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ON COEFFICIENT PROBLEMS OF AN OPERATOR WITH RESPECT TO SYMMETRIC POINT

Abstract. In this research work, we study the properties of a certain differential subordination involving an operator with respect to a symmetric point. We establish coefficient estimates as our main results.

Key words: subordination, starlikeness, convexity, symmetric point, coefficient estimates.

AMS Mathematics Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ denote a class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let S be the subclass of $\mathcal{A}$ consisting of analytic univalent function of the form (1.1). Kanas and Ronning[6] introduced an interesting analytic function $A(\omega)$ defined as follows:

$$f(z) = (z - \omega) + \sum_{k=0}^{\infty} a_k (z - \omega)^k \tag{2}$$

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$ and normalized by the condition

$$f(\omega) = 0 \quad \text{and} \quad f'(\omega) = 1$$

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and ω is fix in U. Using (2) the following classes are accordingly defined as in [5]:

$$ST(\omega) = S^*(\omega) = \left\{ f(z) \in S(\omega) : \Re \frac{(z-\omega)f'(z)}{f(z)} > 0, \ z \in U \right\}$$

$$CV(\omega) = S^c(\omega) = \left\{ f(z) \in S(\omega) : 1 + \Re \frac{(z-\omega)f''(z)}{f'(z)} > 0, \ z \in U \right\}$$

known respectively as ω-starlike and ω-convex functions. Many contributors in the like of Darus[4], Acu and Owa [1], Oladipo [7], Olatunji and Oladipo [8] and many others have worked on these classes mostly viewing them with different mathematical angles of interest.

Let $H(\omega) \in S(\omega)$ be of the form (2) which are analytic and normalized as stated above. Let $f(z)$ be defined as in (2) and $f \in H(\omega)$ satisfy

$$\Re \frac{(z-\omega)f'(z)}{f(z)} > 0.$$

Then $f(\omega) \in T^*(\omega)$ where $T^*(\omega)$ is a subfamily of $S^*(\omega)$ and ω is a fixed point U. Let $f(z)$ be defined as stated above and $f \in H(\omega)$ satisfy

$$\Re \left\{ 1 + \frac{(z-\omega)f''(z)}{f'(z)} > 0, \quad z \in U \right\}$$

then $f \in K^c(\omega) \in S^c(\omega)$ and ω is a fixed point in U. These classes are respectively subfamilies of ω-starlike and ω-convex.

2. Definition of terms and preliminaries

The linear operator $T_\lambda^{m,l}(a,c) : \mathcal{A} \to \mathcal{A}$ defined and studied in [2] is stated as follows:

If $f \in \mathcal{A}$ is of the form (1.1)then

$$T_\lambda^{l,m}(a,c)f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^l \left[\frac{(a)_{k-1}}{(c)_{k-1}} \right]^m a_k z^k \tag{3}$$

$(\lambda \geq 0,\ a \in \mathbb{R},\ c \in \mathbb{R}/ \in \mathbb{Z}_\circ^-;\ \mathbb{Z}_\circ^- = \{0, -1, -2, \cdots\};\ m, l \in \mathbb{N}_\circ = \mathbb{N} \cup \{0\})$.
It is easily seen from (3) that

$$T_\lambda^{0,0}(a,c)f(z) = f(z)$$
$$T_\lambda^{0,1}(a,c)f(z) = \ell(a,c)f(z)$$

where $\ell(a,c)f(z)$ is the familiar Carson-Shaffer operator [3]. Our objectives in this work are to establish coefficient estimates of the stated operator with respect to a fix point.

The following definitions are analogue of the definitions defined in [5], [9] and [8], and we shall modified some of them for the purpose of these work.

Definition 1. *Let $T^*_c(\omega)$ be the subclass of S consisting of*

$$\Re\left\{\frac{f'(z)}{f(z)-f(-z)}\right\}>0,\quad z\in U.$$

This is known as the class of ω-starlike with respect to symmetric point. While the following is known as ω-starlike with respect to conjugate point.

$$\Re\left\{\frac{(z-\omega)f'(z)}{f(z)+\overline{f(\overline{z})}}\right\}>0,\quad z\in U.$$

Moreover, we let $K^c_s(\omega)$ be the subclass of $S(\omega)$ consisting function given by (1.2) satisfying the following condition:

$$\Re\left\{\frac{((z-\omega)f'(z))}{(f(z)-f(-z))'}\right\}>0,\quad z\in U.$$

This class is known the class of ω-convex with respect to symmetric point.

In subordination form, Goel and Mehrok [5], Selvaraj and Vasanthi[9] introduced a subclass S^*_s denoted by $S^*_s(A,B)$ and f is of the form (2). Olatunji and Oladipo [8] considered and viewed it in terms of symmetric point. Analogously going by their definitions, we define the following:

Let $T^*_s(\omega,A,B)$ be the class of functions f of the form (3) defined by an operator stated above satisfying the condition

$$\frac{2(z-\omega)T^{l,m}_\lambda(a,c)f'(z)}{T^{l,m}_\lambda(a,c)f(z)-T^{l,m}_\lambda(a,c)f(-z)}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\ \ -1\le B<A\le 1,\ \ z\in U.$$

Let $T^*_c(\omega,A,B)$ be the subclass of functions of the form (3) and satisfying

$$\frac{2\left((z-\omega)T^{l,m}_\lambda(a,c)f'(z)\right)'}{T^{l,m}_\lambda(a,c)f(z)+\overline{T^{l,m}_\lambda(a,c)f(\overline{z})}}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\quad -1\le B<A\le 1,\ \ z\in U.$$

In this paper we introduce the class $\Psi_s(\omega,A,B)$ consisting of analytic function of the form (3) and satisfying the condition

$$\begin{aligned}&\frac{2(z-\omega)T^{l,m}_\lambda(a,c)f'(z)+2\alpha(z-\omega)^2T^{l,m}_\lambda(a,c)f''(z)}{[(1-\alpha)+\alpha(z-\omega)]\left(T^{l,m}_\lambda(a,c)f(z)-T^{l,m}_\lambda(a,c)f(-z)\right)}\\&\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\quad -1\le B<A\le 1,\quad z\in U.\end{aligned}\tag{4}$$

Also we introduce the class $\Psi_c(\omega, A, B)$ consisting of analytic function f of the form (3) satisfying the condition

$$(5)\quad \frac{2(z-\omega)T_\lambda^{l,m}(a,c)f'(z)+2\alpha(z-\omega)^2T_\lambda^{l,m}(a,c)f''(z)}{[(1-\alpha)+\alpha(z-\omega)]\left(T_\lambda^{l,m}(a,c)f(z)-\overline{T_\lambda^{l,m}(a,c)f(\overline{z})}\right)} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, \quad -1 \leq B < A \leq 1, \quad z \in U.$$

Equivalently, the above can be stated in terms of subordination as follows: $f \in \Psi_s(\omega, A, B)$ if and only if

$$(6)\quad \frac{2(z-\omega)T_\lambda^{l,m}(a,c)f'(z)+2\alpha(z-\omega)^2T_\lambda^{l,m}(a,c)f''(z)}{[(1-\alpha)+\alpha(z-\omega)]\left(T_\lambda^{l,m}(a,c)f(z)-T_\lambda^{l,m}(a,c)f(-z)\right)} = \frac{1+Ah(z)}{1+B(h(z)} = p(z)$$

$h \in U$ and h is of the form

$$h(z) = (z-\omega) + \sum_{b=0}^{\infty} b_k(z-\omega)^k$$

$h(\omega) = 0$ and $|h(z)| < 1$, h is analytic and univalent, and that $f \in \Psi_c(\omega, A, B)$ if and only if

$$(7)\quad \frac{2(z-\omega)T_\lambda^{l,m}(a,c)f'(z)+2\alpha(z-\omega)^2T_\lambda^{l,m}(a,c)f''(z)}{[(1-\alpha)+\alpha(z-\omega)]\left(T_\lambda^{l,m}(a,c)f(z)-\overline{T_\lambda^{l,m}(a,c)f(\overline{z})}\right)} = \frac{1+Ah(z)}{1+Bh(z)} = p(z)$$

where $p(z)$ is given here as

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k(z-\omega)^k$$

and

$$(8)\quad |p_k| \leq \frac{(A-B)}{(1+d)(1-d)^k}, \quad k \geq 1, \quad |\omega| = d.$$

In the preceding section we shall study the classes $\Psi_c(\omega, A, B)$ and $\Psi_s(\omega, A, B)$ in which the coefficient estimate for functions f in these classes are obtained.

3. Coefficient estimate

Theorem 1. *Let f as defined in (3) be in the class $\Psi_s(\omega, A, B)$, then for $k = 2, 3, 4, \cdots,\quad 0 \le \alpha \le 1$, we have*

$$|a_2| \le \frac{A-B}{2(1+d)(1-d)(1+\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m} \tag{9}$$

$$|a_3| \le \frac{A-B}{(1+d)(1-d)^2(1-\alpha)^2 2(1+2\lambda)^l(3+\alpha) \left[\frac{(a)_1}{(c)_1}\right]^m} \tag{10}$$

$$|a_4| \le \frac{A-B}{2(1+d)(1-d)(\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m}. \tag{11}$$

Proof. Considering (5) and (8) we have the following:

$$2(z-\omega) + (1+\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m a_2(z-\omega)^2 + (1+2\alpha)6(1+2\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m a_3(z-\omega)^3 + \cdots = \tag{12}$$

$$2(z-\omega) + 2p_1(z-\omega)^2 + p_2(z-\omega)^3 + 4\alpha(1+2\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m a_3(z-\omega)^3 + \cdots . \tag{13}$$

Equating coefficients

$$2p_1 = (1+\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m a_2$$

$$p_2 + 4\alpha(1+2\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m a_3 = (1+2\alpha)6(1+2\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m a_3$$

and applying (8) we have

$$|a_2| \le \frac{A-B}{2(1+d)(1-d)(1+\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m}, \tag{14}$$

$$|a_3| \le \frac{A-B}{(1+d)(1-d)^2(1-\alpha)^2 2(1+2\lambda)^l(3+\alpha) \left[\frac{(a)_1}{(c)_1}\right]^m}, \tag{15}$$

and

$$|a_4| \leq \frac{A-B}{2(1+d)(1-d)(\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m}. \tag{16}$$

■

Suppose we have $d = 0$ in Theorem 1, we have

Corollary 1. *Let f as defined in (3) be in the class $\Psi_s(\omega, A, B)$, then for $k = 2, 3, 4\cdots$, $0 \leq \alpha \leq 1$, we have*

$$|a_2| \leq \frac{A-B}{2(1+\alpha)4(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m} \tag{17}$$

and

$$|a_3| \leq \frac{A-B}{(1-\alpha)^2 2(1+2\lambda)^l(3+\alpha) \left[\frac{(a)_1}{(c)_1}\right]^m}. \tag{18}$$

If we set $\alpha = 0$ in Corollary 1 we have

Corollary 2. *Let f as defined in (3) be in the class $\Psi_s(\omega, A, B)$, then for $k = 2, 3, 4, \cdots$, $0 \leq \alpha \leq 1$, we have*

$$19.|a_2| \leq \frac{A-B}{8(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m} \tag{19}$$

and

$$|a_3| \leq \frac{A-B}{6(1+2\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m}. \tag{20}$$

Next we continue with the following theorem.

Theorem 2. *Let f as defined in (3) be in the class $\Psi_c(\omega, A, B)$, then for $k = 2, 3, 4, \cdots$, $0 \leq \alpha \leq 1$, we have*

$$|a_2| \leq \frac{A-B}{(1+d)(1-\alpha)^2(1+\alpha)(1+\lambda)^l \left[\frac{(a)_1}{(c)_1}\right]^m} \tag{21}$$

and

$$|a_3| \leq \frac{(A-B)^2}{4(1+d)^2(1-\alpha)^4(1+2\alpha)(1+2\lambda)^l \left[\frac{(a)_2}{(c)_2}\right]^m} + \frac{A-B}{(1+d)(1-\alpha)2(1+2\alpha)(1+2\lambda)^l \left[\frac{(a)_2}{(c)_2}\right]^m}. \tag{22}$$

Proof. Considering (7) and (8) we have

$$(z-\omega)+2(1+\alpha)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m a_2(z-\omega)^2 + 3(1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m a_3(z-\omega)^3+\cdots = \tag{23}$$

$$\left\{(z-\omega)+(1+\omega)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m a_2(z-\omega)^2 +(1+\alpha)(1+\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m a_3(z-\omega)^3+\cdots\right\}\times$$

$$\left\{1+p_1(z-\omega)+p_2(z-\omega)^2+p_3(z-\omega)^3+\cdots\right\}. \tag{24}$$

Expanding and equating coefficients we have the following

$$2(1+\alpha)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m a_2 = p_1+(1+\alpha)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m a_2 \tag{25}$$

and

$$3(1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m a_3 = (1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m a_3 + p_1(1+\alpha)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m a_2+p_2. \tag{26}$$

Solving for a_2 we have

$$|a_2|\le\frac{A-B}{(1+d)(1-\alpha)^2(1+\alpha)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m} \tag{27}$$

and

$$|a_3|\ \le\ \frac{(A-B)^2}{4(1+d)^2(1-\alpha)^4(1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m} + \frac{A-B}{(1+d)(1-\alpha)2(1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m}. \tag{28}$$

■

Set $d=0$ in Theorem 2, we have the following:

Corollary 3. *Let f as defined in (3) be in the class $\Psi_c(\omega,A,B)$, then for $k=2,3,4,\cdots$, $0\le\alpha\le1$, we have*

$$|a_2|\le\frac{A-B}{(1-\alpha)^2(1+\alpha)(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m} \tag{29}$$

and

$$|a_3| \leq \frac{(A-B)^2}{4(1-\alpha)^4(1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m} + \frac{A-B}{(1-\alpha)2(1+2\alpha)(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m}. \tag{30}$$

Setting $\alpha = 0$ *in Corollary 1 we have*

Corollary 4. *Let* f *as defined in (2.1) be in the class* $\Psi_c(\omega, A, B)$*, then for* $k = 2, 3, 4, \cdots$*,* $0 \leq \alpha \leq 1$*, we have*

$$|a_2| \leq \frac{A-B}{(1+\lambda)^l\left[\frac{(a)_1}{(c)_1}\right]^m} \tag{31}$$

and

$$|a_3| \leq \frac{(A-B)^2}{4(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m} + \frac{A-B}{2(1+2\lambda)^l\left[\frac{(a)_2}{(c)_2}\right]^m}. \tag{32}$$

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