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SEMIPRIME NEAR-RINGS WITH MULTIPLICATIVE GENERALIZED $(\theta,\theta)-$DERIVATIONS

Abstract. Let N be a semiprime right near-ring and I a semigroup ideal of N. A map $f : N \to N$ is called a multiplicative generalized $(\theta,\theta)-$derivation if there exists a multiplicative $(\theta,\theta)-$derivation $d : R \to R$ such that $f(xy) = f(x)\theta(y) + \theta(x)d(y)$, for all $x, y \in R$. The purpose of this paper is to investigate the following: (i) $f(uv) = \pm uv$, (ii) $f(uv) = \pm vu$, (iii) $f(u)f(v) = \pm uv$, (iv) $f(u)f(v) = \pm vu$, (v) $d(u)d(v) = \theta([u,v])$, (vi) $d(u)d(v) = \theta(uov)$, (vii) $d(u)\theta(v) = \theta(u)d(v)$.

Key words: semiprime near-ring, semigroup ideal, generalized derivation, multiplicative generalized derivation, multiplicative generalized $(\theta,\theta)-$derivation.

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1. Introduction

An additively written group $(N,+)$ equipped with a binary operation $\cdot : N\times N \to N$, $(x,y) \to xy$ such that $(xy)z = x(yz)$ and $(x+y)z = xz+yz$ for all $x, y, z \in N$ is called a right near-ring. Recall that a near-ring N is called semiprime if for any $x \in N$, $xNx = 0$ implies that $x = 0$. A nonempty subset I of N will be called a semigroup ideal if $IN \subseteq I$ and $NI \subseteq I$. For any $x, y \in N$ the symbol $[x, y]$ will denote $xy - yx$, while the symbol xoy will denote $xy + yx$. Let S be a nonempty subset of N. A mapping g from N to N is called commuting on S if $[g(x), x] = 0$, for all $x \in S$. An additive mapping $d : N \to N$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. In [3], Bresar defined the following: An additive mapping $F : N \to N$ is called a generalized derivation if there exists a derivation $d : N \to N$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

Generalized derivations have been primarily studied on operator algebras.

In [6], the notion of a multiplicative derivation was introduced by Daif and was motivated by Martindale in [12]. $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. These maps are not additive. In [9], Goldman and Semrl gave the complete description of these maps. We have $R = C[0, 1]$, the near ring of all continuous (real or complex valued) functions and define a map $d : R \to R$ such as

$$d(f)(x) = \left\{ \begin{array}{ll} f(x) \log |f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear of that d is a multiplicative derivation, but d is not additive. Inspired by the definition multiplicative derivation, the notion of multiplicative generalized derivation was extended by Daif and Tammam El-Sayiad in [7] as follows:

$F : R \to R$ is called a multiplicative generalized derivation if there exists a multiplicative derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and Ali gave a slight generalization of this definition taking g as any map in [8]. So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. Every generalized derivation is a multiplicative generalized derivation. But the converse is not ture in general. The following example justifies this:

Example 1 ([8])**.** Let S be any ring and

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}.$$

Define the maps d and $F : R \to R$ as follows: $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ve $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is straightforward to verify that F is a multiplicative generalized associated with a multiplicative derivation d, but F is not a generalized derivation of R.

Several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations on appropriate subsets of R. On the other hand, in [2], Ashraf and Rehman showed that if R is prime ring with a nonzero ideal U of R and d is a derivation of R such that $d(xy) \pm xy = 0$, for all $x, y \in U$, then R is commutative. Being inspired by this result, recently Ashraf et al. [1] have studied the situations when derivation d is replaced with a generalized derivation F. More preciesly, they proved that a prime ring R must be commutative, if R satisfies any one of the following conditions: (i) $f(xy) = \pm xy$, (ii) $f(xy) = \pm yx$, (iii) $f(x)f(y) = \pm xy$, (iv) $f(x)f(y) = \pm yx$, where f is a generalized derivation of R and I is a nonzero two-sided ideal of R. Recently many authors have studied commutativity of prime rings with derivations (e.g., [11]). Very recently, Dhara and Ali studied this results for multiplicative generalized derivations on semiprime ring in [8]. The study of derivations of near-rings was initiated by Bell and Mason in 1987 (see [4] for details). In view with of above results it is natural to look for comparable results on near rings. Inspired the definition of multiplicative generalized derivation, the notion of multiplicative generalized $(\theta,\theta)-$derivation was extended as follows [10, Definition 1]:

Definition 1. *Let N be a near-ring and d be a multiplicative $(\theta,\theta)-$derivation of N. A map $f : N \to N$ is called a right multiplicative generalized $(\theta,\theta)-$derivation associated with d if*

$$f(xy) = f(x)\theta(y) + \theta(x)d(y), \text{ for all } x, y \in N,$$

and $f : N \to N$ is called a left multiplicative generalized $(\theta,\theta)-$derivation associated with d if

$$f(xy) = d(x)\theta(y) + \theta(x)f(y), \text{ for all } x, y \in N.$$

f is said to be a multiplicative generalized $(\theta,\theta)-$derivation associated with d if it is both left and right multiplicative generalized $(\theta,\theta)-$derivation associated with d.

In the present paper, we shall extend the above results for semigroup ideals of semiprime near-rings with a multiplicative generalized $(\theta,\theta)-$derivation of R. Also, we aim to prove some commutativity theorems for semiprime near-rings with multiplicative $(\theta,\theta)-$derivations. Throughout this paper, N will denote a zero symetric right near-ring.

2. Results on multiplicative generalized $(\theta,\theta)-$derivations

Lemma 1. *Let N be a right near-ring, θ an automorphism of N and d a multiplicative $(\theta,\theta)-$derivation of N. Then*

$$z(d(x)\theta(y) + \theta(x)d(y)) = zd(x)\theta(y) + z\theta(x)d(y), \text{ for all } x, y, z \in N.$$

Proof. Given $x, y \in N$, we obtain

$$\begin{aligned} (1) \qquad d(z(xy)) &= d(z)\theta(xy) + \theta(z)d(xy) \\ &= d(z)\theta(x)\theta(y) + \theta(z)(d(x)\theta(y) + \theta(x)d(y)). \end{aligned}$$

On the other hand,

$$\begin{aligned} d\left((zx)\,y\right) &= d\left(zx\right)\theta\left(y\right)+\theta\left(zx\right)d\left(y\right) \\ &= d\left(z\right)\theta\left(x\right)\theta\left(y\right)+\theta\left(z\right)d\left(x\right)\theta\left(y\right)+\theta\left(z\right)\theta\left(x\right)d\left(y\right). \end{aligned} \tag{2}$$

Comparing (1) and (2), we conclude that

$$\theta\left(z\right)\left(d\left(x\right)\theta\left(y\right)+\theta\left(x\right)d\left(y\right)\right)=\theta\left(z\right)d\left(x\right)\theta\left(y\right)+\theta\left(z\right)\theta\left(x\right)d\left(y\right),$$

for all $x,y,z\in N$. Since θ is an automorphism of N, we can write this equation as

$$z\left(d\left(x\right)\theta\left(y\right)+\theta\left(x\right)d\left(y\right)\right)=zd\left(x\right)\theta\left(y\right)+z\theta\left(x\right)d\left(y\right),\ \text{for all } x,y,z\in N.$$

■

Theorem 1. *Let N be a semiprime near-ring, (f,d) a right multiplicative generalized $(\theta,\theta)-$derivation of N, I a semigroup ideal of N, θ an automorphism of N, and $\theta d=d\theta$. If $f\left(uv\right)=\pm\theta\left(uv\right)$, for all $u,v\in I$, then $d(N)\in C_N(I)$ and d is a commuting on I.*

Proof. By the hypothesis, we have

$$\pm\theta\left(uv\right)+f\left(uv\right)=0,\ \text{for all } u,v\in I. \tag{3}$$

Replacing v by vw, $w\in I$ in (3), we obtain that

$$\pm\theta\left(uvw\right)+f(uv)\theta\left(w\right)+\theta\left(uv\right)d(w)=0,\ \text{for all } u,v,w\in I.$$

That is,

$$\left\{\pm\theta\left(uv\right)+f(uv)\right\}\theta\left(w\right)+\theta\left(uv\right)d(w)=0,\ \text{for all } u,v,w\in I.$$

Applying equation (3), we get

$$\theta\left(uv\right)d(w)=0,\ \text{for all } u,v,w\in I.$$

Replacing v by $\theta^{-1}\left(d(w)x\right)u$, $x\in N$ in the above equation, we find that

$$\theta\left(u\right)d(w)x\theta\left(u\right)d(w)=0,\ \text{for all } u,w\in I,\ x\in N,$$

and so

$$\theta\left(u\right)d(w)N\theta\left(u\right)d(w)=0,\ \text{for all } u,w\in I.$$

By the semiprimeness of N, we have

$$\theta\left(u\right)d(w)=0,\ \text{for all } u,w\in I. \tag{4}$$

Writing wx, $x \in N$ instead of w in this relation, we get

$$\theta(u)(d(w)\theta(x) + \theta(w)d(x)) = 0, \text{ for all } u, w \in I, \ x \in N.$$

Using Lemma 1 and equation (4), we conclude that

$$\theta(uw)d(x) = 0, \text{ for all } x \in N \text{ and } u, w \in I.$$

Replacing w by $\theta^{-1}(d(x)y)u$, $y \in N$ in the above equation, we arrive at

$$\theta(u)d(x)y\theta(u)d(x) = 0, \text{ for all } x, y \in N \text{ and } u \in I.$$

This implies that

$$\theta(u)d(x)N\theta(u)d(x) = 0, \text{ for all } x \in N \text{ and } u \in I.$$

By the semiprimeness of N, we get

$$\theta(u)d(x) = 0, \text{ for all } x \in N \text{ and } u \in I,$$

and so

$$u\theta^{-1}(d(x)) = 0, \text{ for all } x \in N \text{ and } u \in I.$$

Using $d\theta = \theta d$, we arrive at

$$ud(x) = 0, \text{ for all } x \in N \text{ and } u \in I. \tag{5}$$

Writing u by $d(x)uy$ in (5), we have

$$d(x)uyd(x) = 0, \text{ for all } x, y \in N \text{ and } u \in I. \tag{6}$$

Multiplying (6) on the right by u, we see that

$$d(x)uNd(x)u = 0, \text{ for all } x \in N \text{ and } u \in I.$$

Since N is a semiprime near-ring, we obtain that

$$d(x)u = 0, \text{ for all } x \in N \text{ and } u \in I. \tag{7}$$

Subtracting (5) from (7), we arrive at $[d(x), I] = 0$, for all $x \in N$. That is, $d(N) \in C_N(I)$. In particular, $[d(u), u] = 0$, for all $u \in I$. Moreover, d is commuting on I. This completes the proof. ∎

Theorem 2. *Let N be a semiprime near-ring, (f, d) a right multiplicative generalized $(\theta, \theta)-$derivation of N, I a semigroup ideal of N and θ an automorphism of N. If $f(uv) = \pm\theta(vu)$, for all $u, v \in I$, then d is $(\theta, \theta)-$commuting on I.*

Proof. First we assume that

$$\pm\theta\left(vu\right)+f\left(uv\right)=0,\ \text{for all } u,v\in I.$$

Subsitituting v by vu in the hypothesis, we arrive at

$$\begin{aligned} 0 &= \pm\theta\left(vu^2\right)+f\left(uvu\right)\\ &= \pm\theta(vu)\theta(u)+f(uv)\theta(u)+\theta(uv)d(u)\\ &= \left(\pm\theta(vu)+f(uv)\right)\theta(u)+\theta(uv)d(u). \end{aligned}$$

Again, using the hypothesis, we find that

$$\theta(u)\theta(v)d(u)=0,\ \text{for all } u,v\in I. \tag{8}$$

Replacing v by $\theta^{-1}\left(d(u)x\right)u$, $x\in N$ in the above equation, we get

$$\theta(u)d(u)x\theta(u)d(u)=0,\ \text{for all } u\in I,\ x\in N,$$

and so,

$$\theta(u)d(u)N\theta(u)d(u)=0,\ \text{for all } u\in I,\ x\in N,$$

Since N is a semiprime near-ring, we have

$$\theta(u)d(u)=0,\ \text{for all } u\in I. \tag{9}$$

On the other hand, multiplying the (8) on the left by $d(u)$ and on the right by $\theta(u)$, we see that

$$d(u)\theta(u)\theta(v)d(u)\theta(u)=0,\ \text{for all } u,v\in I.$$

Substituting $v\theta^{-1}(x)$, $x\in N$ for v in the last equation, we get

$$d(u)\theta(u)\theta(v)Nd(u)\theta(u)=0,\ \text{for all } u,v\in I.$$

Again, multiplying the last equation on the right by $\theta(v)$, we have

$$d(u)\theta(u)\theta(v)Nd(u)\theta(u)\theta(v)=0,\ \text{for all } u,v\in I.$$

Since N is a semiprime near-ring, we obtain that

$$d(u)\theta(u)\theta(v)=0,\ \text{for all } u,v\in I.$$

Replacing v by $\theta^{-1}(xd(u))u$ in the above equation, the last expression forces that

$$d(u)\theta(u)Nd(u)\theta(u)=0,\ \text{for all } u\in I.$$

Again, by the semiprimeness, we have

$$d(u)\theta(u)=0,\ \text{for all } u\in I. \tag{10}$$

Comparing (9) from (10), we arrive at $d(u)\theta(u)-\theta(u)d(u)=0$, for all $u\in N$. Therefore, d is $(\theta,\theta)-$commuting on I. This completes the proof. ∎

Theorem 3. *Let N be a semiprime near-ring, (f,d) a left multiplicative generalized $(\theta,\theta)-$derivation of N, I a semigroup ideal of N, θ an automorphism of N, and $\theta d = d\theta$. If $f(u)f(v) = \pm\theta(uv)$ for all $u,v \in I$, then $d(N) \in C_N(I)$ and d is commuting on I.*

Proof. Assume that

$$f(u)f(v) = \pm\theta(uv) \text{ for all } u,v \in I. \tag{11}$$

Substituting u by uw in (11), we obtain that

$$(d(u)\theta(w) + \theta(u)f(w))f(v) = \pm\theta(uwv), \text{ for all } u,v,w \in I,$$

and so,

$$d(u)\theta(w)\,f(v) + \theta(u)f(w)\,f(v) = \pm\theta(uwv), \text{ for all } u,v,w \in I.$$

By the hypothesis, we get

$$d(u)\theta(w)\,f(v) \pm \theta(uwv) = \pm\theta(uwv), \text{ for all } u,v,w \in I.$$

It follows that

$$d(u)\theta(w)\,f(v) = 0, \text{ for all } u,v,w \in I.$$

Multiplying the last equation on the right by $f(k)$, $k \in I$, we have

$$d(u)\theta(w)\,f(v)f(k) = 0, \text{ for all } u,v,w,k \in I.$$

Applying the hypothesis in the above equation yields that

$$d(u)\theta(wvk) = 0, \text{ for all } k,u,v,w \in I.$$

Putting k by $\theta^{-1}(xd(u))wu$, $x \in N$ in the last equation, we get

$$d(u)\theta(wv)xd(u)\theta(wv) = 0, \text{ for all } u,v,w \in I, \ x \in N,$$

and so

$$d(u)\theta(wv)Nd(u)\theta(wv) = 0, \text{ for all } u,v,w \in I.$$

Since N is a semiprime, we find that

$$d(u)\theta(wv) = 0, \text{ for all } u,v,w \in I.$$

Replacing v by $\theta^{-1}(xd(u))w$, $x \in N$ in the last equation, we conclude that

$$d(u)\theta(w)Nd(u)\theta(w) = 0, \text{ for all } u,w \in I.$$

By the semiprimeness, we get

$$d(u)\theta(w) = 0, \text{ for all } u, w \in I. \tag{12}$$

Writing xu, $x \in N$ instead of u in this relation, we get

$$(d(x)\theta(u) + \theta(x)d(u))\theta(w) = 0, \text{ for all } u, w \in I, \ x \in N.$$

Since N is right near-ring, then using equation (12) we find that

$$d(x)\theta(uw) = 0, \text{ for all } x \in N \text{ and } u, w \in I.$$

Subsitituting w by $\theta^{-1}(yd(x))u$, $y \in N$ in the above equation, we arrive at

$$d(x)\theta(u)yd(x)\theta(u) = 0, \text{ for all } x, y \in N \text{ and } u \in I.$$

This implies that

$$d(x)\theta(u)Nd(x)\theta(u) = 0, \text{ for all } x \in N \text{ and } u \in I.$$

Since N is a semiprime near-ring, we get

$$d(x)\theta(u) = 0, \text{ for all } x \in N \text{ and } u \in I,$$

and so

$$\theta^{-1}(d(x))u = 0, \text{ for all } x \in N \text{ and } u \in I.$$

Using $d\theta = \theta d$, we find that

$$d(x)u = 0, \text{ for all } x \in N \text{ and } u \in I. \tag{13}$$

Substituting u by $yud(x)$ in the above equation, we have

$$d(x)yud(x) = 0, \text{ for all } x, y \in N \text{ and } u \in I.$$

Multiplying equation (6) on the left for u, we see that

$$ud(x)yud(x) = 0, \text{ for all } x \in N \text{ and } u \in I.$$

Since N is a semiprime near-ring, we obtain that

$$ud(x) = 0, \text{ for all } x \in N \text{ and } u \in I. \tag{14}$$

Comparing (13) from (14), we arrive at $[d(x), I] = 0$, for all $x \in N$. Thus, $d(N) \in C_N(I)$. Also, d is commuting on I. This completes the proof. ■

Theorem 4. *Let N be a semiprime near-ring, (f,d) a left multiplicative generalized $(\theta,\theta)-$derivation of N, θ an automorphism of N, and I be a semigroup ideal of N. If $f(u)f(v)=\pm\theta(vu)$, for all $u,v\in I$, then d is a $(\theta,\theta)-$commuting map on I.*

Proof. Our hypothesis is

$$f(u)f(v)=\pm\theta(vu) \text{ for all } u,v\in I.$$

Replacing u by vu, we obtain that

$$(d(v)\theta(u)+\theta(v)f(u))f(v)=\pm\theta\left(v^2u\right) \text{ for all } u,v\in I,$$

and so

$$d(v)\theta(u)f(v)+\theta(v)f(u)f(v)=\pm\theta\left(v^2u\right) \text{ for all } u,v\in I.$$

Using the hypothesis, we get

$$d(v)\theta(u)f(v)\pm\theta\left(v^2u\right)=\pm\theta\left(v^2u\right) \text{ for all } u,v\in I.$$

That is,

$$d(v)\theta(u)f(v)=0, \text{ for all } u,v,w\in I.$$

Multiplying the last equation from the right by $f(w)$, $w\in I$, we have

$$d(v)\theta(u)f(v)f(w)=0, \text{ for all } u,v,w\in I.$$

Applying the hypothesis in the above equation, we obtain that

$$d(v)\theta(uwv)=0, \text{ for all } u,v,w\in I.$$

Replacing w by $v\theta^{-1}(xd(v))u$, $x\in N$ in the last equation, we get

$$d(v)\theta(uv)xd(v)\theta(uv)=0, \text{ for all } u,v\in I,\ x\in N,$$

and so

$$d(v)\theta(uv)Nd(v)\theta(uv)=0, \text{ for all } u,v\in I.$$

Since N is a semiprime, we find that

$$d(v)\theta(uv)=0, \text{ for all } u,v\in I. \tag{15}$$

Substituting u by $v\theta^{-1}(xd(v))$, $x\in N$ in the last equation, we conclude that

$$d(v)\theta(v)xd(v)\theta(v)=0, \text{ for all } v\in I.$$

The semiprimeness of N forces that

$$d(v)\theta(v) = 0, \text{ for all } v \in I. \tag{16}$$

On the other hand, multiplying the (15) on the right by $d(v)$ and on the left by $\theta(v)$, we have

$$\theta(v)d(v)\theta(uv)d(v) = 0, \text{ for all } u, w \in I.$$

Replacing u by $u\theta^{-1}(x)$, $x \in N$ in the last equation, we get

$$\theta(v)d(v)\theta(u)x\theta(v)d(v) = 0, \text{ for all } u, v \in I, \ x \in N.$$

Again, multiplying the last equation on the right by $\theta(u)$, we have

$$\theta(v)d(v)\theta(u)x\theta(v)d(v)\theta(u) = 0, \text{ for all } u, v \in I, \ x \in N.$$

Since N is a semiprime near-ring, we get

$$\theta(v)d(v)\theta(u) = 0, \text{ for all } u, v \in I.$$

Substituting u by $\theta^{-1}(x)v\theta^{-1}(d(v))$ in the above expression yields that

$$\theta(v)d(v)x\theta(v)d(v) = 0, \text{ for all } v \in I, \ x \in N.$$

Again, by the semiprimeness of N, we have

$$\theta(v)d(v) = 0, \text{ for all } v \in I. \tag{17}$$

Subtracting (16) from (17), we arrive at $d(v)\theta(v) - \theta(v)d(v) = 0$, for all $v \in N$. That is, d is $(\theta, \theta)-$commuting on I. This completes the proof. ■

Remark 1. Each of the above theorems yields on obvious result for (θ, θ) $-$derivations.

3. Results on multiplicative (θ, θ) $-$derivations

Theorem 5. *Let N be a semiprime near-ring, d a multiplicative $(\theta, \theta)-$derivation of N, I a semigroup ideal of N, θ an automorphism of N, and $\theta d = d\theta$. If $d(u)\,d(v) = \pm\theta([u, v])$ for all $u, v \in I$, then $[I, I] = (0)$, $d(N) \in C_N(I)$, and d is commuting on I.*

Proof. Assume that

$$d(u)\, d(v) = \pm\theta([u,v]), \text{ for all } u,v \in I. \tag{18}$$

Replacing v by vu in (18), we obtain that

$$d(u)\,(d(v)\,\theta(u) + \theta(v)\, d(u)) = \pm\theta([u,v])\,\theta(u), \text{ for all } u,v \in I.$$

By Lemma 1, we have

$$d(u)\, d(v)\,\theta(u) + d(u)\,\theta(v)\, d(u) = \pm\theta([u,v])\,\theta(u), \text{ for all } u,v \in I.$$

Using equation (18), we find that

$$d(u)\,\theta(v)\, d(u) = 0, \text{ for all } u,v \in I. \tag{19}$$

Substituting v by $v\theta^{-1}(x)$, $x \in N$, in the last equation and multiplying this equation on the right by $\theta(v)$, we get

$$d(u)\,\theta(v)\, xd(u)\,\theta(v) = 0, \text{ for all } u,v \in I,\ x \in N.$$

Since N is a semiprime near-ring, we get

$$d(u)\,\theta(v) = 0, \text{ for all } u,v \in I. \tag{20}$$

Now, multipliying the hypothesis on the right by $\theta(w)$, $w \in I$, we have

$$d(u)\, d(v)\,\theta(w) = \theta([u,v])\,\theta(w), \text{ for all } u,v,w \in I.$$

Using equation (20), we obtain that

$$\theta([u,v]\, w) = 0, \text{ for all } u,v,w \in I.$$

Since θ is an automorphism of N, we get

$$[u,v]\, w = 0, \text{ for all } u,v,w \in I,$$

and so

$$[u,v]N[u,v] = (0), \text{ for all } u,v \in I.$$

Again, since N is a semiprime near-ring, we have $[u,v] = 0$, for all $u,v \in I$, that is, $[I,I] = (0)$.

On the other hand, writing $\theta^{-1}(x)v$, $x \in N$ instead of v in (19) and after multipliying this equation from the left by $\theta(v)$, we conclude that

$$\theta(v)d(u)N\theta(v)d(u) = 0, \text{ for all } u,v \in I.$$

By the semiprimeness, we get

$$\theta(v)d(u) = 0, \text{ for all } u, v \in I.$$

Using a similar approach with necessary variations after the equation (4) in the proof of Theorem 1, we can prove that $[d(x), I] = 0$, for all $x \in N$. That is, $d(N) \in C_N(I)$. Moreover, d is commuting on I. This completes the proof. ■

Theorem 6. *Let N be a semiprime near-ring, d a multiplicative $(\theta, \theta)-$derivation of N, I a semigroup ideal of N, θ an automorphism of N, and $\theta d = d\theta$. If $d(u)\, d(v) = \pm\theta(uov)$ for all $u, v \in I$, then $IoI = (0)$, $d(N) \in C_N(I)$ and d is a commuting on I.*

Proof. Substituting vu for v in the hypothesis, we obtain that

$$d(u)\,(d(v)\,\theta(u) + \theta(v)\,d(u)) = \pm\theta(uov)\,\theta(u)\,, \text{ for all } u, v \in I.$$

Application of Lemma 1, gives

$$d(u)\,d(v)\,\theta(u) + d(u)\,\theta(v)\,d(u) = \pm\theta(uov)\,\theta(u)\,, \text{ for all } u, v \in I.$$

Using equation (18), we find that

$$\pm\theta(uov)\,\theta(u) + d(u)\,\theta(v)\,d(u) = \pm\theta(uov)\,\theta(u)\,, \text{ for all } u, v \in I.$$

This implies that

$$d(u)\,\theta(v)\,d(u) = 0, \text{ for all } u, v \in I. \tag{21}$$

Replacing v by $v\theta^{-1}(x)$, $x \in N$ in the above expression and multiplying this equation for the right by $\theta(v)$, we find that

$$d(u)\,\theta(v)\,xd(u)\,\theta(v) = 0, \text{ for all } u, v \in I, \ x \in N.$$

Since N is a semiprime near-ring, we get

$$d(u)\,\theta(v) = 0, \text{ for all } u, v \in I. \tag{22}$$

On the other hand, multiplying the hypothesis on the right by $\theta(w)$, $w \in I$, we have

$$d(u)\,d(v)\,\theta(w) = \theta(uov)\,\theta(w), \text{ for all } u, v, w \in I.$$

Using equation (22), we find that

$$\theta((uov)\,w) = 0, \text{ for all } u, v, w \in I.$$

Since θ is an automorphism of N, we get

$$(uov)\,w = 0, \text{ for all } u, v, w \in I,$$

and so

$$(uov)\,N\,(uov) = (0), \text{ for all } u, v \in I.$$

Since N is a semiprime near-ring, we get $uov = 0$, for all $u, v \in I$, that is, $IoI = (0)$.

Now, replacing v by $\theta^{-1}(x)v$, $x \in N$ in (21) and multiplying the above equation on the left by $\theta\,(v)$, we conclude that

$$\theta(v)d(u)N\theta(v)d(u) = 0, \text{ for all } u, v \in I.$$

By the semiprimeness of N, we get

$$\theta(v)d(u) = 0, \text{ for all } u, v \in I.$$

This equation is the same as (4) in the proof of Theorem 1. By the same arguments, we conclude that $[d(x), I] = 0$, for all $x \in N$. That is $d(N) \in C_N(I)$. Moreover, d is commuting on I. This completes the proof. ∎

Theorem 7. *Let N be a semiprime near-ring, d, h be two multiplicative $(\theta,\theta)-$derivations of N, I a semigroup ideal of N, θ an automorphism of N, and $d\theta = \theta d$, $h\theta = \theta h$. If $d\,(u)\,\theta\,(v) = \theta\,(u)\,h\,(v)$, for all $u, v \in I$, then $d(N), h(N) \in C_N(I)$, and d, h are commuting maps on I.*

Proof. By the assumption

$$d\,(u)\,\theta\,(v) = \theta\,(u)\,h\,(v)\,, \text{ for all } u, v \in I. \tag{23}$$

Replacing v by vw, $w \in I$ in (23), we arrive at

$$d\,(u)\,\theta\,(v)\,\theta\,(w) = \theta\,(u)\,(h\,(v)\,\theta\,(w) + \theta\,(v)\,h\,(w))\,, \text{ for all } u, v, w \in I.$$

By Lemma 1, we have

$$d\,(u)\,\theta\,(v)\,\theta\,(w) = \theta\,(u)\,h\,(v)\,\theta\,(w) + \theta\,(u)\,\theta\,(v)\,h\,(w)\,, \text{ for all } u, v, w \in I.$$

Using equation (23), we find that

$$\theta\,(u)\,h\,(v)\,\theta\,(w) = \theta\,(u)\,h\,(v)\,\theta\,(w) + \theta\,(u)\,\theta\,(v)\,h\,(w)\,, \text{ for all } u, v, w \in I.$$

That is,

$$\theta\,(u)\,\theta\,(v)\,h\,(w) = 0, \text{ for all } u, v, w \in I.$$

Replacing v by $\theta^{-1}(h(w)x)\,u$, $x \in N$ in the above equation, we find that

$$\theta(u)\,h(w)x\theta(u)\,h(w) = 0, \text{ for all } u, w \in I, \ x \in N,$$

and so

$$\theta(u)\,d(w)N\theta(u)\,d(w) = 0, \text{ for all } u, w \in I.$$

Since N is a semiprime near-ring, we have

$$\theta(u)\,h(w) = 0, \text{ for all } u, w \in I. \tag{24}$$

Using the same arguments after the equation (4) in and the proof of Theorem 1, we get $[h(x), I] = 0$, for all $x \in N$. Thus, $h(N) \in C_N(I)$ and h is commuting on I.

Moreover, from the equation (24), we get $\theta(u)\,h(v) = 0$, for all $u, v \in I$. Thus, $d(u)\,\theta(v) = 0$, for all $u, v \in I$ by the hypothesis. By the same argument after the equation (12) in the proof of Theorem 3, we get $[d(x), I] = 0$, for all $x \in N$. That is, $d(N) \in C_N(I)$ and d is commuting on I. This completes the proof. ■

Corollary 1. *Let N be a semiprime near-ring, d be a multiplicative $(\theta, \theta)-$ derivation of N, I a semigroup ideal of N, θ an automorphism of N, and $d\theta = \theta d$. If $d(u)\,\theta(v) = \theta(u)\,d(v)$, for all $u, v \in I$, then $d(N) \in C_N(I)$ and d is a commuting on I.*

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