DE GRUYTER OPEN

FASCICULI MATHEMATICI

Nr 58 2017

DOI:10.1515/fascmath-2017-0008

T.D. Narang * and Sahil Gupta **

BEST APPROXIMATION IN METRIC SPACES

Abstract. The aim of this paper is to prove some results on the existence and uniqueness of elements of best approximation and continuity of the metric projection in metric spaces. For a subset M of a metric space (X, d), the nature of set of those points of X which have at most one best approximation in M has been discussed. Some equivalent conditions under which an M-space is strictly convex have also been given in this paper.

Key words: proximinal set, Chebyshev set, strongly approximatively compact set, boundedly compact set, strictly convex metric space.

AMS Mathematics Subject Classification: 41A50, 41A52, 41A65.

1. Introduction

For a non-empty subset M of a metric space (X, d) and $x \in X$, one of the main problem of approximation theory is to find an element $m_0 \in M$ such that

$$d(x, m_0) = \inf_{m \in M} d(x, m) \equiv d(x, M).$$

The set of all best approximations to x in M is denoted by $P_M(x)$. The set M is called **proximinal** if for every $x \in X$, the set $P_M(x)$ is non-empty. If for each $x \in X$, the set $P_M(x)$ is a singleton then the set M is called **Chebyshev**.

The set-valued mapping $P_M : X \to 2^M (\equiv$ the set of all subsets of $M)$ defined by $P_M(x) = \{y \in M : d(x, y) = d(x, M)\}$ is called **metric projection.**

The problem of finding elements of best approximation have been discussed by many researchers in normed linear linear spaces (see e.g. [2],[3],[12],

* The research work of the author has been supported by U.G.C., India under Emeritus Fellowship.

** The research work of the author has been supported by U.G.C., India under Senior Research Fellowship.

[13] and references cited therein) but only a few have taken up this study in more general abstract spaces viz. metric linear spaces, convex metric spaces and metric spaces (see e.g. [1], [6], [8]-[10] and [13]). In this paper, we also discuss this problem in metric spaces and convex metric spaces.

We denote an open ball with center at x and radius r by $B(x,r)$ and by $B[x,r]$ the corresponding closed ball. For a subset A of a metric space, we denote the set of limit points, the closure and the complement respectively of A by $A', \overline{A}$ and A^c.

A subset M of a metric space (X,d) is said to be **approximatively compact** for $x \in X$ if every minimizing sequence $\{y_n\} \subseteq M$ for x, i.e., $\lim_{n\to\infty} d(x,y_n) = d(x,M)$, has a convergent subsequence in M.

A subset M of a metric space (X,d) is said to be **strongly approximatively compact** for $x \in X$ if every minimizing sequence $\{y_n\} \subseteq M$ for x is convergent in M.

The set M is said to be approximatively compact (strongly approximatively compact) in X if it is approximatively compact (strongly approximatively compact) for every $x \in X$.

It is well-known that an approximatively compact set is proximinal and closed (see [13]–p.382). The notion of strong approximative compactness was introduced and discussed in [2] under the name strongly Chebyshev.

The set M is said to be **boundedly compact** if every bounded sequence in M has a subsequence converging to some point of M.

A metric space (X,d) is said to be **metrically convex** or **convex** (in the sense of Menger [7]) if for any two distinct points x and y of X there exist at least one $z \in X$, $x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

A point z satisfying the above condition is called a **between point** of x and y, and the set of all between points of x and y, denoted by $[x,y]$ is called a **metric segment** joining x and y.

Following Menger [7], one can also define a convex metric space as:

A metric space (X,d) is said to be **convex** (see Khalil [6]) if

$$B[x,r] \cap B[y,\lambda - r] \neq \emptyset, \lambda = d(x,y), \quad r \in [0,\lambda].$$

Another form of convexity was introduced in metric spaces by Takahashi [15] as under:

For a metric space (X,d) and closed interval $I = [0,1]$, a continuous mapping $W : X \times X \times I \to X$ is said to be a **convex structure** on X if for all $x,y \in X$, $\lambda \in I$

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y) \ \ for\ all \ \ u \in X.$$

The metric space (X, d) together with a convex structure W, denoted by (X, d, W), is called a **convex metric space**.

A metric space (X, d) is said to be an M**-space** (see [6]) if for any two distinct points x and y of X with $d(x, y) = \lambda$, and for every $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that

$$B[x, r] \bigcap B[y, \lambda - r] = \{z_r\}.$$

An M-space (X, d) is called **externally convex** (see [4], [6]) if for every distinct $x, y \in X$, the equalities

$$d(x, z_1) = d(x, y) + d(y, z_1) = d(x, y) + d(y, z_2) = d(x, z_2)$$

for $z_1, z_2 \in X$ imply $z_1 = z_2$.

In normed linear spaces, this property is equivalent to strict convexity.

A convex metric space (X, d) is said to be **strictly convex** (see Khalil [6]) if for all $z \in X$ and for all $x, y \in B[z, r]$ with $d(x, y) = \lambda$, we have $B[x, (1 - t)\lambda] \cap B[y, t\lambda] \subseteq B(z, r)$ for all $0 < t < 1$.

A convex metric space (X, d) is said to be **strictly convex** (see Narang [8]) if for every pair $x, y \in X$ and $r > 0$,

$$d(x, p) \leq r, \quad d(y, p) \leq r \text{ imply } d(z, p) < r$$

unless $x = y$, where p is arbitrary but fixed point of X and $z \neq x \neq y$ is any point between x and y.

It is known (see [6]) that a strictly convex metric space is an M-space but the converse is not true.

The convexity of Menger and that of Khalil, and strict convexity of Narang and that of Khalil are equivalent in complete metric spaces (see [5]–p.24, [11]).

A non-empty subset M of a metric space (X, d) is said to be

(1) dense in X if the closure of M is the space X.

(2) nowhere dense in X if the interior of the closure of M is empty.

(3) an F_σ**-set** if M can be written as a countable union of closed sets.

(4) a $G_\delta - set$ if M can be written as a countable intersection of open sets.

Clearly, if a set is $F_\sigma - set$ then its complement is a $G_\delta - set$ and conversely.

A metric space (X, d) is said to be of **Baire's first category** if it can be written as a countable union of nowhere dense subsets of X.

In this paper, we use strong approximative compactness to prove some results on the existence and uniqueness of best approximation, and continuity of the metric projection in metric spaces. Some equivalent conditions under which an M-space is strictly convex have been given. For a subset M

of a metric space (X, d), the nature of the set of those points of X which have at most one best approximation in M are also discussed in this paper. The results proved in this paper generalize and extend some results of [2], [3] and [14].

2. Main results

We start with the following theorem which deals with the continuity of the metric projection.

Theorem 1. *Let M be a closed subset of a metric space (X, d) and $x \in X$. If every minimizing sequence for x converges then $P_M(x)$ is a singleton and P_M is continuous at x.*

Proof. Let $x \in X$. Since $d(x, M) = \inf_{m \in M} d(x, m)$, there exist a sequence $\{y_n\} \subseteq M$ such that

$$\lim_{n \to \infty} d(x, y_n) = d(x, M) \tag{1}$$

i.e., $\{y_n\}$ is a minimizing sequence for x. Then by the hypothesis, $\{y_n\} \to y_0 \in M$. Therefore using (1), $y_0 \in P_M(x)$. Suppose $y_1, y_2 \in P_M(x)$, $y_1 \neq y_2$. Then the sequence $\{y_n\}$ defined by $y_{2n} = y_1$ and $y_{2n+1} = y_2$ is a minimizing sequence for x which is not convergent, a contradiction. Therefore, $P_M(x)$ is a singleton. Now to prove the continuity of P_M, let $\{x_n\}$ be a sequence in X such that $x_n \to x$ and $\{y_n\} = P_M(x_n)$; $n = 1, 2, 3....$ Then

$$d(x_n, y_n) = d(x_n, M), \quad n = 1, 2, 3, \ldots . \tag{2}$$

Since $d(x, M) \leq d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) = d(x, x_n) + d(x_n, M)$, on taking limit, we get

$$\lim_{n \to \infty} d(x, y_n) = d(x, M) \tag{3}$$

i.e., $\{y_n\} \subseteq M$ is a minimizing sequence for x and so by the hypothesis $\{y_n\}$ converges to some $y \in M$. Then it follows from (3) that $d(x, y) = d(x, M)$ and so $\{y\} = P_M(x)$. Hence P_M is continuous at x. ■

Remarks. A strongly approximatively compact subset of a metric space is Chebyshev and the metric projection P_M is continuous.

The following lemma (see [1]) shows that in a convex metric space (X, d, W), the set $P_M(x)$ is a part of the boundary of M.

Lemma 1 ([1]). *Let M be a closed subset of a convex metric space (X, d, W) then for any $x \in X$, $P_M(x) \subset bd(M)$.*

The following example (see [13]) shows that an element of best approximation may be an interior point of the set and so the above lemma does not hold in non convex metric spaces.

Example. Let $X = \{x, y\} \subset \mathbb{R}$, with $x \neq y$ and $M = \{y\}$. Then M is convex and $P_M(x) = \{y\} = Int(M)$. Since $bd(M) = \phi$, $P_M(x) \nsubseteq bd(M)$.

Using Lemma 1, we prove the following:

Proposition 1. *Let M be a closed subset of a convex metric space (X, d, W) such that $x_0 \in X \backslash M$ has a best approximation in M then $d(x_0, M)$ $= d(x_0, bd(M))$.*

Proof. Suppose $y_0 \in P_M(x)$ then by Lemma 1, $y_0 \in bd(M)$. Therefore, $d(x_0, M) \leq d(x_0, bd(M)) \leq d(x_0, y_0) = d(x_0, M)$ gives $d(x_0, M) = d(x_0, bd(M))$. ■

The following theorem shows that strict convexity of the space is closely related to approximation properties of the space.

Theorem 2. *Let (X, d) be a complete M-space. Then the following are equivalent:*

(i) *(X, d) is strictly convex.*
(ii) *Every non-empty convex subset of X is semi-Chebyshev.*
(iii) *Every non-empty closed convex subset of X is semi-Chebyshev.*
(iv) *Every non-empty proximinal convex subset of X is Chebyshev.*

Proof. $(i) \Rightarrow (ii)$. Let M be a non-empty convex subset of X which is not semi-Chebyshev i.e. there exists some $x \in X$ for which there are $m_1, m_2 \in M$ such that $d(x, m_1) = d(x, m_2) = d(x, M)$. Let $m \in M$ be the mid point of m_1, m_2. Then $d(x, M) \leq d(x, m)$. Since the space is strictly convex, $d(x, m) < d(x, M)$, a contradiction. Hence M is semi-Chebyshev.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (i)$ follows from ([6]–Theorem 2.6). ■

We require the following result for our next theorems.

Lemma 2. *Let (X, d) be a convex metric space, $M \subseteq X$, $x \in X \backslash M$ and $m_0 \in P_M(x)$.*

1. *$m_0 \in P_M(y)$ for every $y \in (m_0, x)$.*
2. *If the metric space (X, d) is externally convex , then $P_M(y) = \{m_0\}$ for every $y \in (m_0, x)$.*

Proof. 1. If for some $y \in (m_0, x)$, $m_0 \notin P_M(y)$, then it would exist $m_1 \in M$ such that $d(y, m_1) < d(y, m_0)$, yielding the contradiction

$$\begin{aligned} d(x, m_1) &\leq d(x, y) + d(y, m_1) \\ &\leq d(x, y) + d(y, m_0) \\ &= d(x, m_0) \end{aligned}$$

2. Suppose that (X, d) is externally convex, and suppose that, for some $y \in (m_0, x)$, there exists $m_1 \neq m_0$ in M such that $d(y, m_1) = d(y, m_0) = d(y, M)$. Then

$$\begin{aligned} d(x, m_0) = d(x, M) &\leq d(x, m_1) \leq d(x, y) + d(y, m_1) \\ &= d(x, y) + d(y, m_0) = d(x, m_0). \end{aligned}$$

It follows

$$\begin{aligned} d(x, m_0) &= d(x, y) + d(y, m_0) \\ &= d(x, y) + d(y, m_1) = d(x, m_1). \end{aligned}$$

Taking into account the external convexity of (X, d), this implies $m_1 = m_0$. ■

Next two theorems deal with the uniqueness set of best approximation in strictly convex metric spaces. For strictly convex normed linear spaces, these results are well known (see [3], [14]).

Theorem 3. *Let M be a non-empty subset of a externally convex metric space (X, d) then the set $U(M) \equiv \{x \in X : cardP_M(x) \leq 1\}$ is dense in* X *for any subset $M \subseteq X$.*

Proof. We have to prove that $\overline{U(M)} = X$, i.e., $X = U(M) \bigcup [U(M)]'$. Let $x \in X$ be arbitrary. If $x \in U(M)$ we are done. Suppose $x \notin U(M)$ then $P_M(x)$ contain at least two distinct points, say m_1, m_2 and so $P_M(x)$ is non-empty and $d(x, m_1) = d(x, M) = d(x, m_2)$. Then by using Lemma 2, we have $P_M(y) = \{m_1\}$ for every $y \in [m_1, x)$, i.e., $[m_1, x) \subseteq U(M)$. Therefore, every neighbourhood of x intersects $U(M)$ in a point other than x and so $x \in U(M)'$. Therefore, $x \in \overline{U(M)}$ and so $X \subseteq \overline{U(M)}$. But $\overline{U(M)} \subseteq X$. Therefore $\overline{U(M)} = X$. ■

Theorem 4. *Let M be an approximatively compact subset of an externally convex metric space (X, d) then $X \backslash U(M)$ is of first Baire's category and $U(M)$ is a $G_\delta -$ set.*

Proof. Let $D(x) = diam(P_M(x)) = \sup\{d(m', m'') : m', m'' \in P_M(x)\}$. Put $Y = X \backslash U(M)$ and let $Y_n = \{x \in X : D(x) \geq \frac{1}{n}\}$, $n \in N$. Since $Y = \bigcup_{n=1}^{\infty} Y_n$, it is sufficient to prove that each Y_n is closed and nowhere dense.

Y_n **is closed:** Let x be a limit point of Y_n. Then there exist a sequence $\{x_k\}$ in Y_n such that $x_k \to x \in X$. Let $m_k', m_k' \in P_M(x_k)$ be such that

$$d(m_k', m_k'') \geq \frac{k}{(k+1)n} , \tag{4}$$

for all $k \in \mathbb{N}$. Then

$$\begin{aligned} d(x, M) &\leq d(x, m_k') \leq d(x, x_k) + d(x_k, m_k') \\ &= d(x, x_k) + d(x_k, M) \to d(x, M). \end{aligned}$$

It follows that $\{m_k'\}$ is a minimizing sequence for x, and so it contains a subsequence convergent to some $m' \in M$. Similarly, $\{m_k'\}$ contains a subsequence convergent to some point $m'' \in M$, so that we can suppose, without loss of generality, that the sequences $\{m_k'\}$ and $\{m_k''\}$ converge to m', $m'' \in M$, respectively, implying $m', m'' \in P_M(x)$. Letting $k \to \infty$ in (4), one obtains $d(m', m'') \geq 1/n$, and so $diam\ P_M(x) \geq 1/n$, i.e., $x \in Y_n$. Hence Y_n is closed.

Y_n **is nowhere dense:** As the set Y_n is closed, it is sufficient to show that $int(Y_n) = \emptyset$. If $x \in Y_n$ then there exist $m', m'' \in P_M(x)$ with $d(m', m'') \geq \frac{1}{n}$. By using Lemma 2, $P_M(x_0) = \{m'\}$ for every $x_0 \in [m', x) \Rightarrow [m', x) \subset U(M) \subset X \backslash Y_n$ Therefore, Y_n does not contain any ball with center at x. Hence $int(Y_n) = \emptyset$.

Therefore, $Y = X \backslash U(M) = \bigcup_{n=1}^{\infty} Y_n$ where each Y_n is closed and nowhere dense, i.e., Y is a countable union of closed and nowhere dense sets and hence Y is an $F_\sigma - set$ and is of Baire's first category. Consequently, Y^c is a $G_\delta - set$, i.e., $U(M)$ is a $G_\delta - set$. ∎

Since an approximatively compact set in a metric space is proximinal, we obtain that the set $U(M) = EU(M) \equiv \{x \in X : \ cardP_M(x) = 1\}$.

Corollary 1. *Let M be an approximatively compact subset of an externally convex metric space (X, d) then $X \backslash EU(M)$ is of first Baire's category and $EU(M)$ is a $G_\delta - set$.*

For a subset $M \subseteq X$ and $x \in X$, we denote by $\min(x, M)$ the problem of best approximation of x by elements of M, i.e., to find $m_0 \in M$ such that $d(x, m_0) = d(x, M)$. We say that the problem $\min(x, M)$ is well-posed if it has unique solution $m_0 \in M$ and every minimizing sequence converges to m_0.

Theorem 5. *If M is a non-empty complete subset of a metric space (X,d) and $x \in X$ then the problem* $\min(x,M)$ *is well-posed if and only if* $\lim_{\delta\to 0} diam P_M(x,\delta) = 0$.

Proof. Suppose that the set M is complete and $\lim_{\delta\to 0} diam P_M(x,\delta) = 0$. Let $\{y_n\} \subseteq M$ be any minimizing sequence for x, i.e., $\lim_{n\to\infty} d(x,y_n) = d(x,M)$. Then for any $\delta > 0$, $y_n \in P_M(x,\delta)$ after some stage. This implies that for any $\delta > 0$, $d(y_n, y_{n+p}) \leq diam P_M(x,\delta)$ after some stage. As $\lim_{\delta\to 0} diam P_M(x,\delta) = 0$, it follows that the sequence $\{y_n\}$ is Cauchy. Since M is complete, $\{y_n\} \to y_0 \in M$, i.e., every minimizing sequence for x is convergent. Then $d(x,y_0) = d(x,M)$, i.e., $y_0 \in P_M(x)$. If x has two distinct elements of best approximation say $y_1, y_2 \in M$, then the sequence $\{y_n\}$ defined by $y_{2n} = y_1$ and $y_{2n+1} = y_2$ is a minimizing sequence for x which is not convergent, a contradiction to the result we have just proved.

Conversely, suppose that the problem $\min(x,M)$ is well posed. We first prove that $\lim_{n\to\infty} diam P_M(x,1/n) = 0$. Put $d_n = diam P_M(x,1/n)$ and choose $y_n, y_n' \in P_M(x,1/n)$ such that

$$d(y_n, y_n') \geq \frac{n}{n+1} d_n,$$

for all $n \in \mathbb{N}$. Then

$$d(x,M) \leq d(x,y_n) \leq \frac{1}{n} + d(x,M)$$

showing that $\{y_n\}$ is a minimizing sequence. The situation is same for $\{y_n'\}$, so that, by hypothesis, both sequences $\{y_n\}$, $\{y_n'\}$ are convergent to some $y_0 \in M$, implying $\lim_{n\to\infty} d(y_n, y_n') = 0$. The equality

$$d_n \leq \frac{n+1}{n} d(y_n, y_n'),$$

valid for every $n \in \mathbb{N}$, implies $\lim_{n\to\infty} d_n = 0$. Now, given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that $d_{n0} \leq \varepsilon$.

Since $P_M(x,\delta_1) \subset P_M(x,\delta_2)$ for $0 < \delta_1 \leq \delta_2$, it follows $P_M(x,\delta) \subset P_M(x, \frac{1}{n_0})$, and so $diam P_M(x,\delta) \leq d_{n0} \leq \varepsilon$ for all $0 < \delta \leq \frac{1}{n0}$. Since $\varepsilon > 0$ is arbitrary, it follows that $diam P_M(x,\delta) \to 0$ as $\delta \to 0$. ■

Acknowledgements. The authors are thankful to the learned referee for careful reading, critical comments and valuable suggestions leading to an improvement of the paper.

References

[1] Al-Thagafi M.A., Best approximation and fixed points in strong m-star-shaped metric spaces, *Internet. J. Math. Math. Sci.,* 18(1995), 613-616.

[2] Bandyopadhyay P., Li Y., Lin B-L. and Narayana D., Proximinality in Banach spaces, *J. Math. Anal. Appl.,* 341(2008), 309-317.

[3] Cobzas S., Geometric properties of Banach spaces and the existence of nearest and farthest points, *Abstr. Appl. Anal.,* 3(2005), 259-285.

[4] Freese R., Murphy G., Andalafte E., Strict convexity in M-space, *J. Geom.,* 34(1989), 42-49.

[5] Goebel K., Kirk W.A., *Topics in Metric Fixed Point Theory,* Cambridge University Press, 1990.

[6] Khalil R., Best approximation in metric spaces, *Proc. Amer. Math. Soc.,* 103(1988), 579-586.

[7] Menger K., Untersuchungen uber allegemeine Metrik, *Math. Ann.,* 100 (1928), 75-163.

[8] Narang T.D., Best approximation and strict convexity of metric spaces, *Arch. Math. Scripta Fac. Sci. Nat. Ujeb Brunensis,* 17(1981), 87-90.

[9] Narang T.D., Gupta S., Proximinality and coproximinality in metric linear spaces, *Ann. Univ. Mariae Curie-Sklodowska Sect. A,* 69(1)(2015), 83-90.

[10] Narang T.D., Gupta S., On best approximation and best coapproximation, *Thai J. Math.,* 14(2016), 505-516.

[11] Narang T.D., Tejpal S., Equivalence of various types of convexities in metric spaces, *Bull. Gauhati University Math. Association,* 9(2006), 54-61.

[12] Rao G.S., Tools and techniques in approximation theorey, *Math. Stud.,* 81 (2012), 115-133.

[13] Singer I., *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces,* Springer-Verlag, New York, 1970.

[14] Steckin S.B., Approximation properties of sets in linear normed space, *Rev. Roumaine Math. Pures Appl.,* 8(1963), 5-18.

[15] Takahashi W., A convexity in metric spaces and non-expansive mappings I, *Kodai Math. Sem. Rep.,* 22(1970), 142-149.

T.D. Narang
Department of Mathematics
Guru Nanak Dev University
Amritsar-143005, India
e-mail: tdnarang1948@yahoo.co.in

Sahil Gupta
Department of Mathematics
Guru Nanak Dev University
Amritsar-143005, India
e-mail: sahilmath@yahoo.in

Received on 27.05.2016 and, in revised form, on 26.04.2017.