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Luong Quoc Tuyen

A PARTIAL ANSWER TO A QUESTION OF Y. TANAKA AND Y. GE

Abstract. In this paper, we give a partial answer to the problem posed by Y. Tanaka and Y. Ge in [9].

Key words: Cauchy symmetric; cs^*-network; σ-strong network; σ-point-finite strong cs^*-network; Quotient map.

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1. Introduction and preliminaries

One of the central problems in general topology is to establish relationships between various topological spaces and metric spaces by means of various maps. In 2002, Y. Ikeda, C. Liu and Y. Tanaka introduced the notion of σ-strong networks, and considered certain quotient images of metric spaces in terms of σ-strong networks. By means of σ-strong networks, some characterizations for certain quotient compact images of metric spaces are obtained ([1, 2, 5, 9, 11]). In 2006, Y. Tanaka and Y. Ge gave some characterizations around sequence-covering quotient compact images of metric spaces in terms of symmetric spaces ([9]). Also, the authors posed the following question.

Question. [Question 3.9, [9]] Let X be a symmetric space having a σ-point-finite cs^*-network. Then, is X a space having σ-point-finite strong cs^*-network? ■

In this paper, we give a partial answer to the Question .

We assume that all spaces are T_1 and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X, we denote

$$(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\};$$

$$\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

For a sequence $\{x_n\}$ converging to x, we say that $\{x_n\}$ is *eventually* in P, if there exists $m \in \mathbb{N}$ such that

$$\{x\}\bigcup\{x_n : n \geq m\} \subset P,$$

and $\{x_n\}$ is *frequently* in P, if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1. *Let $\mathcal{P}$ be a family of subsets of a space X.*

(a) *$\mathcal{P}$ is a* network at x in X*, if $x \in P$ for every $P \in \mathcal{P}$, and whenever $x \in U$ with U is open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.*

(b) *$\mathcal{P}$ is a* cs^*-network *for X [9], if for each sequence S converging to a point $x \in U$ with U is open in X, S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.*

(c) *$\mathcal{P}$ is a* cs^*-cover *[12], if every convergent sequence is frequently in some $P \in \mathcal{P}$.*

Definition 2. *Let (X, d) be a symmetric space.*

(a) *X is* Cauchy symmetric *[7], if every convergent sequence is d-Cauchy.*

(b) *For each $x \in X$ and $n \in \mathbb{N}$, let*

$$S_n(x) = \Big\{y \in X : d(x, y) < \frac{1}{n}\Big\}.$$

(c) *Let $P \subset X$, we put*

$$d(P) = \sup\{d(x, y) : x, y \in P\}.$$

Remark 1. (a) Let X is a symmetric space. Then $\{S_n(x) : n \in \mathbb{N}\}$ is a weak neighborhood base at x in X for all $x \in X$.

(b) X is Cauchy symmetric if and only if for each $x \in X$, $d\big(S_n(x)\big)$ converges to 0, see [7].

Definition 3. *Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$.*

(a) *$\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a* σ-strong network *for X [5], if $\{St(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at each point $x \in X$.*

(b) *$\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a* σ-point-finite strong network *for X, if it is a σ-strong network and each $\mathcal{P}_n$ is point-finite.*

Definition 4. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-point-finite strong network for a space X.*

(a) *$\mathcal{P}$ is a* σ-point-finite strong cs^*-network*, if $\mathcal{P}$ is a cs^*-network.*

(b) *$\mathcal{P}$ is a* σ-point-finite strong network consisting of cs^*-covers*, if each $\mathcal{P}_n$ is a cs^*-cover.*

For some undefined or related concepts, we refer the reader to [4], [8] and [9].

2. Main results

Lemma 1. *Let $\{x_n\}$ be a sequence converging to $x \in X$, and $\mathcal{P}$ be a point-countable cs^*-network for a symmetric space X. Then, for each $n \in \mathbb{N}$, there exists a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ such that $\bigcup \mathcal{H} \subset S_n(x)$, and $\{x_n\}$ is eventually in $\bigcup \mathcal{H}$.*

Proof. Let $\{x_n\}$ be a sequence converging to $x \in X$, and $\mathcal{P}$ be a point-countable cs^*-network for a symmetric space X, we can assume that $\mathcal{P}$ is closed under finite intersections. Since $\mathcal{P}$ is a point-countable cs^*-network, it follows from Lemma 3 in [10] that there is a finite subfamily $\mathcal{F} \subset (\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{F}$. So, the set

$$\big\{\mathcal{F} \subset (\mathcal{P})_x : \mathcal{F} \text{ is finite and} \{x_n\} \text{is eventually in} \bigcup \mathcal{F}\big\}$$

is non-empty. Furthermore, since $(\mathcal{P})_x$ is countable, we can write

$$\big\{\mathcal{F} \subset (\mathcal{P})_x : \mathcal{F} \text{ is finite and } \{x_n\} \text{ is eventually in} \bigcup \mathcal{F}\big\} = \{\mathcal{F}_n : n \in \mathbb{N}\}.$$

For each $k \in \mathbb{N}$, put

$$H_k = \bigcap \big\{ \bigcup \mathcal{F}_i : i \leq k \big\}.$$

Then, for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $H_m \subset S_n(x)$. Let

$$\mathcal{H} = \Big\{ \bigcap_{i \leq m} P_i : P_i \in \mathcal{F}_i, i \leq m \Big\}.$$

Then, $\mathcal{H}$ is finite, $\mathcal{H} \subset (\mathcal{P})_x$ and $\{x_n\}$ is eventually in $\bigcup \mathcal{H} = H_m \subset S_n(x)$. ■

Theorem 1. *Let X be a Cauchy symmetric space having a point-finite cs^*-network, then X has a σ-point-finite strong network consisting of cs^*-covers. So, X has a σ-point-finite strong cs^*-network.*

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-point-finite cs^*-network for a Cauchy symmetric space X. We can assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$ and each $\mathcal{P}_n$ is closed under finite intersections. So, $\mathcal{P}$ is closed under finite intersections. For each $m, n, k \in \mathbb{N}$, put

$$\mathcal{Q}_{m,n} = \Big\{ P \in \mathcal{P}_m : d(P) < \frac{1}{n} \Big\};$$

$$A_{m,n,k} = \big\{ x \in X : S_k(x) \subset \mathtt{St}(x, \mathcal{Q}_{m,n}) \big\};$$

$$B_{m,n,k} = X - A_{m,n,k};$$

$$\mathcal{F}_{m,n,k} = \mathcal{Q}_{m,n} \bigcup \{B_{m,n,k}\}.$$

Then, each $\mathcal{F}_{m,n,k}$ is point-finite. Furthermore, we have

(i) Each $\mathcal{F}_{m,n,k}$ is a cs^*-cover. Let $S = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to $x \in X$, then

Case 1. If $x \in A_{m,n,k}$, then $S_k(x) \subset \mathtt{St}(x, \mathcal{Q}_{m,n})$. Since S is eventually in $S_k(x)$, S is eventually in $\mathtt{St}(x, \mathcal{Q}_{m,n})$. On the other hand, since $\mathcal{Q}_{m,n}$ is point-finite, S is frequently in P for some $P \in \mathcal{Q}_{m,n}$. Therefore, S is frequently in P for some $P \in \mathcal{F}_{m,n,k}$.

Case 2. If $x \notin A_{m,n,k}$ and $S \cap B_{m,n,k}$ is infinite, then S is frequently in $B_{m,n,k} \in \mathcal{F}_{m,n,k}$.

Case 3. If $x \notin A_{m,n,k}$ and $S \cap B_{m,n,k}$ is finite, then there exists $k_0 \in \mathbb{N}$ such that

$$\{x_i : i \geq k_0\} \subset S \cap A_{m,n,k}.$$

Since $x_i \in A_{m,n,k}$, we have

$$x_i \in S_k(x_i) \subset \mathtt{St}(x_i, \mathcal{Q}_{m,n}) \text{ for each } i \geq k_0.$$

Furthermore, since S converges to x, there exists $i_0 \geq k_0$ such that

$$d(x, x_i) < \frac{1}{k} \text{ for every } i \geq i_0.$$

This implies that

$$\{x, x_i\} \subset S_k(x_i) \subset \mathtt{St}(x_i, \mathcal{Q}_{m,n}) \text{ for every } i \geq i_0.$$

Hence, for each $i \geq i_0$, there exists $P_i \in \mathcal{Q}_{m,n}$ such that $\{x, x_i\} \subset P_i$. Since $\mathcal{Q}_{m,n}$ is point-finite, the set $\{P_i : i \geq i_0\}$ is finite. Thus, S is frequently in $P_{i_1} \in \mathcal{F}_{m,n,k}$ for some $i_1 \geq i_0$.

Therefore, each $\mathcal{F}_{m,n,k}$ is a cs^*-cover for X.

(ii) $\{\mathtt{St}(x, \mathcal{F}_{m,n,k}) : m, n, k \in \mathbb{N}\}$ is a network at x.

Let $x \in U$ with U is open in X. Then, there exists $n_0 \in \mathbb{N}$ such that $S_{n_0}(x) \subset U$. Since X is Cauchy symmetric, there exists $i \in \mathbb{N}$ such that $d\big(S_i(x)\big) < \frac{1}{n_0}$. Firstly, we prove that there exist $m_0, k_0 \in \mathbb{N}$ such that $S_{k_0}(x) \subset \mathtt{St}(x, \mathcal{Q}_{m_0,n_0})$. In fact, if not, for each $k, m \in \mathbb{N}$, there exists $x_{k,m} \in S_k(x) - \mathtt{St}(x, \mathcal{Q}_{m,n_0})$. For each $k \geq m$, let $x_{k,m} = y_j$, where $j = m + \frac{k(k-1)}{2}$. Then, the sequence $\{y_j\}$ converges to x. Since $\mathcal{P}$ is a point-countable cs^*-network, it follows from Lemma 1 that $\{y_j\}$ is eventually in $\bigcup \mathcal{H} \subset S_i(x)$ for some finite subfamily $\mathcal{H}$ of $(\mathcal{P})_x$. Since $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$, it follows that $\mathcal{H} \subset (\mathcal{P}_{i_1})_x$ for some $i_1 \in \mathbb{N}$. Hence, $\bigcup \mathcal{H} \subset \mathtt{St}(x, \mathcal{Q}_{i_1,n_0})$. So, there exists $j_0 \in \mathbb{N}$ such that

$$\{x\} \bigcup \{y_j : j \geq j_0\} \subset \mathtt{St}(x, \mathcal{Q}_{i_1,n_0}).$$

Take $j \geq j_0$ with $y_j = x_{k,i_1}$ for some $k \geq i_1$. Then, $x_{k,i_1} \in \mathtt{St}(x, \mathcal{Q}_{i_1,n_0})$. This is a contradiction.

Now, we shall show that $\mathtt{St}(x, \mathcal{F}_{m_0,n_0,k_0}) \subset U$. In fact, let $y \in \mathtt{St}(x, \mathcal{F}_{m_0,n_0,k_0})$, then $\{x, y\} \subset F$ for some $F \in \mathcal{F}_{m_0,n_0,k_0}$. Since $S_{k_0}(x) \subset \mathtt{St}(x, \mathcal{Q}_{m_0,n_0})$, we get $x \notin B_{m_0,n_0,k_0}$, this implies that $F \neq B_{m_0,n_0,k_0}$. Thus, $F \in \mathcal{Q}_{m_0,n_0}$, and $d(F) < \dfrac{1}{n_0}$. Hence, $d(x,y) < \dfrac{1}{n_0}$. This implies that $y \in S_{n_0}(x) \subset U$. Therefore, $\{\mathtt{St}(x, \mathcal{F}_{m,n,k}) : m, n, k \in \mathbb{N}\}$ is a network at x.

Next, we write

$$\{\mathcal{F}_{m,n,k} : m, n, k \in \mathbb{N}\} = \{\mathcal{H}_i : i \in \mathbb{N}\},$$

and for each $i \in \mathbb{N}$, put

$$\mathcal{G}_i = \bigwedge\{\mathcal{H}_j : j \leq i\}.$$

Then, $\bigcup\{\mathcal{G}_i : i \in \mathbb{N}\}$ is a σ-point-finite strong network consisting of cs^*-covers for X. Hence, X has a σ-point-finite strong cs^*-network. ■

Remark 2. By Theorem 1, we get a partial answer to the Question.

By Theorem 1 and Lemma 2.2 in [9], we obtain the following.

Corollary 1. *The following are equivalent for a Cauchy symmetric space X.*

(a) *X has a σ-point-finite cs^*-network;*
(b) *X has a σ-point-finite strong cs^*-network;*
(c) *X is a quotient compact image of a metric space.*

Remark 3. In [5], the authors posed the following question that lets X be a symmetric space with a σ-point-finite cs-network. Is X a quotient compact image of a metric space? By Corollary 1, we get a partial answer to this Question.

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LUONG QUOC TUYEN
DEPARTMENT OF MATHEMATICS
DA NANG UNIVERSITY, VIETNAM
e-mail: tuyendhdn@gmail.com

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