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HERMITE-HADAMARD TYPE INEQUALITIES FOR MT$_m$-PREINVEX FUNCTIONS

Abstract. In the present paper, the notion of MT_m-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving MT_m-preinvex functions along with beta function are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via classical integrals and Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given. These results not only extend the results appeared in the literature (see [13]), but also provide new estimates on these types.

Key words: Hermite-Hadamard type inequality, MT-convex function, Hölder's inequality, power mean inequality, fractional integral, m-invex, P-function.

AMS Mathematics Subject Classification: 26A33, 26A51, 33B15, 26B25, 26D07, 26D10, 26D15.

1. Introduction and preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [9], [13], [6], [21], [5], [11], [10], [4], [17], [3]) and the references cited therein. In (see [19], [14]) and the references cited therein, Tunç and Yildirim defined the following so-called MT-convex function:

Definition 1. *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of $MT(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{2}$$

Fractional calculus (see [13]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. *Here* $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13]) and the references cited therein.

Now, let us recall some definitions of various convex functions.

Definition 3 (see [7])**.** *A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\circ}$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

Definition 4 (see [1])**.** *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please see (see [1],[20]) and the references therein.

Definition 5 (see [16]). *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have*

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y-x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [18]).

Recently, Liu (see [12]) obtained several integral inequalities for the left-hand side of (3) under the Definition 3 of P-function. Also in (see [15]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of MT_m-preinvex function is introduced and some new integral inequalities for the left-hand side of (3) involving MT_m-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via classical integrals are given. In Section 4, some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via fractional integrals are given. In Section 5, some applications to special means are given. These results given in Sections 3-4 not only extend the results appeared in the literature (see [13]), but also provide new estimates on these types.

2. New integral inequalities for MT_m-preinvex functions

Definition 6 (see [8]). *A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1. In Definition 6, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as MT_m-preinvex function.

Definition 7. *Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$. For $f : K \longrightarrow \mathbb{R}$ and any fixed $m \in (0,1]$, if*

$$f(my + t\eta(x,y,m)) \leq \frac{m\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y), \tag{4}$$

is valid for all $x, y \in K$ and $t \in (0,1)$, then we say that $f(x)$ belong to the class of $MT_m(K)$ with respect to η.

Remark 2. In Definition 7, it is worthwhile to note that the class $MT_m(K)$ is a generalization of the class MT(I) given in Definition 1 on $K = I$ with respect to $\eta(x,y,1) = x - y$ and $m = 1$.

Example 1. $f, g : (1,\infty) \longrightarrow \mathbb{R}$, $f(x) = x^p$, $g(x) = (1+x)^p$, $p \in \left(0, \frac{1}{1000}\right)$; $h : [1, 3/2] \longrightarrow \mathbb{R}$, $h(x) = (1+x^2)^k$, $k \in \left(0, \frac{1}{100}\right)$, are simple examples of the new class of MT_m-preinvex functions with respect to $\eta(x,y,m) = x - my$ for any fixed $m \in (0,1]$, but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving MT_m-preinvex functions along with beta function, we need the following new lemma:

Lemma 1. *Let $f : K = [ma, ma + \eta(b,a,m)] \longrightarrow \mathbb{R}$ be a continuous function on the interval of real numbers K° with $a < b$ and $ma < ma + \eta(b,a,m)$. Then for any fixed $m \in (0,1]$ and any fixed $p, q > 0$, we have*

$$\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x)dx$$
$$= \eta(b,a,m)^{p+q+1} \int_0^1 t^p (1-t)^q f(ma + t\eta(b,a,m))dt.$$

Proof. It is easy to observe that

$$\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x)dx$$
$$= \eta(b,a,m) \int_0^1 (ma + t\eta(b,a,m) - ma)^p (ma + \eta(b,a,m) - ma - t\eta(b,a,m))^q f(ma + t\eta(b,a,m))dt$$
$$= \eta(b,a,m)^{p+q+1} \int_0^1 t^p (1-t)^q f(ma + t\eta(b,a,m))dt.$$

■

The following definition will be used in the sequel.

Definition 8. *The Euler Beta function is defined for* $x, y > 0$ *as*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2. *Let* $f : K = [ma, ma + \eta(b,a,m)] \longrightarrow \mathbb{R}$ *be a continuous function on the interval of real numbers* K°, $a < b$ *with* $ma < ma+\eta(b,a,m)$. *If* $|f|$ *is a* MT_m*-preinvex function on* K *for any fixed* $m \in (0,1]$, *then for any fixed* $p, q > 0$, *we have*

$$\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$
$$\leq \frac{m}{2}\eta(b,a,m)^{p+q+1}\left[|f(a)|\beta\left(p+\frac{1}{2}, q+\frac{3}{2}\right) + |f(b)|\beta\left(p+\frac{3}{2}, q+\frac{1}{2}\right)\right].$$

Proof. Since $|f|$ is a MT_m-preinvex function on K, we have

$$\Big|f(ma+t\eta(b,a,m))\Big| \leq \frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|$$

for all $t \in (0,1)$ and for any fixed $m \in (0,1]$. By Lemma 1 and the fact that $|f|$ is a MT_m-preinvex function on K, we get

$$\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$
$$\leq \eta(b,a,m)^{p+q+1}\int_0^1 t^p(1-t)^q\Big|f(ma+t\eta(b,a,m))\Big|dt$$
$$\leq \eta(b,a,m)^{p+q+1}\int_0^1 t^p(1-t)^q\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|\right]dt$$
$$= \frac{m}{2}\eta(b,a,m)^{p+q+1}\left[|f(a)|\beta\left(p+\frac{1}{2}, q+\frac{3}{2}\right) + |f(b)|\beta\left(p+\frac{3}{2}, q+\frac{1}{2}\right)\right].$$

■

Theorem 3. *Let* $f : K = [ma, ma + \eta(b,a,m)] \longrightarrow \mathbb{R}$ *be a continuous function on the interval of real numbers* K°, $a < b$ *with* $ma < ma+\eta(b,a,m)$. *Let* $k > 1$ *and* $|f|^{\frac{k}{k-1}}$ *be a* MT_m*-preinvex function on* K *for any fixed* $m \in (0,1]$. *Then for any fixed* $p, q > 0$, *we have*

$$\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$
$$\leq \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}}\eta(b,a,m)^{p+q+1}\Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}}$$
$$\times\left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a MT_m-preinvex function on K, combining with Lemma 1 and Hölder inequality for all $t \in (0,1)$ and for any fixed $m \in (0,1]$, we get

$$\begin{aligned}
&\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx \\
&\leq \eta(b,a,m)^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}} \\
&\quad \times \left[\int_0^1 \left|f(ma+t\eta(b,a,m))\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}} \\
&\leq \eta(b,a,m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \\
&\quad \times \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)|^{\frac{k}{k-1}} + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}} \\
&= \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} \eta(b,a,m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \\
&\quad \times \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.
\end{aligned}$$

■

Theorem 4. *Let $f : K = [ma, ma+\eta(b,a,m)] \longrightarrow \mathbb{R}$ be a continuous function on the interval of real numbers K°, $a < b$ with $ma < ma+\eta(b,a,m)$. Let $l \geq 1$ and $|f|^l$ be a MT_m-preinvex function on K for any fixed $m \in (0,1]$. Then for any fixed $p, q > 0$, we have*

$$\begin{aligned}
&\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx \\
&\leq \left(\frac{m}{2}\right)^{\frac{1}{l}} \eta(b,a,m)^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}} \\
&\quad \times \left[|f(a)|^l\beta\left(p+\frac{1}{2}, q+\frac{3}{2}\right) + |f(b)|^l\beta\left(p+\frac{3}{2}, q+\frac{1}{2}\right)\right]^{\frac{1}{l}}.
\end{aligned}$$

Proof. Since $|f|^l$ is a MT_m-preinvex function on K, combining with Lemma 1 and Hölder inequality for all $t \in (0,1)$ and for any fixed $m \in (0,1]$, we get

$$\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$

$$= \eta(b,a,m)^{p+q+1} \int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}$$
$$\times \Big[t^p(1-t)^q\Big]^{\frac{1}{l}} f(ma+t\eta(b,a,m))dt$$
$$\le \eta(b,a,m)^{p+q+1} \left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$
$$\times \left[\int_0^1 t^p(1-t)^q \big|f(ma+t\eta(b,a,m))\big|^l dt\right]^{\frac{1}{l}}$$
$$\le \eta(b,a,m)^{p+q+1} \Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\times \left[\int_0^1 t^p(1-t)^q \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)|^l + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|^l\right) dt\right]^{\frac{1}{l}}$$
$$= \left(\frac{m}{2}\right)^{\frac{1}{l}} \eta(b,a,m)^{p+q+1} \Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\times \left[|f(a)|^l \beta\left(p+\frac{1}{2}, q+\frac{3}{2}\right) + |f(b)|^l \beta\left(p+\frac{3}{2}, q+\frac{1}{2}\right)\right]^{\frac{1}{l}}.$$

■

Remark 3. In Theorem 4, if we choose $l = 1$, we get Theorem 2.

3. Hermite-Hadamard type classical integral inequalities for MT_m-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via classical integrals, we need the following new lemma:

Lemma 2. *Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma+\eta(b,a,m)]$. Then, for each $x \in [ma, ma+\eta(b,a,m)]$, we have*

$$\frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)} - \frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right] \tag{5}$$

$$= \frac{\eta(x,a,m)^2}{\eta(b,a,m)} \int_0^1 (t-1) f'(ma + t\eta(x,a,m)) dt + \frac{\eta(x,b,m)^2}{\eta(b,a,m)} \int_0^1 (1-t) f'(mb + t\eta(x,b,m)) dt.$$

Proof. Denote

$$I = \frac{\eta(x,a,m)^2}{\eta(b,a,m)} \int_0^1 (t-1) f'(ma + t\eta(x,a,m)) dt + \frac{\eta(x,b,m)^2}{\eta(b,a,m)} \int_0^1 (1-t) f'(mb + t\eta(x,b,m)) dt.$$

Integrating by parts, we get

$$\begin{aligned} I &= \frac{\eta(x,a,m)^2}{\eta(b,a,m)} \left[(t-1) \frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)} \Big|_0^1 - \int_0^1 \frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)} dt \right] \\ &+ \frac{\eta(x,b,m)^2}{\eta(b,a,m)} \left[(1-t) \frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)} \Big|_0^1 + \int_0^1 \frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)} dt \right] \\ &= \frac{\eta(x,a,m) f(ma) - \eta(x,b,m) f(mb)}{\eta(b,a,m)} \\ &- \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u) du - \int_{mb}^{mb+\eta(x,b,m)} f(u) du \right]. \end{aligned}$$

■

Remark 4. Clearly, if we choose $m = 1$ and $\eta(x,y,1) = x - y$ in Lemma 2, we get (see [9], Lemma 1).

Using the Lemma 2 the following results can be obtained.

Theorem 5. *Let $A \subseteq \mathbb{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}_0$ for any fixed $m \in (0,1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|$ is a MT_m-preinvex function on $[ma, ma + \eta(b,a,m)]$ and $|f'(x)| \le M$, then for each $x \in [ma, ma + \eta(b,a,m)]$, we have*

$$\begin{aligned} &\left| \frac{\eta(x,a,m) f(ma) - \eta(x,b,m) f(mb)}{\eta(b,a,m)} \right. \\ &\left. - \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u) du - \int_{mb}^{mb+\eta(x,b,m)} f(u) du \right] \right| \\ &\le \frac{Mm\pi}{4|\eta(b,a,m)|} \left[\eta(x,a,m)^2 + \eta(x,b,m)^2 \right]. \end{aligned} \tag{6}$$

Proof. Using Lemma 2, MT_m-preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b,a,m)]$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)} - \frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 |t-1||f'(ma+t\eta(x,a,m))|dt + \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 |1-t||f'(mb+t\eta(x,b,m))|dt$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 (1-t)\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|\right]dt$$
$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 (1-t)\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|\right]dt$$
$$\leq \frac{Mm\pi}{4|\eta(b,a,m)|}\left[\eta(x,a,m)^2+\eta(x,b,m)^2\right].$$

■

Remark 5. In Theorem 5, if we choose $m=1$ and $\eta(x,y,1)=x-y$ then we get (see [13], Theorem 2.2).

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

Theorem 6. *Let $A \subseteq \mathbb{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}_0$ for any fixed $m \in (0,1]$ and let $a,b \in A$, $a<b$ with $ma < ma+\eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma+\eta(b,a,m)]$, $q>1$, $p^{-1}+q^{-1}=1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma+\eta(b,a,m)]$, we have*

$$\left|\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)} - \frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right]. \tag{7}$$

Proof. Suppose that $q > 1$. Using Lemma 2, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b,a,m)]$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)}\right.$$
$$\left. - \frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 |t-1||f'(ma + t\eta(x,a,m))|dt$$
$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 |1-t||f'(mb + t\eta(x,b,m))|dt$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ma + t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(mb + t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q \right.\right.$$
$$\left.\left. + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$
$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q \right.\right.$$
$$\left.\left. + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|}\right].$$

■

Remark 6. In Theorem 6, if we choose $m = 1$ and $\eta(x,y,1) = x - y$ then we get (see [13], Theorem 2.4).

Theorem 7. *Let $A \subseteq \mathbb{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}_0$ for any fixed $m \in (0,1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma + \eta(b,a,m)]$, $q \geq 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b,a,m)]$, we have*

$$\left|\frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)}\right. \tag{8}$$

$$- \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \Bigg|$$
$$- \leq M \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (m\pi)^{\frac{1}{q}} \left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|} \right].$$

Proof. Using Lemma 2, MT_m-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b,a,m)]$, and taking the modulus, we have

$$\Bigg| \frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)}$$
$$- \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \Bigg|$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \int_0^1 |t-1||f'(ma+t\eta(x,a,m))|dt$$
$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \int_0^1 |1-t||f'(mb+t\eta(x,b,m))|dt$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 (1-t)dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)|f'(ma+t\eta(x,a,m))|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 (1-t)dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)|f'(mb+t\eta(x,b,m))|^q dt \right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 (1-t)dt \right)^{1-\frac{1}{q}}$$
$$\times \left[\int_0^1 (1-t) \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}}$$
$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 (1-t)dt \right)^{1-\frac{1}{q}}$$
$$\times \left[\int_0^1 (1-t) \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}}$$
$$\leq M \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (m\pi)^{\frac{1}{q}} \left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|} \right].$$

■

Remark 7. In Theorem 7, if we choose $m = 1$ and $\eta(x,y,1) = x - y$ then we get (see [13], Theorem 2.6). Also, in Theorem 7, if we choose $q = 1$, we get Theorem 5.

4. Hermite-Hadamard type fractional integral inequalities for MT_m-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via fractional integrals, we need the following new lemma:

Lemma 3. *Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma+\eta(b,a,m)]$. Then, for each $x \in [ma, ma+\eta(b,a,m)]$ and $\alpha > 0$, we have*

$$
\begin{aligned}
(9)\qquad & \frac{\eta(x,a,m)^\alpha f(ma) - \eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)} \\
& - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^\alpha_{(ma+\eta(x,a,m))-} f(ma) - J^\alpha_{(mb+\eta(x,b,m))-} f(mb)\right] \\
& = \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (t^\alpha - 1) f'(ma + t\eta(x,a,m))dt \\
& + \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (1 - t^\alpha) f'(mb + t\eta(x,b,m))dt,
\end{aligned}
$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the Euler Gamma function.

Proof. Denote

$$
\begin{aligned}
I \;=\; & \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (t^\alpha - 1) f'(ma + t\eta(x,a,m))dt \\
& + \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (1 - t^\alpha) f'(mb + t\eta(x,b,m))dt.
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
I \;=\; & \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \Bigg[(t^\alpha - 1) \frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)} \Big|_0^1 \\
& - \alpha \int_0^1 \frac{t^{\alpha-1} f(ma + t\eta(x,a,m))}{\eta(x,a,m)} dt \Bigg] \\
& + \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \Bigg[(1 - t^\alpha) \frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)} \Big|_0^1 \\
& + \alpha \int_0^1 \frac{t^{\alpha-1} f(mb + t\eta(x,b,m))}{\eta(x,b,m)} dt \Bigg]
\end{aligned}
$$

$$= \frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)}$$
$$- \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \Big[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb) \Big].$$

■

Remark 8. Clearly, if we choose $m = 1$ and $\eta(x,y,1) = x - y$ in Lemma 3, we get (see [13], Lemma 3.1).

By using Lemma 3, one can extend to the following results.

Theorem 8. *Let $A \subseteq \mathbb{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}_0$ for any fixed $m \in (0,1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|$ is a MT_m-preinvex function on $[ma, ma + \eta(b,a,m)]$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma+\eta(b,a,m)]$ and $\alpha > 0$, we have*

$$\Bigg| \frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)} - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \Big[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb) \Big] \Bigg|$$
$$\leq \frac{Mm}{2} \left[\frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \right] \left[\pi - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)} \right]. \tag{10}$$

Proof. Using Lemma 3, MT_m-preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b,a,m)]$, $\alpha > 0$, and taking the modulus, we have

$$\Bigg| \frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)} - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \Big[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb) \Big] \Bigg|$$
$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 |t^{\alpha} - 1| |f'(ma + t\eta(x,a,m))| dt$$
$$+ \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 |1 - t^{\alpha}| |f'(mb + t\eta(x,b,m))| dt$$
$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 (1 - t^{\alpha}) \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)| \right] dt$$
$$+ \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 (1 - t^{\alpha}) \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)| \right] dt$$

$$\leq \frac{Mm}{2}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\pi-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right].$$

■

Remark 9. In Theorem 8, if we choose $m=1$ and $\eta(x,y,1)=x-y$ then we get (see [13], Theorem 3.2). Also, in Theorem 8, if we choose $\alpha=1$, we get the inequality in Theorem 5.

Theorem 9. *Let $A\subseteq\mathbb{R}_0$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow\mathbb{R}_0$ for any fixed $m\in(0,1]$ and let $a,b\in A$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f:A\longrightarrow\mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma,ma+\eta(b,a,m)]$, $q>1$, $p^{-1}+q^{-1}=1$ and $|f'(x)|\leq M$, then for each $x\in[ma,ma+\eta(b,a,m)]$ and $\alpha>0$, we have*

$$\begin{aligned}(11)\quad &\Bigg|\frac{\eta(x,a,m)^\alpha f(ma)-\eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)}\\ &\quad-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^\alpha_{(ma+\eta(x,a,m))-}f(ma)-J^\alpha_{(mb+\eta(x,b,m))-}f(mb)\right]\Bigg|\\ &\leq M\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}.\end{aligned}$$

Proof. Suppose that $q>1$. Using Lemma 3, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)|\leq M$ for each $x\in[ma,ma+\eta(b,a,m)]$, $\alpha>0$, and taking the modulus, we have

$$\begin{aligned}&\Bigg|\frac{\eta(x,a,m)^\alpha f(ma)-\eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)}\\ &\quad-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^\alpha_{(ma+\eta(x,a,m))-}f(ma)-J^\alpha_{(mb+\eta(x,b,m))-}f(mb)\right]\Bigg|\\ &\leq\frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1|t^\alpha-1||f'(ma+t\eta(x,a,m))|dt\\ &\quad+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1|1-t^\alpha||f'(mb+t\eta(x,b,m))|dt\\ &\leq\frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^\alpha)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}\\ &\quad+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^\alpha)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^\alpha)^p dt\right)^{\frac{1}{p}}$$
$$\times \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$
$$+ \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^\alpha)^p dt\right)^{\frac{1}{p}}$$
$$\times \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq M\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}.$$

■

Remark 10. In Theorem 9, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 3.5). Also, in Theorem 9, if we choose $\alpha = 1$, we get the inequality in Theorem 6.

Theorem 10. *Let $A \subseteq \mathbb{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma + \eta(b, a, m)]$, $q \geq 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have*

$$\left|\frac{\eta(x,a,m)^\alpha f(ma) - \eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)} - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^\alpha_{(ma+\eta(x,a,m))-}f(ma) - J^\alpha_{(mb+\eta(x,b,m))-}f(mb)\right]\right|$$
$$\leq M\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\pi - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}}$$
$$\times\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]. \tag{12}$$

Proof. Using Lemma 3, MT_m-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b, a, m)]$, $\alpha > 0$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)^\alpha f(ma) - \eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)}\right.$$

$$
\begin{aligned}
&- \frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]\Bigg| \\
&\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 |t^{\alpha}-1||f'(ma+t\eta(x,a,m))|dt \\
&\quad + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 |1-t^{\alpha}||f'(mb+t\eta(x,b,m))|dt \\
&\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 (1-t^{\alpha})|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 (1-t^{\alpha})|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}} \\
&\qquad \times \left[\int_0^1 (1-t^{\alpha})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}} \\
&\quad + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}} \\
&\qquad \times \left[\int_0^1 (1-t^{\alpha})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}} \\
&\leq M\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\pi - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}} \\
&\qquad \times \left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right].
\end{aligned}
$$

■

Remark 11. In Theorem 10, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get (see [13], Theorem 3.8). Also, in Theorem 10, if we choose $\alpha = 1$, we get Theorem 7.

5. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 9 (see [2]). *A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:*

1. *Homogeneity:* $M(ax, ay) = aM(x, y)$, *for all* $a > 0$,
2. *Symmetry:* $M(x, y) = M(y, x)$,
3. *Reflexivity:* $M(x, x) = x$,
4. *Monotonicity: If* $x \leq x'$ *and* $y \leq y'$, *then* $M(x, y) \leq M(x', y')$.
5. *Internality:* $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\log(\beta) - \log(\alpha)}, \quad \alpha \neq \beta.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad \alpha \neq \beta.$$

8. The weighted p-power mean:

$$M_p \begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, therefore one can obtain various inequalities using the results of Sections (-) for these means as follows:

Replace $\eta(x,y,m)$ with $\eta(x,y)$ and setting $\eta(a,x) = M(a,x)$ and $\eta(b,x) = M(b,x)$, $\forall x \in A$, for value $m = 1$ in (7), (8), (11) and (12) one can obtain the following interesting inequalities involving means:

$$\left| \frac{\eta(a,x)f(a) - \eta(b,x)f(b)}{M(a,b)} - \frac{1}{M(a,b)} \left[\int_a^{a+M(a,x)} f(u)du - \int_b^{b+M(b,x)} f(u)du \right] \right| \leq \frac{M}{(p+1)^{1/p}} \left(\frac{\pi}{2}\right)^{\frac{1}{q}} \left[\frac{M(a,x)^2 + M(b,x)^2}{M(a,b)} \right], \tag{13}$$

$$\left| \frac{\eta(a,x)f(a) - \eta(b,x)f(b)}{M(a,b)} - \frac{1}{M(a,b)} \left[\int_a^{a+M(a,x)} f(u)du - \int_b^{b+M(b,x)} f(u)du \right] \right| \leq M \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (\pi)^{\frac{1}{q}} \left[\frac{M(a,x)^2 + M(b,x)^2}{M(a,b)} \right], \tag{14}$$

$$\left| \frac{M(a,x)^\alpha f(a) - M(b,x)^\alpha f(b)}{M(a,b)} - \frac{\Gamma(\alpha+1)}{M(a,b)} \left[J^\alpha_{(a+M(a,x))-} f(a) - J^\alpha_{(b+M(b,x))-} f(b) \right] \right| \leq M \left(\frac{\pi}{2}\right)^{\frac{1}{q}} \left[\frac{M(a,x)^{\alpha+1} + M(b,x)^{\alpha+1}}{M(a,b)} \right] \left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)} \right]^{\frac{1}{p}}, \tag{15}$$

$$\left| \frac{M(a,x)^\alpha f(a) - M(b,x)^\alpha f(b)}{M(a,b)} - \frac{\Gamma(\alpha+1)}{M(a,b)} \left[J^\alpha_{(a+M(a,x))-} f(a) - J^\alpha_{(b+M(b,x))-} f(b) \right] \right| \tag{16}$$

$$\leq M\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\pi-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}}$$
$$\times\left[\frac{M(a,x)^{\alpha+1}+M(b,x)^{\alpha+1}}{M(a,b)}\right].$$

Letting $M(a,x)$ and $M(b,x)$ equal to $A, G, H, P_r, I, L, L_p, M_p$, $\forall x \in A$ in (13), (14), (15) and (16), we get the inequalities involving means for a particular choices of a differentiable MT_1-preinvex function f. The details are left to the interested reader.

References

[1] Antczak T., Mean value in invexity analysis, *Nonlinear Anal.,* 60(2005), 1473-1484.

[2] Bullen P.S., *Handbook of Means and Their Inequalities,* Kluwer Academic Publishers, Dordrecht, (2003).

[3] Chen F.X., Wu S.H., Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions, *J. Nonlinear Sci. Appl.,* 9(2)(2016), 705-716.

[4] Chu Y.M., Khan M.A., Khan T.U., Ali T., Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.,* 9(2016), 4305-4316.

[5] Chu Y.M., Khan M.A., Khan T.U., Ali T., Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.,* 9(5)(2016), 4305-4316.

[6] Chu Y.M., Wang G.D., Zhang X.H., Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.,* 13(4)(2010), 725-731.

[7] Dragomir S.S., Pečarić J., Persson L.E., Some inequalities of Hadamard type, *Soochow J. Math.,* 21(1995), 335-341.

[8] Du T.S., Liao J.G., Li Y.J., Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions, *J. Nonlinear Sci. Appl.,* 9(2016), 3112-3126.

[9] Kavurmaci H., Avci M., Özdemir M.E, New inequalities of Hermite-Hadamard type for convex functions with applications, *arXiv:1006.1593v1 [math. CA],* (2010), 1-10.

[10] Khan M.A., Khurshid Y, Ali T., Hermite-Hadamard inequality for fractional integrals via α-convex functions, *Acta Math. Univ. Comenianae,* 79(1)(2017), 153-164.

[11] Khan M.A., Khurshid Y, Ali T., Rehman N., Inequalities for three times differentiable functions, *J. Math., Punjab Univ.,* 48(2)(2016), 35-48.

[12] Liu W., New integral inequalities involving beta function via P-convexity, *Miskolc Math. Notes,* 15(2)(2014), 585-591.

[13] Liu W., Wen W., Park J., Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.,* 9(2016), 766-777.

[14] LIU W., WEN W., PARK J., Ostrowski type fractional integral inequalities for MT-convex functions, *Miskolc Mathematical Notes,* 16(1)(2015), 249-256.
[15] ÖZDEMIR M.E., SET E., ALOMARI M., Integral inequalities via several kinds of convexity, *Creat. Math. Inform.,* 20(1)(2011), 62-73.
[16] PINI R., Invexity and generalized convexity, *Optimization,* 22(1991), 513-525.
[17] SHI H.N., Two Schur-convex functions related to Hadamard-type integral inequalities, *Publ. Math. Debrecen,* 78(2)(2011), 393-403.
[18] STANCU D.D., COMAN G., BLAGA P., *Analiză numerică şi teoria aproximării,* Cluj-Napoca: Presa Universitară Clujeană., 2(2002).
[19] TUNÇ M., YILDIRIM H., On MT-Convexity, *arXiv: 1205.5453v1* [*math. CA*], 2012(2012), 1-6.
[20] YANG X.M., YANG X.Q., TEO K.L., Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory Appl.,* 117(2003), 607-625.
[21] ZHANG X.M., CHU Y.M., ZHANG X.H., The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.,* 2010), Article ID 507560, 11 pages.

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