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Kuldip Raj, Charu Sharma and Suruchi Pandoh

MULTIPLICATION OPERATORS ON CESÀRO-ORLICZ SEQUENCE SPACES

Abstract. In this paper, we characterize the compact, invertible, Fredholm and closed range multiplication operators on Cesàro-Orlicz sequence spaces.

Key words: multiplication operator, Fredholm multiplication operator, invertible operator, compact operator, isometry, Cesàro sequence space, Cesàro-Orlicz sequence space.

AMS Mathematics Subject Classification: 47B38, 46A06.

1. Introduction and preliminaries

Cesàro sequence spaces Ces_p, $1 \leq p < \infty$, appeared for the first time in 1968 as a problem of finding their duals [1]. Some basic properties of these spaces were studied in the early seventies by Shiue [23] and Leibowitz [13]. In 1974 Jagers [9] found the dual space of Ces_p (see also [12]). In the late nineties mathematicians became interested in geometric properties of these spaces. Cui and Płuciennik studied Local Uniform Nonsquareness [6] and Banach-Saks Property and Property β [7], Cui and Hudzik studied Fixed Point Property [5] and obtained the Packing Constant [3].

An Orlicz function $\varphi : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. An Orlicz function is said to satisfy the δ_2- condition if their exists $K > 0$ such that $\varphi(Lx) \leq KL\varphi(x)$, for all $x \geq 0$ and for $L > 1$. By w we shall denote the space of all complex sequences. The Cesàro-Orlicz sequence space $Ces_\varphi(\mathbb{N})$ is defined as

$$Ces_\varphi(\mathbb{N}) = \Big\{x = (x_k) \in w : \sum_{m=1}^{\infty} \varphi\Big(\frac{1}{m}\sum_{k=1}^{m} |\lambda x_k|\Big) < \infty\Big\}.$$

The space $Ces_\varphi(\mathbb{N})$ is a Banach space under the norm

$$\|x\| = \inf\Big\{\lambda > 0 : \sum_{m=1}^{\infty} \varphi\Big(\frac{\frac{1}{m}\sum_{k=1}^{m}|x_k|}{\lambda}\Big) \leq 1\Big\} \text{ (see [17]).}$$

The Cesàro-Orlicz sequence spaces $Ces_\varphi(\mathbb{N})$ appeared for the first time in 1988, when Lim and Lee [11] found their dual spaces. Recently, Cui et al. [4] obtained important properties of the spaces. In 2007 Maligranda et al. [14] showed that $Ces_\varphi(\mathbb{N})$ is not B-convex, if $\varphi \in \delta_2$ and $Ces_\varphi(\mathbb{N}) \neq \{0\}$. The extreme points and strong U-points of $Ces_\varphi(\mathbb{N})$ have been characterized by Foralewski et al. in [8]. Although the spaces $Ces_\varphi(\mathbb{N})$ have been studied by several mathematicians, some essential and basic properties remain still unknown but recently some of its properties have been discussed by Damian in [10].

In the case when $\varphi(x) = |x|^p, (p > 1)$ the Cesàro-Orlicz sequence space $Ces_\varphi(\mathbb{N})$ becomes the Cesàro sequence space Ces_p.

Let $u : \mathbb{N} \to \mathbb{C}$ be a function such that $u.f \in Ces_\varphi(\mathbb{N})$ for every $f \in Ces_\varphi(\mathbb{N})$, then we can define a multiplication transformation $M_u : Ces_\varphi(\mathbb{N}) \to Ces_\varphi(\mathbb{N})$ by

$$M_u f = uf\ , \quad \forall\, f \in Ces_\varphi(\mathbb{N}).$$

If M_u is continuous, we call it a multiplication operator induced by u. These operators received considerable attention over the past several decades especially on L^p-spaces and Bergman spaces. From the recent literature available in Operator theory we find that multiplication operators are very much intimately connected with the composition operators as most of the properties of composition operators on L^p-spaces can be stated in terms of properties of multiplication operators. For example Singh and Kumar [21] proved that a composition operator on $L^p(X, \mathbb{C})$ is compact if and only if the multiplication operator M_u is compact, where $u = \frac{d\mu T^{-1}}{d\mu}$, the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ. Infact the multiplication operators play an important role in the theory of Hilbert space operators. One of the main application is that every normal operator on a separable Hilbert space is unitarily equivalent to a multiplication operator. Moreover, multiplication operators has its roots in the spectral theory and is being pursued today in such guises as the theory of subnormal operators and the theory of Toeplitz operators. For more details on multiplication operators we refer to ([2], [18], [19], [20], [22], [24]) and refrences therein. Moreover, Compact operators on sequence spaces have recently been studied by Mursaleen and Noman in ([15], [16]). By $B(Ces_\varphi(\mathbb{N}))$ we denote the set of all bounded linear operators from $Ces_\varphi(\mathbb{N})$ into itself.

A bounded linear operator $A : E \to E$ (where E is a Banach space) is called compact if $A(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E.

A bounded linear operator $A : E \to E$ is called Fredholm if A has closed range, dim(ker$A$) and co-dim(ran$A$) are finite. The main purpose of this paper is to characterize the boundedness, compactness, closed range and Fredholmness of multiplication operators on Cesàro-Orlicz sequence spaces.

2. Multiplication operators

Theorem 1. *Let $u : \mathbb{N} \to \mathbb{C}$ be a mapping. Then $M_u : Ces_\varphi(\mathbb{N}) \to Ces_\varphi(\mathbb{N})$ is bounded if and only if u is a bounded function.*

Proof. Suppose u is a bounded function. Then there exists $M > 0$ such that $|u_n| \le M$, $\forall\, n \in \mathbb{N}$. Let $x \in Ces_\varphi(\mathbb{N})$, we have

$$\begin{aligned}
\|M_u x\| &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |(ux)_k|}{\lambda}\right) \\
&= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u_k||x_k|}{\lambda}\right) \\
&\le M \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |x_k|}{\lambda}\right) \\
&= M\|x\|.
\end{aligned}$$

Thus, $\|M_u x\| \le M\|x\|$, $\forall\, x \in Ces_\varphi(\mathbb{N})$ which implies that M_u is a bounded operator.

Conversely, suppose that M_u is a bounded operator. We show that u is a bounded function. For, if u is not a bounded function, then for every $n \in \mathbb{N}$, there exists some $p_n \in \mathbb{N}$ such that $|u(p_n)| > n$. Now $\|e^{p_n}\| = \displaystyle\sum_{m=p_n}^{\infty} \frac{1}{m\lambda\varphi^{-1}(1)}$ and

$$\begin{aligned}
\|M_u e^{p_n}\| &= \left(\sum_{m=p_n}^{\infty} \frac{|u_m|}{m\lambda\varphi^{-1}(1)}\right) \\
&> n\left(\sum_{m=p_n}^{\infty} \frac{1}{m\lambda\varphi^{-1}(1)}\right) \\
&= n\|e^{p_n}\|.
\end{aligned}$$

This shows that M_u is not a bounded operator. Hence, u must be a bounded function. ■

Theorem 2. *M_u is an isometry if and only if $|u_n| = 1$, $\forall\, n \in \mathbb{N}$.*

Proof. Suppose first that $|u_n| = 1$, $\forall\, n \in \mathbb{N}$. Then

$$\begin{aligned}\|M_u x\| &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m}|u_k x_k|}{\lambda}\right)\\ &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m}|x_k|}{\lambda}\right)\\ &= \|x\|.\end{aligned}$$

Therefore, $\|M_u x\| = \|x\|, \forall\, x \in Ces_{\varphi}(\mathbb{N})$ and hence M_u is an isometry.

Conversely, suppose that $|u_n| \neq 1$ for some $n = n_0$. Then $\|e^{n_0}\| = \sum_{m=n_0}^{\infty} \frac{1}{m\lambda\varphi^{-1}(1)}$. Suppose $|u_{n_0}| > 1$. Then

$$\begin{aligned}\|M_u e^{n_0}\| &= \left(\sum_{m=n_0}^{\infty} \frac{|u_{n_0}|}{m\lambda\varphi^{-1}(1)}\right)\\ &> \sum_{m=n_0}^{\infty} \frac{1}{m\lambda\varphi^{-1}(1)}\\ &= \|e^{n_0}\|.\end{aligned}$$

Similarly if $|u_{n_0}| < 1$, then we can show that $\|M_u e^{n_0}\| < \|e^{n_0}\|$. In both cases, when $|u_{n_0}| < 1$ or $|u_{n_0}| > 1$, we get contradiction. Hence, $|u_n| = 1\, \forall\, n \in \mathbb{N}$. ■

3. Compact multiplication operators

Theorem 3. *Let $M_u \in B(Ces_{\varphi}(\mathbb{N}))$. Then M_u is a compact operator if and only if $u(n) \to 0$ as $n \to \infty$.*

Proof. We first assume that M_u is a compact operator. We show that $u(n) \to 0$ as $n \to \infty$. For if this were not true, then there exists $\epsilon > 0$ such that the set $A_\epsilon = \{k \in \mathbb{N} : |u_k| \geq \epsilon\}$ is an infinite set. Let $p_1, p_2, ..., p_n, ..$ be in A_ϵ. Then $\{e^{p_n} : p_n \in A_\epsilon\}$ is an infinite bounded set in $Ces_{\varphi}(\mathbb{N})$. Consider

$$\begin{aligned}\|M_u e^{p_n} - M_u e^{p_s}\| &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m}|u(k)e^{p_n}(k) - u(k)e^{p_s}(k)|}{\lambda}\right)\\ &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m}|u(k)||e^{p_n}(k) - e^{p_s}(k)|}{\lambda}\right)\\ &\geq \epsilon\|e^{p_n} - e^{p_s}\|, \forall\, p_n, p_s \in A_\epsilon.\end{aligned}$$

This proves that $\{e^{p_n} : p_n \in A_\epsilon\}$ is a bounded sequence which cannot have a convergent subsequence under M_u. This shows that M_u cannot be a compact operator, which is a contradiction. Hence, $u(n) \to 0$ as $n \to \infty$.
Conversely, suppose $u(n) \to 0$ as $n \to \infty$. Then for every $\epsilon > 0$, the set $A_\epsilon = \{n \in \mathbb{N} : |u(n)| \geq \epsilon\}$ is a finite set. Then $Ces_\varphi(A_\epsilon)$ is a finite dimensional space for each $\epsilon > 0$. Therefore, $M_u|Ces_\varphi(A_\epsilon)$ is a compact operator. For each $n \in \mathbb{N}$, define $u_n : \mathbb{N} \to \mathbb{C}$ by

$$u_n(m) = \begin{cases} u(m), & \forall m \in A_{\frac{1}{n}} \\ 0, & \forall m \notin A_{\frac{1}{n}}. \end{cases}$$

Clearly, M_{u_n} is a compact operator as the space $Ces_\varphi(A_{\frac{1}{n}})$ is finite dimensional for each $n \in \mathbb{N}$. Now

$$\begin{aligned} \|(M_{u_n} - M_u)x\| &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u_n(k)x_k - u(k)x_k|}{\lambda}\right) \\ &= \sum_{m \in A_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u_n(k)x_k - u(k)x_k|}{\lambda}\right) \\ &\quad + \sum_{m \in A'_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u_n(k)x_k - u(k)x_k|}{\lambda}\right) \\ &= \sum_{m \in A'_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u(k)x_k|}{\lambda}\right) \\ &< \frac{1}{n} \sum_{m \in A'_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |x_k|}{\lambda}\right) \\ &\leq \frac{1}{n}\|x\| \end{aligned}$$

or $\|(M_{u_n} - M_u)(x)\| \leq \frac{1}{n}\|x\|$. This proves that $\|(M_{u_n} - M_u)\| \leq \frac{1}{n}$ and that M_u is a limit of compact operators and hence, M_u is a compact operator. ■

Theorem 4. *Let $M_u \in B(Ces_\varphi(\mathbb{N}))$. Then M_u has closed range if and only if u is bounded away from zero on $\mathbb{N} \setminus keru = S$.*

Proof. Suppose u is bounded away from zero on S. Then there exists $\epsilon > 0$ such that $|u_n| \geq \epsilon \ \forall \ n \in \mathbb{N} \setminus keru$. We have to prove that $ranM_u$

is closed. Let z be a limit point of $ranM_u$. Then there exists a sequence $M_u x^n \to z$. Clearly, the sequence $\{M_u x^n\}$ is a Cauchy sequence. Now,

$$\begin{aligned}
\|M_u x^n - M_u x^m\| &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u_k x_k^n - u_k x_k^m|}{\lambda}\right) \\
&= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{\substack{k=1\\k\in S}}^{m} |u_k||x_k^n - x_k^m|}{\lambda}\right) \\
&\geq \epsilon \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{\substack{k=1\\k\in S}}^{m} |x_k^n - x_k^m|}{\lambda}\right) \\
&= \epsilon \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |\tilde{x_k^n} - \tilde{x_k^m}|}{\lambda}\right) \\
&= \epsilon||\tilde{x^n} - \tilde{x^m}||,
\end{aligned}$$

where

$$\tilde{x_k^n} = \begin{cases} x_k^n, & if \;\; k \in S \\ 0, & if \;\; k \notin S. \end{cases}$$

This proves that $\{\tilde{x_n}\}$ is a Cauchy sequence in $Ces_\varphi(\mathbb{N})$. But $Ces_\varphi(\mathbb{N})$ is complete. Therefore, there exists $x \in Ces_\varphi(\mathbb{N})$ such that $\tilde{x^n} \to x$ as $n \to \infty$. Inview of continuity of M_u, $M_u\tilde{x^n} \to M_u x$. But $M_u x^n = M_u \tilde{x^n} \to z$. Therefore, $M_u x = z$. Hence, $z \in ranM_u$. This proves that M_u has closed range.

Conversely, suppose that M_u has closed range. Then M_u is bounded away from zero on $(kerM_u)^\perp = Ces_\varphi(\mathbb{N} \setminus keru)$. That is, there exists $\epsilon > 0$ such that

$$\|M_u x\| \geq \epsilon \|x\| \; \forall \; x \in Ces_\varphi(\mathbb{N} \setminus keru). \tag{1}$$

Let $G = \{k \in \mathbb{N} \setminus keru : |u_k| < \frac{\epsilon}{2}\}$. If $G \neq \phi$, then for $n_0 \in G$, we have

$$\begin{aligned}
\|M_u e^{n_0}\| &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m} |u(k)e^{n_0}(k)|}{\lambda}\right) \\
&= \sum_{m=n_0}^{\infty} \left(\frac{|u(n_0)|}{m\lambda\varphi^{-1}(1)}\right) \\
&< \epsilon \sum_{m=n_0}^{\infty} \frac{1}{m\lambda\varphi^{-1}(1)} \\
&= \epsilon\|e^{n_0}\|.
\end{aligned}$$

That is, $\|M_u e^{n_0}\| < \epsilon\|e^{n_0}\|$ which contradicts (1). Hence, $G = \phi$ so that $|u_k| \geq \epsilon$, $\forall \; k \in \mathbb{N} \setminus keru$. This proves the theorem. ■

4. Invertible and Fredholm multiplication operators

Theorem 5. *Let* $u : \mathbb{N} \to \mathbb{C}$ *be a mapping. Then* $M_u : Ces_\varphi(\mathbb{N}) \to Ces_\varphi(\mathbb{N})$ *is invertible if and only if there exist* $m > 0$ *and* $M > 0$ *such that* $m < u_n < M$, $\forall\, n \in \mathbb{N}$.

Proof. Suppose that the condition is true. Define $\gamma : \mathbb{N} \to \mathbb{C}$ by $\gamma_n = \frac{1}{u_n}$. Then M_u and M_γ are bounded linear operators in view of Theorem 1. Also $M_u.M_\gamma = M_\gamma.M_u = I$. Hence, M_γ is the inverse of M_u.

Conversely, suppose that M_u is invertible. Then $ranM_u = Ces_\varphi(\mathbb{N})$. Therefore, $ranM_u$ is closed. Hence, by Theorem 4, there exists $\epsilon > 0$ such that $|u_n| \geq \epsilon \; \forall\, n \in \mathbb{N} \setminus keru$. Now $keru = \phi$, otherwise $u_{n_0} = 0$, for some $n_0 \in \mathbb{N}$, in which case $e^{n_0} \in kerM_u$ which is a contradiction, since $kerM_u$ is trivial. Hence, $|u_n| \geq \epsilon \; \forall\, n \in \mathbb{N}$. Since M_u is bounded, so by Theorem 1, there exists $M > 0$ such that $|u_n| \leq M$, $\forall\, n \in \mathbb{N}$. Thus, we have proved that $\epsilon \leq |u_n| \leq M$, $\forall\, n \in \mathbb{N}$. ■

Theorem 6. *Let* $M_u : Ces_\varphi(\mathbb{N}) \to Ces_\varphi(\mathbb{N})$ *be a bounded operator. Then* M_u *is Fredholm operator if and only if*

(i) $keru$ *is a finite subset of* $\mathbb{N}$.

(ii) $|u_n| \geq \epsilon$, $\forall\, n \in \mathbb{N} \setminus keru$.

Proof. Suppose M_u is Fredholm. If $keru$ is an infinite subset of $\mathbb{N}$, then $e^n \in kerM_u \; \forall\, n \in keru$. But e^n's are linearly independent, which shows that $kerM_u$ is an infinite dimensional which is a contradiction. Hence, $keru$ must be a finite subset of $\mathbb{N}$. The condition (ii) follows from Theorem 4.

Conversely, If the conditions (i) and (ii) are true, then we prove that M_u is Fredholm. In view of Theorem 4, the condition (ii) implies that M_u has closed range. The condition (i) implies that $kerM_u$ and $kerM_u^\star$ are finite dimensional. This proves that M_u is Fredholm. ■

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KULDIP RAJ
DEPARTMENT OF MATHEMATICS
SHRI MATA VAISHNO DEVI UNIVERSITY
KATRA-182320, J & K, INDIA
e-mail: kuldipraj68@gmail.com

Charu Sharma
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J & K, India
e-mail: charu145.cs@gmail.com

Suruchi Pandoh
Department of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J & K, India
e-mail: suruchi.pandoh87@gmail.com

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