DE GRUYTER OPEN

FASCICULI MATHEMATICI

Nr 58 2017

DOI:10.1515/fascmath-2017-0005

Waghamore P. Harina and S. Rajeshwari

# NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING SMALL FUNCTION WITH FINITE WEIGHT

Abstract. The purpose of the paper is to study the uniqueness of entire and meromorphic functions sharing a small function with finite weight. The results of the paper improve and extend some recent results due to Abhijit Banerjee and Pulak Sahoo [3].

Key words: entire and meromorphic function, weighted sharing, nonlinear differential polynomials.

*AMS Mathematics Subject Classification*: 30D35.

## 1. Introduction

In this paper by meromorphic functions we will always mean meromorphic function in the complex plane.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a finite complex number. We say that $f$ and $g$ share $a$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that $f$ and $g$ share $\infty$ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that $f$ and $g$ share $\infty$ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM.

We adopt the standard notations of value distribution theory (see [8]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

Throughout this paper, we need the following definition.

$$\Theta(a;f)=1-\lim_{r\to\infty}\sup\frac{\bar{N}(r,a;f)}{T(r,f)},$$

where $a$ is a value in the extended complex plane.

In 1959, Hayman [7] proved the following result.

**Theorem A.** *Let $f$ be a transcendental entire function, and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many zeros.*

In 2002, Fang and Fang [6] proved the following result.

**Theorem B.** *Let $f$ and $g$ be two non-constant entire functions, and let $n(\geq 8)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In the same year Fang [5] investigated the value sharing of more general non-linear differential polynomial than that was considered in Theorem B and obtained the following result.

**Theorem C.** *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $k$ be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM then $f \equiv g$.*

In 2004, Lin and Yi [14] considered the case of meromorphic function in Theorem B and obtained the following.

**Theorem D.** *Let $f$ and $g$ be two non-constant meromorphic functions with $\Theta(\infty, f) > \frac{2}{n+1}$, and let $n(\geq 12)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

Natural inquisition would be to investigate the situation for meromorphic function in Theorem C. In this direction in 2008, Zhang [20] proved the following result.

**Theorem E.** *Suppose that $f$ is a transcendental meromorphic function with finite number of poles, $g$ is a transcendental entire function, and let $n, k$ be two positive integers with $n \geq 2k+6$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

To proceed further we require the following definition known as weighted sharing of values introduced by I. Lahiri [9] which measure how close a shared value is to being shared CM or to being shared IM.

**Definition 1.** *Let $k$ be a non negative integer or infinity. For $a \in \mathbb{C}\bigcup\{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.*

*The definition implies that if $f$, $g$ share a value $a$ with weight $k$, then $z_0$ is an $a$-point of $f$ with multiplicity $m(\leq k)$ if and only if it is an $a$-point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m(> k)$ if and only if it is an $a$-point of $g$ with multiplicity $n(> k)$, where $m$ is not necessarily equal to $n$.*

*We write $f$, $g$ share $(a, k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a, k)$ then $f$, $g$ share $(a, p)$ for any integer*

*$p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a, 0)$ or $(a, \infty)$ respectively.*

In 2009, using the notion of weighted sharing of values, Xu, Yi and Cao [15] proved the following result.

**Theorem F.** *Let $f$ and $g$ be two non-constant meromorphic functions, and $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 5k + 11$ or if $l = 1$ and $n > 7k + \frac{23}{2}$, then $f \equiv g$.*

Recently, Li [13] proved the following result which rectify and at the same time improve Theorem F.

**Theorem G.** *Let $f$ and $g$ be two non-constant meromorphic functions, and $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 3k + 11$ or if $l = 1$ and $n > 5k + 14$, then $f = g$ or $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} = 1$.*

In this direction recently Abhijith Banerjee [1] proved the following results first one of which improves Theorem G.

**Theorem H.** *Let $f$ and $g$ be two transcendental meromorphic function and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose for two nonzero constants $a$ and $b$, $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 3k+9$ or if $l = 1$ and $n \geq 4k+10$, or if $l = 0$ and $n \geq 9k+18$, then $f = g$ or $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} = 1$. The possibility $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} = 1$ does not occur for $k = 1$.*

**Theorem I.** *Let $f$ and $g$ be two transcendental entire functions, and let $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers. Suppose for two nonzero constants $a$ and $b$, $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 2k+6$ or if $l = 1$ and $n \geq \frac{5k}{2} + 7$, or if $l = 0$ and $n \geq 5k + 12$, then $f = g$.*

In 2015, Abhijith Banerjee and Pulak Sahoo [3] obtained the following result.

**Theorem J.** *Let $f$ and $g$ be two non-entire transcendental meromorphic functions and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose for two nonzero constants $a$ and $b$, $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\not\equiv 0)$ is a polynomial. If $l \geq 2$ and $n \geq 3k+9$ or if $l = 1$ and $n \geq 4k+10$ or if $l = 0$ and $n \geq 9k+18$, then $f = g$.*

**Theorem K.** *Let $f$ and $g$ be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants $a$ and $b$, $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\not\equiv 0)$*

*is a polynomial. If* $l \geq 2$ *and* $n \geq 2k+6$ *or if* $l = 1$ *and* $n \geq \frac{5k}{2} + 7$ *or if* $l = 0$ *and* $n \geq 5k + 12$, *then* $f = g$.

The following questions are inevitable.

**Quation 1.** *What can be said if the sharing value zero is replaced by a small function* $a$ *in the above Theorems J and K?*

**Quation 2.** *Are the Theorems J and K also true for non-constant entire and meromorphic functions?*

In this paper, taking the possible answer of the above questions into background we obtain the following results.

**Theorem 1.** *Let* $f$ *and* $g$ *be two non-constant meromorphic functions, let* $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$, *for a positive integer* $m$ *or* $P(w) \equiv c_0$ *where* $a_0 (\neq 0), a_1 ... a_{m-1}, a_m (\neq 0)$, $c_0 (\neq 0)$ *are complex constants. Also we suppose that* $[f^n P(f)]^{(k)}$ *and* $[g^n P(g)]^{(k)}$ *share* $(a, l)$, *and* $n (\geq 1)$, $k (\geq 1)$, $l (\geq 0)$ *are positive integers. Now*
*(I) when* $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$, *and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > 3k + m + 8$,
(b) $l = 1$ *and* $n > 4k + \frac{3m+8}{2}$,
(c) $l = 0$ *and* $n > 9k + 4m + 14$,
*then one of the following three cases holds:*
(I1) $f(z) \equiv t g(z)$ *for a constant* $t$ *such that* $t^{d_1} = 1$, *where* $d_1 = gcd(n + m, n + m - i, ..., n), a_{m-i} \neq 0$ *for some* $i = 0, 1, 2, \ldots, m$,
(I2) $f$ *and* $g$ *satisfy the algebraic equation* $R(f, g) \equiv 0$, *where* $R(w_1, w_2) = w_1^n (a_m w_1^m + a_{m-1} w_1^{m-1} + \ldots + a_1 w_1 + a_0) - w_2^n (a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_1 w_2 + a_0)$, *except for* $P(w) = a_1 w + a_2$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$,
(I3) $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv a^2$, *except for* $k = 1$,
*(II) when* $P(w) \equiv c_0$, *and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > 3k + 8$,
(b) $l = 1$ *and* $n > 4k + 9$,
(c) $l = 0$ *and* $n > 9k + 14$,
*then one of the following two cases holds:*
(II1) $f \equiv t g$ *for some constant* $t$ *such that* $t^n = 1$,
(II2) $c_0^2 [f^n]^{(k)} [g^n]^{(k)} \equiv a^2$. *In particular when* $n > 2k$ *and* $a(z) = d_2 =$ *constant, we get* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* $c_1$, $c_2$ *and* $c$ *are constants satisfying* $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

**Theorem 2.** *Let* $f$ *and* $g$ *be two non-constant entire functions, let* $P(w) = a_m w^m + a_{m-1} w^{m-1} + ... + a_1 w + a_0$, *for a positive integer* $m$ *or* $P(w) \equiv c_0$ *where* $a_0 (\neq 0), a_1 ... a_{m-1}, a_m (\neq 0)$, $c_0 (\neq 0)$ *are complex con-*

*stants. Also we suppose that* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* $(a,l)$, *and* $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ *are positive integers. Now*
$(I)$ *when* $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$, *and one of the following conditions holds:*
 $(a)$ $l \geq 2$ *and* $n > 2k + m + 5$,
 $(b)$ $l = 1$ *and* $n > \frac{5k}{2} + 2m + 5$,
 $(c)$ $l = 0$ *and* $n > 5k + 4m + 8$,
*then the conclusion of Theorem 1 holds:*
$(II)$ *when* $P(w) \equiv c_0$, *and one of the following conditions holds:*
 $(a)$ $l \geq 2$ *and* $n > 2k + 5$,
 $(b)$ $l = 1$ *and* $n > \frac{5k}{2} + 5$,
 $(a)$ $l = 0$ *and* $n > 5k + 8$,
*then the conclusion of Theorem 1 holds.*

**Theorem 3.** *Let* $f$ *and* $g$ *be two non-constant meromorphic functions and* $a(z)(\not\equiv 0,\infty)$ *be a small function of* $f$ *and* $g$, *let* $P(w) = a_mw^m + a_{m-1}w^{m-1} + ... + a_1w + a_0$, *for a positive integer* $m$ *or* $P(w) \equiv c_0$ *where* $a_0(\neq 0), a_1...a_{m-1}, a_m(\neq 0)$, $c_0(\neq 0)$ *are complex constants. Also we suppose that* $f^nP(f)f'$ *and* $g^nP(g)g'$ *share* $(a,l)$, *and* $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ *are positive integers and one of the following conditions holds:*
 $(a)$ $l \geq 2$ *and* $n > m + 10$,
 $(b)$ $l = 1$ *and* $n > \frac{3m}{2} + 12$,
 $(c)$ $l = 0$ *and* $n > 4m + 22$,
*then one of the following two cases holds:*
$(I)$ $f(z) \equiv tg(z)$ *for a constant* $t$ *such that* $t^{d_3} = 1$, *where* $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ *for some* $i = 0, 1, 2, \ldots, m$,
$(II)$ $f$ *and* $g$ *satisfy the algebraic equation* $R(f,g) \equiv 0$, *where* $R(w_1,w_2) = w_1^{n+1}(\frac{a_mw_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1}) - w_2^{n+1}(\frac{a_mw_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1})$,

**Theorem 4.** *Let* $f$ *and* $g$ *be two non-constant entire functions and* $a(z)(\not\equiv 0,\infty)$ *be a small function of* $f$ *and* $g$, *let* $P(w) = a_mw^m + a_{m-1}w^{m-1} + ... + a_1w + a_0$, *for a positive integer* $m$ *or* $P(w) \equiv c_0$ *where* $a_0(\neq 0), a_1...a_{m-1}$, $a_m(\neq 0)$, $c_0(\neq 0)$ *are complex constants. Also we suppose that* $f^nP(f)f'$ *and* $g^nP(g)g'$ *share* $(a,l)$, *and* $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ *are positive integers and one of the following conditions holds:*
 $(a)$ $l \geq 2$ *and* $n > m + 4$,
 $(b)$ $l = 1$ *and* $n > \frac{3m}{2} + 6$,
 $(c)$ $l = 0$ *and* $n > 4m + 11$,
*then the conclusion of Theorem 3 holds.*

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by $H$ the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

**Lemma 1** ([16]). *Let $f$ be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in $f$ of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + ... + a_1 f(z) + a_0\,,$$

*where $a_n(\neq 0), a_{n-1}...a_1, a_0$ are complex numbers. Then*

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

**Lemma 2** ([21]). *Let $f$ be a non-constant meromorphic function, and $p, k$ be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{1}$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2}$$

**Lemma 3** ([9]). *Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following cases holds:*

*(i)* $T(r) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r)$,

*(ii)* $F = G$ ,

*(iii)* $FG = 1$. *where $T(r)$ denotes the maximum of $T(r, F)$ and $T(r, G)$ and $S(r) = o\{T(r)\}$ as $r \to \infty$, possibly outside a set of finite linear measure.*

**Lemma 4** ([2]). *Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1, 1)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ & + \frac{1}{2}\overline{N}(r, 0; F) + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G). \end{aligned}$$

**Lemma 5** ([2]). *Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1, 0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ & + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) \\ & + \overline{N}(r, \infty; G) + S(r, F) + S(r, G). \end{aligned}$$

**Lemma 6** ([4])**.** *Let $f, g$ be two non-constant meromorphic functions, let $n, k$ be two positive integers such that $n > 2k$. Suppose $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $d_2$ CM. If $[f^n]^{(k)}[g^n]^{(k)} \equiv d_2^2$, then $f = c_1e^{cz}$, $g = c_2e^{-cz}$, where $c_1$, $c_2$ and $c$ are constants such that $(-1)^k(c_1c_2)^n(nc)^{2k} = d_2^2$.*

**Lemma 7** ([18])**.** *If $H \equiv 0$, then $F$, $G$ share 1CM. If further $F$, $G$ share $\infty$ IM then $F, G$ share $\infty$ CM.*

**Lemma 8.** *Let $f$ and $g$ be two non-constant meromorphic(entire) functions. Let $P(w)$ be defined as in Theorem 1 and $k$, $m$, $n > 3k+m (> 2k+m)$ be three positive integers. If $[f^nP(f)]^{(k)} \equiv [g^nP(g)]^{(k)}$, then $f^nP(f) \equiv g^nP(g)$.*

**Proof.** By the assumption $[f^nP(f)]^{(k)} \equiv [g^nP(g)]^k$.

When $k \geq 2$, integrating we get

$$[f^nP(f)]^{(k-1)} \equiv [g^nP(g)]^{(k-1)} + C_{k-1}.$$

If possible we suppose $C_{k-1} \neq 0$.

Now in the view of the Lemma 2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,f) &\leq T(r,[f^nP(f)]^{(k-1)}) - \overline{N}(r,0;[f^nP(f)]^{(k-1)}) \\
&\quad + N_k(r,0;f^nP(f)) + S(r,f) \\
&\leq \overline{N}(r,0;[f^nP(f)]^{(k-1)}) + \overline{N}(r,\infty;f) \\
&\quad + \overline{N}(r,C_{k-1};[f^nP(f)]^{(k-1)}) \\
&\quad - \overline{N}(r,0;[f^nP(f)]^{(k-1)}) \\
&\quad + N_k(r,0;f^nP(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;[g^nP(g)]^{(k-1)}) + k\overline{N}(r,0;f) \\
&\quad + N(r,0;P(f)) + S(r,f) \\
&\leq (k+m+1)T(r,f) + (k-1)\overline{N}(r,\infty;g) \\
&\quad + N_k(r,0;g^nP(g)) + S(r,f) \\
&\leq (k+m+1)T(r,f) + (k-1)\overline{N}(r,\infty;g) \\
&\quad + k\overline{N}(r,0;g) + N(r,0;P(g)) + S(r,f) \\
&\leq (k+m+1)T(r,f) + (2k+m-1)T(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\leq (3k+2m)T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)T(r,g) \leq (3k+2m)T(r) + S(r).$$

Where $T(r) = \max\{T(r,f), T(r,g)\}$ and $S(r) = \max\{S(r,f), S(r,g)\}$. Combining these we get

$$(n-m-3k)T(r) \leq S(r).$$

Which is a contradiction since $n > 3k+m$.

Therefore $C_{k-1} = 0$ and so $[f^nP(f)]^{(k-1)} \equiv [g^nP(g)]^{(k-1)}$, Repeating $k-1$ times, we obtain

$$f^nP(f) = g^nP(g) + c_0.$$

If $k = 1$, clearly integrating one we obtain the above. If possible suppose $c_0 \neq 0$.

Now using the second fundamental theorem we get

$$\begin{aligned}(n+m)T(r,f) &\leq \overline{N}(r,0;f^nP(f)) + \overline{N}(r,\infty;f^nP(f)) \\ &\quad + \overline{N}(r,c_0;f^nP(f)) + S(r,f) \\ &\leq \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,\infty;f) \\ &\quad + \overline{N}(r,0;g^nP(g)) + S(r,f) \\ &\leq (m+2)T(r,f) + \overline{N}(r,0;g) + mT(r,g) \\ &\quad + S(r,f) + S(r,g) \\ &\leq (m+2)T(r,f) + (m+1)T(r,g) + S(r,f) + S(r,g) \\ &\leq (2m+3)T(r) + S(r).\end{aligned}$$

similarly we get

$$(n+m)T(r,g) \leq (2m+3)T(r) + S(r)$$

combining these we get

$$(n-m-3)T(r) \leq S(r).$$

which is a contradiction, since $n > m+3$. Therefore $c_0 = 1$ and so

$$f^nP(f) \equiv g^nP(g).$$

This completes the lemma. ■

**Lemma 9.** *Let $f$, $g$ be two nonconstant meromorphic (entire functions) and $F = \frac{[f^nP(f)]^{(k)}}{a(z)}$, $G = \frac{[g^nP(g)]^{(k)}}{a(z)}$, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are positive integers such that $n > 3k+m+3(> 2k+m+2)$ and $P(w)$ be defined as in Theorem 1. If $H \equiv 0$ then*
*(I) when $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$, one of the following three cases holds:*

*(I1)* $f \equiv tg$ *for a constant* $t$ *such that* $t^{d_1} = 1$, *where* $d_1 = gcd(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ *for some* $i = 1, 2, \ldots, m$,

*(I2)* $f$ *and* $g$ *satisfy the algebraic equation* $R(f,g) \equiv 0$, *where* $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \ldots + a_0) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + \ldots + a_0)$, *except for* $P(w) = a_1 w + a_2$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$,

*(I3)* $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv a^2$,

*(II) when* $P(w) \equiv c_0$,, *one of the following two case holds:*

*(II1)* $f \equiv tg$ *for some constant* $t$ *such that* $t^n = 1$,

*(II2)* $c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$. *In particular, when* $n > 2k$ *and* $a(z) = d_2$ *we get* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$, *where* $c_1$, $c_2$ *and* $c$ *are constants satisfying* $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

**Proof.** Since $H \equiv 0$, by Lemma 7, we get $F$ and $G$ share 1 CM. On integration we get,

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G-1}, \tag{3}$$

where $a, b$ are constants and $a \neq 0$. We now consider the following cases.

**Case 1.** Let $b \neq 0$ and $a \neq b$. If $b = -1$, then from (3) we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,g) &\leq T(r,G) + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) \\
&\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) \\
&\quad + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n P(g)) + \overline{N}(r,\infty;f) + S(r,g) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) \\
&\quad + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) \\
&\quad + T(r,P(g)) + S(r,g) \\
&\leq T(r,f) + (k+m+2)T(r,g) + S(r,f) + S(r,g)
\end{aligned}$$

without loss of generality, we suppose that there exists a set $I$ with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.

So for $r \in I$ we have

$$(n-k-3)T(r,g) \leq S(r,g)$$

which is a contradiction since $n > k + 3$.

If $b \neq -1$, from (3) we obtain that

$$F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^2[G + \frac{a-b}{b}]}.$$

So,

$$\overline{N}(r, \frac{(b-a)}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

Using Lemma 2 and the same argument as used in the case when $b = -1$ we can get a contradiction.

**Case 2.** Let $b \neq 0$ and $a = b$. If $b = -1$, then from (3) we have

$$FG \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv a^2(z),$$

where $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $a(z)$ CM.

Note that if $P(w) \equiv c_0$ then we have

$$c_0^2 [f^n]^{(k)} [g^n]^{(k)} \equiv a^2(z).$$

In particular when $n > 2k$ and $a(z) = d_2$ then we get by Lemma 6 that $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are constants satisfying $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

If $b = -1$, from (3) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore,

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,g) &\leq T(r,G) + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{1}{a+b}; G) \\
&\quad + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) \\
&\quad + \overline{N}(r,0;F) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) \\
&\quad + (k+1)\overline{N}(r,0;f) + T(r,P(f)) + k\overline{N}(r,\infty;f) \\
&\quad + S(r,f) + S(r,g) \\
&\leq (k+m+2)T(r,g) + (2k+m+1)T(r,f) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(n - 3k - 3 - m)T(r, g) \leq S(r, g).$$

Which is a contradiction, since $n > 3k + m + 3$.

**Case 3.** Let $b = 0$. From (3) we obtain

$$F \equiv \frac{G + a - 1}{a}. \tag{4}$$

If $a \neq 1$ then from (4) we obtain

$$\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in case 2. Therefore $a = 1$ and from (4) we obtain

$$F \equiv G,$$

i.e.,

$$[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}.$$

Note that $n > 3k + m + 3 > 3k + m$.

So by Lemma 8, we have

$$f^n P(f) \equiv g^n P(g). \tag{5}$$

Let $h = \frac{f}{g}$. If $h$ is a constant, putting $f = gh$ in (5) we get

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + ... + a_0 g^n(h^n - 1) = 0,$$

which implies $h^{d_1} = 1$, where $d_1 = gcd(n + m, ..., n + m - i, ..., n + 1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$. Thus $f = tg$ for a constant $t$ such that $t^{d_1} = 1$. where $d_1 = gcd(n+m, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If $h$ is not a constant, then from (5) we can say that $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^n + a_{m-1} w_1^{m-1} + ... + a_0) - w_2^n(a_m w_2^n + a_{m-1} w_2^{m-1} + ... + a_0).$$

In particular when $P(w) = a_1 w + a_2$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ then $f \equiv g$. Note that when $P(w) \equiv c_0$ then we must have $f \equiv tg$ for some constant $t$ such that $t^n = 1$. ■

**Lemma 10.** *Let $f$ and $g$ be two non constant meromorphic functions and $a(z)(\not\equiv 0, \infty)$ be a small function of $f$ and $g$. Let $n$ and $m$ be two positive integers such that $n > \frac{4m}{t} - (m - 1)$, $t$ denotes the number of distinct roots of the equation $P(w) \equiv 0$, where $P(w)$ is defined as in Theorem 3. Then*

$$f^n P(f) f' g^n P(g) g' \not\equiv a^2.$$

**Proof.** First suppose that

$$f^nP(f)f'g^nP(g)g' \equiv a^2(z). \tag{6}$$

Let $d_1$ be the distinct zeros of $P(w) = 0$ and multiplicity $P_i$, where $i = 1, 2, ..., t$, $1 \leq t \leq m$ and $\sum_{i=1}^{t} p_i = m$.

Now by the second fundamental theorem for $f$ and $g$ we get respectively

$$\begin{aligned} tT(r,f) \ &\leq \ \overline{N}(r,0;f) + \overline{N}(r,\infty;f) \\ &+ \sum_{i=1}^{t} \overline{N}(r,d_i;f) - \overline{N_0}(r,0;f') + S(r,f) \end{aligned} \tag{7}$$

and

$$\begin{aligned} tT(r,g) \ &\leq \ \overline{N}(r,0;g) + \overline{N}(r,\infty;g) \\ &+ \sum_{i=1}^{t} \overline{N}(r,d_i;g) - \overline{N_0}(r,0;g') + S(r,g), \end{aligned} \tag{8}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function of those zeros of $f'$ which are not the zeros of $f$ and $f - d_i$, $i = 1, 2, \ldots, t$ and $\overline{N}_0(r,0;g')$ can be similarly defined.

Let $z_0$ be a zero of $f$ with multiplicity $p$ but $a(z_0) \neq 0, \infty$. Clearly $z_0$ must be a pole of $g$ with multiplicity $q$. Then from (6) we get $np + p - 1 = nq + mq + q + 1$. This gives

$$mq + 2 = (n+1)(p-q). \tag{9}$$

From (9) we get $p - q \geq 1$ and so $q \geq \frac{n-1}{m}$. Now $np+p-1 = nq+mq+q+1$ gives $p \geq \frac{n+m-1}{m}$. Thus we have

$$\overline{N}(r,0;f) \leq \frac{m}{n+m-1}N(r,0;f) \leq \frac{m}{n+m-1}T(r,f). \tag{10}$$

Let $z_1(a(z_1) \neq 0, \infty)$ be a zero of $f - d_i$ with multiplicity $q_i$, $i = 1, 2, ...t$. Obviously $z_1$ must be a pole of $g$ with multiplicity $r(\geq 1)$. Then from (6) we get $p_iq_i + q_i - 1 = (n+m+1)r + 1 \leq n+m+2$. This gives $q_i \geq \frac{n+m+2}{p_i+1}$ for $i = 1, 2, \ldots, t$ and so we get

$$\overline{N}(r,d_i;f) \leq \frac{p_i+1}{n+m+3}N(r,d_i;f) \leq \frac{p_i+1}{n+m+3}T(r,f).$$

Clearly

$$\sum_{i=1}^{d} \overline{N}(r,d_i;f) \leq \frac{m+t}{n+m+3}T(r,f). \tag{11}$$

Similarly we have

$$\overline{N}(r,0;g) \leq \frac{m}{n+m-1}T(r,g) \tag{12}$$

and

$$\sum_{i=1}^{t}\overline{N}(r,d_i;g) \leq \frac{m+t}{n+m+3}T(r,g). \tag{13}$$

Also it is clear that

$$\begin{aligned}\overline{N}(r,\infty;f) \;&\leq\; \overline{N}(r,0;g)+\sum_{i=1}^{t}\overline{N}(r,d_i;g)\\ &\quad+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\ &\leq \left(\frac{m}{n+m-1}+\frac{m+t}{n+m+3}\right)T(r,g)+\overline{N}_0(r,0;g')\\ &\quad+S(r,f)+S(r,g)\end{aligned} \tag{14}$$

then by (7), (10), (11) and (14) we get

$$\begin{aligned}tT(r,f) \;&\leq\; \left(\frac{m}{n+m-1}+\frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad+\overline{N}_0(r,0;g')-\overline{N}_0(r,0;f')+S(r,f)+S(r,g).\end{aligned} \tag{15}$$

Similarly we have

$$\begin{aligned}tT(r,g) \;&\leq\; \left(\frac{m}{n+m-1}+\frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad+\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned} \tag{16}$$

Then from (15) and (16) we get

$$\begin{aligned}t\{T(r,f)+T(r,g)\} \;&\leq\; 2\left(\frac{m}{n+m-1}+\frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad+S(r,f)+S(r,g)\end{aligned}$$

i.e.,

$$\left(t-\frac{2m}{n+m-1}-\frac{2(m+t)}{n+m+3}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g). \tag{17}$$

Since

$$\begin{aligned}&\left(t-\frac{2m}{n+m-1}-\frac{2(m+t)}{n+m+3}\right)\\ &\qquad=\frac{(n+m-1)^2t+2(n+m-1)(t-2m)-8m}{(n+m-1)(n+m+3)}.\end{aligned}$$

We note that when $n+m-1 > \frac{4m}{t}$ i.e., when $n > \frac{4m}{t} - (m-1)$, then clearly $t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3} > 0$ and so (17) leads to a contradiction. This completes the proof. ■

**Lemma 11.** *Let $f$, $g$ be two nonconstant meromorphic functions,let $F = \frac{f^n P(f) f'}{a}$, $G = \frac{g^n P(f) g'}{a}$, where $P(w)$ is defined as in Theorem 3, $a = a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f$ and $g$, and $n$ is a positive integer such that $n > m+5$. If $H \equiv 0$ then one of the following three cases holds:*

*(I) $f^n P(f) f' g^n P(g) g' \equiv a^2(z)$,*

*(II) $f(z) \equiv tg(z)$ for a constant $t$ such that $t^{d_3} = 1$, where $d_3 = gcd(n+m+1, ..., n+m+1-i, ..., n+1), a_{m-i} \neq 0$ for some $i = 1, 2, ..., m$,*

*(III) $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^{n+1}(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1} w_1^{m-1}}{n+m} + ... + \frac{a_0}{n+1}) - w_2^{n+1}(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1} w_2^{m-1}}{n+m} + ... + \frac{a_0}{n+1})$.*

**Proof.** Clearly

$$F = \frac{[f^{n+1}\{\frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + ... + \frac{a_0}{n+1}\}]'}{a}$$

and

$$G = \frac{[g^{n+1}\{\frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + ... + \frac{a_0}{n+1}\}]'}{a},$$

where

$$P_1(w) = \{\frac{a_m}{n+m+1} w^m + \frac{a_{m-1}}{n+m} w^{m-1} + ... + \frac{a_0}{n+1}\}$$

proceeding in the same way as the proof of Lemma 9, taking $k = 1$ and considering $n+1$ instead of $n$ we get either

$$f^n P(f) f' g^n P(g) g' \equiv a^2(z)$$

or

$$f^n P(f) f' \equiv g^n P(g) g'.$$

Let $h = \frac{f}{g}$. If $h$ is a constant, by putting $f = gh$ in the above equation we get

$$a_m g^m (h^{n+m+1} - 1) + a_{m-1} g^{m-1} (h^{n+m} - 1) + ... \\ + a_1 g (h^{n+2} - 1) + a_0 (h^{n+1} - 1) \equiv 0,$$

which implies that $h^{d_3} = 1$, where $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$. Thus $f \equiv tg$ for a constant $t$ such that $t^{d_3} = 1$, where $d_3 = gcd(n+m+1, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$.

If $h$ is not constant then $f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1,w_2) = w_1^{n+1}(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1}) - w_2^{n+1}(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + ... + \frac{a_0}{n+1})$. ■

## 3. Proof of the theorems

**Proof of Theorem 1.** Let $F(z)$ and $G(z)$ be given as in Lemma 9. It follows that $F$ and $G$ share $(1,l)$ except for the zeros and poles of $P(z)$. So from (1) we obtain

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;[f^nP(f)]^{(k)}) + S(r,f) \\ &\leq T(r,[f^nP(f)]^{(k)}) - (n+m)T(r,f) \\ &\quad + N_{k+2}(r,0;f^nP(f)) + S(r,f) \\ &\leq T(r,F) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) \\ &\quad + O\{\log r\} + S(r,f). \end{aligned} \tag{18}$$

Again by (2) we have

$$N_2(r,0;G) \leq k\overline{N}(r,\infty;f) + N_{k+2}(r,0;g^nP(g)) + S(r,g). \tag{19}$$

From (18) we get

$$\begin{aligned} (n+m)T(r,f) &\leq T(r,F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F) \\ &\quad + O\{\log r\} + S(r,f). \end{aligned} \tag{20}$$

**Case 1.** Let $H \not\equiv 0$.

*Subcase* 1. Let $l \geq 2$. Let $(i)$ of Lemma 3 holds. Then using (19) we obtain from (20),

$$\begin{aligned} (n+m)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ &\quad + N_{k+2}(r,0;g^nP(f)) + O\{\log r\} \\ &\quad + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;f^nP(f)) + N_{k+2}(r,0;g^nP(g)) \\ &\quad +2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} \\ &\quad + S(r,f) + S(r,g) \\ &\leq (k+m+2)\{T(r,f) + T(r,g)\} + 2\overline{N}(r,\infty;f) \\ &\quad + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} \\ &\quad + S(r,f) + S(r,g) \\ &\leq [(k+m+4) - 2\Theta(\infty;f) + \epsilon]T(r,f) \\ &\quad + [(2k+m+4) - (k+2)\Theta(\infty,g) + \epsilon]T(r,g) \\ &\quad + S(r,f) + S(r,g) \end{aligned} \tag{21}$$

$$\leq [(3k+m+8) - 2\Theta(\infty,f) - 2\Theta(\infty,g) \\ - k\min\{\Theta(\infty,f),\Theta(\infty,g)\} + 2\epsilon]T(r) + S(r).$$

In a similar way we can obtain

$$(n+m)T(r,g) \leq [(3k+m+8) - 2\Theta(\infty,f) - 2\Theta(\infty,g) \\ - k\min\{\Theta(\infty,f),\Theta(\infty,g)\} + 2\epsilon]T(r) + S(r). \tag{22}$$

From (21) and (22) we obtain

$$[n - 3k - m - 8 + 2\Theta(\infty,f) + 2\Theta(\infty,g) \\ + k\min\{\Theta(\infty,f)\Theta(\infty,g)\} - 2\epsilon]T(r) \leq S(r)$$

contradicting with the fact that $n \geq 3k+m+8$.

*Subcase* 2. Let $l = 1$, using Lemma 4 and (19) we obtain from (20),

$$\begin{aligned}(n+m)T(r,f) \leq\ & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ & + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) \\ & + N_{k+2}(r,0;f^nP(f)) + O\{\log r\} \\ & + S(r,f) + S(r,g) \\ \leq\ & N_{k+2}(r,0;f^nP(f)) + N_{k+2}(r,0;g^nP(g)) \\ & + \frac{1}{2}N_{k+1}(r,0;f^nP(f)) + \frac{k+5}{2}\overline{N}(r,\infty;f) \\ & + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} \\ & + S(r,f) + S(r,g) \\ \leq\ & (k+m+2)\{T(r,f) + T(r,g)\} \\ & + \frac{k+m+1}{2}T(r,f) + \frac{k+5}{2}\overline{N}(r,\infty;f) \\ & + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} \\ & + S(r,f) + S(r,g) \\ \leq\ & [(2k + \frac{3m+10}{2}) - (\frac{k}{2}+3)\Theta(\infty,f) \\ & - \frac{1}{2}\Theta(\infty,f) + \epsilon]T(r,f) + [(2k+m+4) \\ & - (\frac{k}{2} + 2g\Theta(\infty,g) - \frac{k}{2}\Theta(\infty,f) + \epsilon]T(r,g) \\ & + O\{\log r\} + S(r,f) + S(r,g) \\ \leq\ & [4k + \frac{5m+18}{2} - (\frac{k+5}{2})\,(\Theta(\infty,f) \\ & +\ \Theta(\infty,g)) + 2\epsilon]T(r) + S(r).\end{aligned} \tag{23}$$

Similarly

$$(24)\quad (n+m)T(r,g) \le [4k+\frac{5m+18}{2}-(\frac{k+5}{2})(\Theta(\infty,f) + \Theta(\infty,g))+2\epsilon]T(r)+S(r).$$

Combining (23) and (24) we obtain

$$[n-4k-\frac{5m+18}{2}+m+\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))+2\epsilon]T(r)\le S(r),$$

contradiction. Since $n \ge 4k+\frac{3m+18}{2}$.

*Subcase 3.* Let $l=0$, using Lemma 5 and (19) we obtain from (20),

$$\begin{aligned}(25)\ (n+m)T(r,f) \le\ & N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\ &+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;f^nP(f))\\ &+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &+O\{\log r\}+S(r,f)+S(r,g)\\ \le\ & N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ &+2N_{k+2}(r,0;f^nP(f))+N_{k+1}(r,0;g^nP(g))\\ &+(2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)\\ &+O\{log\ r\}+S(r,f)+S(r,g)\\ \le\ & [(5k+3m+8)-(2k+4)\Theta(\infty;f)-\epsilon]T(r,f)\\ &+[(4k+2m+6)-(2k+3)\Theta(\infty;g)-\epsilon]T(r,g)\\ &+O\{log\ r\}+S(r,f)+S(r,g)\\ &+[(9k+5m+14)-(2k+3)[\Theta(\infty;f)\\ &+\Theta(\infty,g)]-\ \min\{\Theta(\infty,f)\Theta(\infty;g)\}\\ &+2\epsilon]T(r)+S(r).\end{aligned}$$

Similarly

$$\begin{aligned}(26)\ (n+m)T(r,g) \le\ & [(9k+5m+14)-(2k+3)[\Theta(\infty;f)+\Theta(\infty,g)]\\ &-\min\{\Theta(\infty,f)\Theta(\infty;g)\}+2\epsilon]T(r)+S(r).\end{aligned}$$

From (25) and (26) we get

$$\begin{aligned}&[n-9k-4m-14]+(2k+3)(\Theta(\infty,f)+\Theta(\infty;g))\\ &\qquad+\min\{\Theta(\infty;f)\Theta(\infty;g)\}-2\epsilon]T(r)\le S(r),\end{aligned}$$

contradicts with the facts that $n \ge 9k+4m+14$.

**Case 3.** Let $H \equiv 0$. Then the Theorem follows from Lemma 9. ■

**Proof of Theorem 2.** Noting that $\overline{N}(r,\infty;f)=0$, $\overline{N}(r,\infty;g)=0$ and proceeding in the like manner as the proof of Theorem 1 we obtain the result of the Theorem 2. ■

**Proof of Theorem 3.** Let $F=\frac{f^nP(f)f'}{a(z)}$ and $G=\frac{g^nP(g)g'}{a(z)}$. Then $F$, $G$ share $(1,l)$, except the zeros and poles of $a(z)$. Clearly

$$F=\frac{[f^{n+1}\{\frac{a_m}{n+m+1}f^m+\frac{a_{m-1}}{n+m}f^{m-1}+...+\frac{a_0}{n+1}\}]'}{a}$$

and

$$G=\frac{[g^{n+1}\{\frac{a_m}{n+m+1}g^m+\frac{a_{m-1}}{n+m}g^{m-1}+...+\frac{a_0}{n+1}\}]'}{a},$$

where

$$P_1(w)=\{\frac{a_m}{n+m+1}w^m+\frac{a_{m-1}}{n+m}w^{m-1}+...+\frac{a_0}{n+1}\}.$$

**Case 1.** Let $H \not\equiv 0$. Now following the same procedure as adopted in the proof of Case 1 of Theorem 1 we can easily deduce a contradiction.

**Case 2.** Let $H \equiv 0$. Since $n > k_1$ and $n > m+5$ the theorem follows from Lemma 10 and 11. ■

**Proof of Theorem 4.** Noting that $\overline{N}(r,\infty;f)=0$, $\overline{N}(r,\infty;g)=0$ and proceeding in the like manner as the proof of Theorem 3 we obtain the result of the Theorem 4. ■

## References


[1] Banerjee A., Uniqueness of certain non-linear differential polynomials sharing 1-points, *Kyungpook Math. J.,* 51(1)(2011), 43-58.

[2] Banerjee A., Meromorphic functions sharing one value, *Int. J. Math. Math. Sci.,* 22(2005), 3587-3598.

[3] Banerjee A., Sahoo P., Certain nonlinear differential polynomials sharing a nonzero polynomial with finite weight, *Kyungpook Math. J.,* 55(2015), 653-666.

[4] Banerjee A., Majumder S., Non-linear differential polynomials sharing small function with finite weight, *Arab. J. Math.* (*Springer*)*,* 4(1)(2015), 7-28.

[5] Fang M.-L., Uniqueness and value-sharing of entire functions, *Comput. Math. Appl.,* 44(5-6)(2002), 823-831.

[6] Fang C.-Y., Fang M.-L., Uniqueness of meromorphic functions and differential polynomials, *Comput. Math. Appl.,* 44(5-6)(2002), 607-617.

[7] Hayman W.K., Picard values of meromorphic functions and their derivatives, *Ann. of Math.,* 70(2)(1959), 9-42.

[8] Hayman W.K., *Meromorphic functions,* Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[9] Lahiri I., Weighted value sharing and uniqueness of meromorphic functions, *Complex Variables Theory Appl.,* 46(3)2001), 241-253.
[10] Lahiri I., Value distribution of certain differential polynomials, *Int. J. Math. Math. Sci.,* 28(2)(2001), 83-91.
[11] Lahiri I., On a question of Hong Xun Yi, *Arch. Math. (Brno),* 38(2)(2002), 119-128.
[12] Lahiri I., Dewan S., Value distribution of the product of a meromorphic function and its derivative, *Kodai Math. J.,* 26(1)(2003), 95-100.
[13] Li J.-D., Uniqueness of meromorphic functions and differential polynomials, *Int. J. Math. Math. Sci.,* (2011), Art. ID 514218, 12 pp.
[14] Lin W., Yi H., Uniqueness theorems for meromorphic functions concerning fixed-points, *Complex Var. Theory Appl.,* 49(11)(2004), 793-806.
[15] Xu H.-Y., Yi C.-F., Cao T.-B., Uniqueness of meromorphic functions and differential polynomials sharing one value with finite weight, *Ann. Polon. Math.,* 95(1)(2009), 51-66.
[16] Yang C.C., On deficiencies of differential polynomials II, *Math. Z.,* 125 (1972), 107-112.
[17] Yang L., *Value distribution theory, translated and revised from the 1982 Chinese original,* Springer, Berlin, 1993.
[18] Yi H.-X., Meromorphic functions that share one or two values II, *Kodai Math. J.,* 22(2)(1999), 264-272.
[19] Yi H.X., Yang C.C., *Uniqueness theory of meromorphic functions,* Science Press, Beijing, 1995.
[20] Zhang J., Uniqueness theorems for entire functions concerning fixed points, *Comput. Math. Appl.,* 56(12)(2008), 3079-3087.
[21] Zhang J.-L., Yang L.-Z., Some results related to a conjecture of R. Brück, *JIPAM. J. Inequal. Pure Appl. Math.,* 8(1)(2007), Article 18, 11 pp.

Waghamore P. Harina
Department of Mathematics
Central College Campus
Bangalore University
Bangalore-560 001, India
*e-mail:* harinapw@gmail.com

S. Rajeshwari
Department of Mathematics
Central College Campus
Bangalore University
Bangalore-560 001, India
*e-mail:* rajeshwaripreetham@gmail.com

*Received on 05.12.2016 and, in revised form, on 23.05.2017.*