)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \bar{z}^j\right| \leq \sum_{j=0}^{n} |a_j| |\alpha|^j \left(\left|z^j - \frac{f(\alpha z)}{f(\alpha)}\right|\right) |z|^j \tag{24}$$

for any $\alpha, z \in \mathbb{C}$.

Making use of the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality we have

$$\sum_{j=0}^{n} |a_j| |\alpha|^j \left(\left|z^j - \frac{f(\alpha z)}{f(\alpha)}\right|\right) |z|^j \leq \left(\sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j}\right)^{1/2} \left[\sum_{j=0}^{n} |a_j| |\alpha|^j \left|z^j - \frac{f(\alpha z)}{f(\alpha)}\right|^2\right]^{1/2} \tag{25}$$

for any $\alpha, z \in \mathbb{C}$.

We also have

$$\begin{aligned} \sum_{j=0}^{n} |a_j| |\alpha|^j \left|z^j - \frac{f(\alpha z)}{f(\alpha)}\right|^2 &= \sum_{j=0}^{n} |a_j| |\alpha|^j \left[|z|^{2j} - 2\mathrm{Re}\left(z^j \overline{\left(\frac{f(\alpha z)}{f(\alpha)}\right)}\right) + \left|\frac{f(\alpha z)}{f(\alpha)}\right|^2\right] \\ &= \sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j} - 2\mathrm{Re}\left(\sum_{j=0}^{n} |a_j| |\alpha|^j z^j \overline{\left(\frac{f(\alpha z)}{f(\alpha)}\right)}\right) \\ &\quad + \left|\frac{f(\alpha z)}{f(\alpha)}\right|^2 \sum_{j=0}^{n} |a_j| |\alpha|^j \end{aligned} \tag{26}$$

for any $\alpha, z \in \mathbb{C}$.

By (24)-(26) we get

$$\left|\sum_{j=0}^{n} a_j \alpha^j |z|^{2j} - \frac{f(\alpha z)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \bar{z}^j\right| \leq \left(\sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j}\right)^{1/2} \left[\sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j}\right. \tag{27}$$

$$- 2\mathrm{Re}\left(\overline{\left(\frac{f(\alpha z)}{f(\alpha)}\right)}\sum_{j=0}^{n}|a_j|\,|\alpha|^j z^j\right) + \left|\frac{f(\alpha z)}{f(\alpha)}\right|^2\sum_{j=0}^{n}|a_j|\,|\alpha|^j\Bigg]^{1/2}$$

for any $\alpha, z \in \mathbb{C}$.

Since all series whose partial sums involved in the inequality (27) are convergent, then by letting $n \to \infty$ in (27) we deduce

$$\left|f\left(\alpha|z|^2\right) - \frac{f(\alpha z) f(\alpha\overline{z})}{f(\alpha)}\right| \le \left[f_A\left(|\alpha|\,|z|^2\right)\right]^{1/2}$$
$$\times\left[f_A\left(|\alpha|\,|z|^2\right) - 2\mathrm{Re}\left(f_A(|\alpha|\,z)\overline{\left(\frac{f(\alpha z)}{f(\alpha)}\right)}\right) + \left|\frac{f(\alpha z)}{f(\alpha)}\right|^2 f_A(|\alpha|)\right]^{1/2},$$

which is equivalent to the desired result (22). ■

Corollary 2. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $x, y \in \mathbb{C}$ are such that $|x|^2$, $|y|^2 < R$, then*

$$\left|f\left(\sigma(x)|y|^2\right) f\left(x^2\right) - f(xy) f(\sigma(x) x\overline{y})\right|^2 \tag{28}$$
$$\le f_A\left(|y|^2\right)\Big[f_A\left(|y|^2\right)\left|f\left(x^2\right)\right|^2 + |f(xy)|^2 f_A\left(\left|x^2\right|\right)$$
$$- 2\left|f\left(x^2\right)\right|^2 Re\left(f_A(\overline{x}y)\overline{\left(f(xy)/f\left(x^2\right)\right)}\right)\Big].$$

Proof. If we take in (22) $\alpha = x^2$ and $z = \frac{y}{x}$, then we have

$$\left|f\left(x^2\left|\frac{y}{x}\right|^2\right) f\left(x^2\right) - f\left(x^2\frac{y}{x}\right) f\left(x^2\frac{\overline{y}}{\bar{x}}\right)\right|^2$$
$$\le f_A\left(\left|x^2\right|\left|\frac{y}{x}\right|^2\right)\Bigg[f_A\left(\left|x^2\right|\left|\frac{y}{x}\right|^2\right)\left|f\left(x^2\right)\right|^2$$
$$- 2\left|f\left(x^2\right)\right|\mathrm{Re}\left(f_A\left(\left|x^2\right|\frac{y}{x}\right)\overline{\left(\frac{f\left(x^2\frac{y}{x}\right)}{f\left(x^2\right)}\right)}\right)$$
$$+ \left|f\left(x^2\frac{y}{x}\right)\right|^2 f_A\left(\left|x^2\right|\right)\Bigg],$$

which is equivalent to (28). ■

We have the following result:

Theorem 4. *Assume that $f(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ $(a_0 \ne 0)$ is a function defined by power series with nonnegative coefficients and convergent on the* *open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $x \in \mathbb{R}$ with $0 \le x \le 1$ and $0 \le \alpha < R$, then*

$$0 \le f(\alpha) f\left(\alpha x^2\right) - f^2(\alpha x) \le \frac{1}{4} f^2(\alpha). \tag{29}$$

Proof. Let $n \ge 1$. Observe that

$$\begin{aligned}\sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\left(x^j - \frac{1}{2}\right) \\ = \sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\end{aligned} \tag{30}$$

for any $\alpha, x \in \mathbb{R}$.

Taking the modulus and utilizing the triangle inequality we have

$$\begin{aligned}\left|\sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\right| \\ \le \sum_{j=0}^{n} a_j \alpha^j \left|x^j - \frac{f(\alpha x)}{f(\alpha)}\right| \left|x^j - \frac{1}{2}\right|\end{aligned} \tag{31}$$

for any $\alpha, x \in \mathbb{R}$.

Since $0 \le x \le 1$, then $0 \le x^j \le 1$ for $j \in \{0, ..., n\}$, which implies that

$$\left|x^j - \frac{1}{2}\right| \le \frac{1}{2} \quad \text{for} \quad j \in \{0, ..., n\}.$$

Then by (31) we get

$$\begin{aligned}\left|\sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\right| \\ \le \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left|x^j - \frac{f(\alpha x)}{f(\alpha)}\right|\end{aligned} \tag{32}$$

for $0 \le x \le 1$ and $n \ge 1$.

Utilising the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\sum_{j=0}^{n} a_j \alpha^j \left|x^j - \frac{f(\alpha x)}{f(\alpha)}\right| \le \left(\sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)^2\right)^{1/2} \left(\sum_{j=0}^{n} a_j \alpha^j\right)^{1/2}. \tag{33}$$

Observe that

$$\begin{aligned}(34)\qquad &\sum_{j=0}^{n} a_j\alpha^j\left(x^j-\frac{f(\alpha x)}{f(\alpha)}\right)^2\\ &=\sum_{j=0}^{n} a_j\alpha^j\left[x^{2j}-2\left(x^j\frac{f(\alpha x)}{f(\alpha)}\right)+\frac{f^2(\alpha x)}{f^2(\alpha)}\right]\\ &=\sum_{j=0}^{n} a_j\alpha^j x^{2j}-2\left(\frac{f(\alpha x)}{f(\alpha)}\sum_{j=0}^{n} a_j\alpha^j x^j\right)+\frac{f^2(\alpha x)}{f^2(\alpha)}\sum_{j=0}^{n} a_j\alpha^j.\end{aligned}$$

From (31)-(34) we can state that

$$\begin{aligned}(35)\qquad &\left|\sum_{j=0}^{n} a_j\alpha^j x^{2j}-\frac{f(\alpha x)}{f(\alpha)}\sum_{j=0}^{n} a_j\alpha^j x^j-\frac{1}{2}\sum_{j=0}^{n} a_j\alpha^j\left(x^j-\frac{f(\alpha x)}{f(\alpha)}\right)\right|\\ &\le\frac{1}{2}\left(\sum_{j=0}^{n} a_j\alpha^j\right)^{1/2}\left[\sum_{j=0}^{n} a_j\alpha^j x^{2j}\right.\\ &\quad\left.-2\left(\frac{f(\alpha x)}{f(\alpha)}\sum_{j=0}^{n} a_j\alpha^j x^j\right)+\frac{f^2(\alpha x)}{f^2(\alpha)}\sum_{j=0}^{n} a_j\alpha^j\right]^{1/2}\end{aligned}$$

for any $0\le x\le 1$ and $n\ge 1$.

Since all series whose partial sums involved in the inequality (35) are convergent, then by letting $n\to\infty$ in (35) we deduce

$$\begin{aligned}&\left|f\left(\alpha x^2\right)-\frac{f(\alpha x)}{f(\alpha)}f(\alpha x)\right|\\ &\le\frac{1}{2}[f(\alpha)]^{1/2}\left[f\left(\alpha x^2\right)-2\left(\frac{f(\alpha x)}{f(\alpha)}f(\alpha x)\right)+\frac{f^2(\alpha x)}{f^2(\alpha)}f(\alpha)\right]^{1/2},\end{aligned}$$

namely

$$\left|f\left(\alpha x^2\right)-\frac{f^2(\alpha x)}{f(\alpha)}\right|\le\frac{1}{2}[f(\alpha)]^{1/2}\left[f\left(\alpha x^2\right)-\frac{f^2(\alpha x)}{f(\alpha)}\right]^{1/2},$$

which is equivalent to the desired result (29). ■

Corollary 3. *Let* $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ $(a_0\ne 0)$ *be a function defined by power series with nonnegative coefficients and convergent on the open disk* $D(0,R)\subset\mathbb{C}$, $R>0$. *If* $u,v\in\mathbb{R}$ *with* $0\le u\le v$ *and* $0<v^2<R$, *then*

$$(36)\qquad 0\le f\left(v^2\right)f\left(u^2\right)-f^2(uv)\le\frac{1}{4}f^2\left(v^2\right).$$

Proof. If we take in (29) $\alpha = v^2$ and $x = \frac{u}{v}$, then we have the desired inequality (36). ■

3. Applications

If we write the above inequalities for the exponential function $\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n$, $\lambda \in \mathbb{C}$, then we have:

$$\begin{aligned} &\left|\exp\left[\alpha\left(1+z^2\right)\right] - \exp(2\alpha z)\right| \\ &\quad \le \exp\left[|\alpha|\left(1+|z|^2\right)\right] - |\exp(2|\alpha| z)|, \quad \alpha, z \in \mathbb{C}, \end{aligned} \tag{37}$$

$$\left|\exp\left(x^2+y^2\right) - \exp(2xy)\right| \le \exp\left(|x|^2+|y|^2\right) - |\exp(2\bar{x}y)|, \tag{38}$$

$x, y \in \mathbb{C}$,

$$\begin{aligned} &\left|\exp\left(x^2+\sigma^2(x)\,y^2\right) - \exp(2\sigma(x)\,xy)\right| \\ &\quad \le \exp\left(|x|^2+|y|^2\right) - |\exp(2xy)|, \quad x, y \in \mathbb{C} \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\left|\exp\left[\alpha\left(1+|z|^2\right)\right] - \exp\left[2\alpha \mathrm{Re}(z)\right]\right|^2 \\ &\quad \le \exp\left(|\alpha|\,|z|^2\right)\left[\exp\left(|\alpha|\,|z|^2\right)|\exp(\alpha)|^2 + |\exp(\alpha z)|^2 \exp(|\alpha|)\right. \\ &\qquad \left. - 2\,|\exp(\alpha)|^2\,\mathrm{Re}\left(\exp\left(|\alpha|\,z + \overline{\alpha z} - \bar{\alpha}\right)\right)\right]. \end{aligned} \tag{40}$$

If we take $\alpha = 1$ in (37), then we get

$$\left|\exp\left(1+z^2\right) - \exp(2z)\right| \le \exp\left(1+|z|^2\right) - |\exp(2z)|, \quad z \in \mathbb{C}. \tag{41}$$

If we take $\alpha = 1$ in (40), then we get

$$\begin{aligned} &\left(\exp\left(1+|z|^2\right) - \exp\left[2\mathrm{Re}(z)\right]\right)^2 \\ &\quad \le \exp\left(|z|^2+1\right)\left[\exp\left(|z|^2+1\right) + |\exp(z)|^2 - 2\exp\left(2\mathrm{Re}(z)\right)\right]. \end{aligned} \tag{42}$$

If $x \in \mathbb{R}$ with $0 \le x \le 1$ and $0 \le \alpha$, then

$$0 \le \exp\left(\alpha\left(1+x^2\right)\right) - \exp(2\alpha x) \le \frac{1}{4}\exp(2\alpha). \tag{43}$$

If $0 \le u \le v$, then

$$0 \le \exp\left(v^2+u^2\right) - \exp(2uv) \le \frac{1}{4}\exp\left(2v^2\right). \tag{44}$$

If we write the above inequalities for the functions $\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$ and $\sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}$, $\lambda \in D(0,1)$, then we have

$$\left|(1 \pm \alpha)^{-1} \left(1 \pm \alpha z^2\right)^{-1} - (1 \pm \alpha z)^{-2}\right| \leq (1 - |\alpha|)^{-1} \left(1 - |\alpha| |z|^2\right)^{-1} - |1 - |\alpha| z|^{-2}, \quad |\alpha|, |\alpha| |z|^2 < 1, \tag{45}$$

$$\left|\left(1 \pm x^2\right)^{-1} \left(1 \pm y^2\right)^{-1} - (1 \pm xy)^{-2}\right| \leq \left(1 - |x|^2\right)^{-1} \left(1 - |y|^2\right)^{-1} - |1 - \bar{x}y|^{-2}, \quad |x|, |y| < 1 \tag{46}$$

and

$$\left|\left(1 \pm x^2\right)^{-1} \left(1 \pm \sigma^2(x) y^2\right)^{-1} - (1 \pm \sigma(x) xy)^{-2}\right| \leq \left(1 - |x|^2\right)^{-1} \left(1 - |y|^2\right)^{-1} - |1 - xy|^{-2}, \quad |x|, |y| < 1. \tag{47}$$

If $u, v \in \mathbb{R}$ with $0 \leq u \leq v < 1$, then

$$0 \leq \left(1 - v^2\right)^{-1} \left(1 - u^2\right)^{-1} - (1 - uv)^{-2} \leq \frac{1}{4} \left(1 - v^2\right)^{-2}. \tag{48}$$

If we write the above inequalities for $\sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}$, $\lambda \in D(0,1)$, then we have

$$\left|\ln (1 \pm \alpha)^{-1} \ln \left(1 \pm \alpha z^2\right)^{-1} - \left[\ln (1 \pm \alpha z)^{-1}\right]^2\right| \leq \ln (1 - |\alpha|)^{-1} \ln \left(1 - |\alpha| |z|^2\right)^{-1} - \left|\ln (1 - |\alpha| z)^{-1}\right|^2, \quad |\alpha|, |\alpha| |z|^2 < 1, \tag{49}$$

$$\left|\ln \left(1 \pm x^2\right)^{-1} \ln \left(1 \pm y^2\right)^{-1} - \left[\ln (1 \pm xy)^{-1}\right]^2\right| \leq \ln \left(1 - |x|^2\right)^{-1} \ln \left(1 - |y|^2\right)^{-1} - \left|\ln (1 - \bar{x}y)^{-1}\right|^2, \quad |x|, |y| < 1 \tag{50}$$

and

$$\left|\ln \left(1 \pm x^2\right)^{-1} \ln \left(1 \pm \sigma^2(x) y^2\right)^{-1} - \left[\ln (1 \pm \sigma(x) xy)^{-1}\right]^2\right| \leq \ln \left(1 - |x|^2\right)^{-1} \ln \left(1 - |y|^2\right)^{-1} - \left|\ln (1 - xy)^{-1}\right|^2, \quad |x|, |y| < 1. \tag{51}$$

If $u,v\in\mathbb{R}$ with $0\leq u\leq v<1$, then

$$0\leq \ln\left(1-v^{2}\right)^{-1}\ln\left(1-u^{2}\right)^{-1}-\left[\ln\left(1-uv\right)^{-1}\right]^{2} \leq \frac{1}{4}\left[\ln\left(1-v^{2}\right)^{-1}\right]^{2}. \tag{52}$$

The *polylogarithm* $Li_{n}(z)$, also known as the *de Jonquières function* is the function defined by

$$Li_{n}(z):=\sum_{k=1}^{\infty}\frac{z^{k}}{k^{n}} \tag{53}$$

defined in the complex plane over the unit disk $D(0,1)$ for all complex values of the order n.

The special case $z=1$ reduces to $Li_{s}(1)=\zeta(s)$, where ζ is the *Riemann zeta function*.

The polylogarithm of nonnegative integer order arises in the sums of the form

$$\sum_{k=1}^{\infty}k^{n}r^{k}=Li_{-n}(r)=\frac{1}{(1-r)^{n+1}}\sum_{i=0}^{n}\left\langle {n \atop i}\right\rangle k^{n-i}$$

where $\left\langle {n \atop i}\right\rangle$ is an *Eulerian number*, namely, we recall that

$$\left\langle {n \atop k}\right\rangle :=\sum_{j=0}^{k+1}(-1)^{j}\binom{n+1}{i}(k-j+1)^{n}.$$

Polylogarithms also arise in sums of generalized harmonic numbers $H_{n,r}$ as

$$\sum_{n=1}^{\infty}H_{n,r}z^{n}=\frac{Li_{r}(z)}{1-z}\quad\text{for}\quad z\in D(0,1),$$

where, we recall that

$$H_{n,r}:=\sum_{k=1}^{n}\frac{1}{k^{r}}\quad\text{and}\quad H_{n,1}:=H_{n}=\sum_{k=1}^{n}\frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$Li_{-2}(z)=\frac{z(z+1)}{(1-z)^{3}},\qquad Li_{-1}(z)=\frac{z}{(1-z)^{2}},$$
$$Li_{0}(z)=\frac{z}{1-z}\quad\text{and}\quad Li_{1}(z)=-\ln(1-z),\quad z\in D(0,1).$$

At argument $z=-1$, the general polylogarithms become $Li_x(-1)=-\eta(x)$, where $\eta(x)$ is the *Dirichlet eta function.*

If we use the inequality (16) for *polylogarithm* $Li_n(z)$ we can state that

$$\left|Li_n(\alpha)\,Li_n\left(\alpha z^2\right)-Li_n^2(\alpha z)\right| \leq Li_n(|\alpha|)\,Li_n\left(|\alpha|\,|z|^2\right)-|Li_n(|\alpha|\,z)|^2 \tag{54}$$

for $\alpha, z\in\mathbb{C}$ with $|\alpha|, |z|<1$ and n is a negative or a positive integer.

If $u, v\in\mathbb{R}$ with $0\leq u\leq v<1$, then

$$0\leq Li_n\left(v^2\right)Li_n\left(u^2\right)-\left[Li_n(uv)\right]^2\leq\frac{1}{4}Li_n\left(v^2\right), \tag{55}$$

where n is a negative or a positive integer.

Similar inequalities can be stated for *hypergeometric functions* or for *modified Bessel functions of the first kind*, see [4]-[6]. The details are omitted.

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S.S. Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
and
School of Computer Science & Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
e-mail: sever.dragomir@vu.edu.au

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