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SOME FIXED POINT THEOREMS FOR ALMOST (GF, δ_b)-CONTRACTIONS AND APPLICATION

Abstract. We propose a new notion of multi-valued almost (GF, δ_b)-contractions involving rational terms under δ-distance on b-metric spaces and give its relevance to fixed point results in orbitally complete b-metric spaces. An ordered version of our main result is also proved with some weaker contractive conditions. Some examples are given to show the usability of the results proved herein. Moreover, application of our result to the nonlinear integral equation is given.

Key words: almost contraction, multivalued mapping, b-Metric space, F-contraction, partial order.

AMS Mathematics Subject Classification: 47H10, 54H25, 45B99.

1. Introduction

Nonlinear analysis is a remarkable confluence of topology, analysis and applied mathematics. Indeed, the fixed point theory is one of the most rapidly growing topic of nonlinear functional analysis. It is a vast and inter-disciplinary subject whose study belongs to several mathematical domains. Most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations. It can be formulated in terms of finding the fixed points of a given nonlinear mapping on an infinite dimensional function space X into itself. Fixed point theory is an important and actual topic of nonlinear analysis. Moreover, it is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920 which was published in 1922 is one of the most important theorems in classical functional analysis. The Banach Contraction Principle is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions-modifying the basic contractive condition or changing the ambiental space. Due to its simplicity and generality, the contraction principle has drawn attention of a very large number of mathematicians.

The advancement of geometric fixed point theory for multivalued mappings was initiated in the work of Nadler, Jr. in 1969 [14]. He used the concept of Hausdorff-Pompeiu metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case. Since then, this discipline has been more developed, and many profound concepts and results have been set up in more generalized spaces. Many fixed point theorems have been proved by various authors as generalizations of Nadler's theorem (see, e.g., [20, 21] and references cited therein).

In this paper, we propose a new notion−almost GF-contraction involving rational terms for multivalued mappings under δ-distance in the setting of b-metric spaces [3, 5, 8, 9, 12, 24] and a concept of F-contractions in the sense of Cosentino et al. [7]. Also, some fixed point results in ordered spaces with weaker contractive conditions are proved. We designed the paper as follows. Section 2 is introductory in character wherein we have discussed the past development of fixed point theory and visited significant preliminary concepts, definitions and important results relevant to our following discussions. In Section 3, we introduce the notion of almost F-contraction for a multivalued mapping $\mathcal{T}$ under δ-distance in a b-metric space and originate fixed point results in orbitally complete b-metric spaces, while in Section 4, we prove an ordered version of the main result of Section 3 with some weaker contractive conditions and with some additional conditions on ordered space. In the final Section 5, application of our result to nonlinear integral equation is discussed. Some suitable examples are furnished to demonstrate the validity of our results and to distinguish them from some known ones.

The present work improves and extends the works done in the papers [1, 2, 4, 7, 11, 17, 18, 19] by taking into account of orbitally complete b-metric space, and endowed with ordered spaces under weaker contractive conditions.

2. Preliminaries

Bakhtin [5] introduced the notion of b-metric spaces as an extension of metric spaces and then extensively used by Czerwik in [8, 9, 10]. After that, a lot of work have been done on the fixed point theory of various classes of single-valued and multi-valued operators in this type of spaces. We recall here just some basic definitions and notation that we are going to use. $\mathbb{R}^+$ and $\mathbb{R}_0^+$ will denote the set of all positive, resp. nonnegative real numbers and $\mathbb{N}$ will be the set of positive integers.

A b-metric on a nonempty set $\mathcal{E}$ is a function $d_b : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_0^+$ such that

for a constant $s \geq 1$ and for all $x, y, z \in \mathcal{E}$ the following three conditions hold true:
(M1) $d_b(x,y) = 0 \iff x = y$,
(M2) $d_b(x,y) = d_b(y,x)$,
(M3) $d_b(x,y) \leq s[d_b(x,z) + d_b(z,y)]$.
The triple $(\mathcal{E}, d_b, s)$ is called a b-metric space.

Obviously, each metric space is a b-metric space (for $s = 1$), but the converse need not be true. Standard examples of b-metric spaces that are not metric spaces are the following:

1. $\mathcal{E} = \mathbb{R}$ and $d_b : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ defined by $d_b(x,y) = |x-y|^2$ for all $x, y \in \mathcal{E}$, with $s = 2$.
2. $\mathcal{E} = \ell^p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, $0 < p < 1$, $d_b : \ell^p(\mathbb{R}) \times \ell^p(\mathbb{R}) \to \mathbb{R}$ given by
$$d_b(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$
for all $\{x_n\}$, $\{y_n\} \in \ell^p(\mathbb{R})$; here $s = 2^{1/p}$.
3. $\mathcal{E} = L^p([0,1]) \ni f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(t)|^p dt < \infty$, $p > 1$, $d_b : L^p([0,1]) \times L^p([0,1]) \to \mathbb{R}$ given by
$$d_b(f,g) = \int_0^1 |f(t) - g(t)|^p$$
for all $f, g \in L^p([0,1])$; here $s = 2^{p-1}$.

The topology on b-metric spaces and the notions of convergent and Cauchy sequences, as well as the completeness of the space are defined in a similar way as for standard metric spaces. However, one has to be aware of some differences. For instance, a b-metric need not be a continuous mapping in both variables (see, e.g., [15]).

Further, we give a brief background for the δ-distance and multivalued mappings defined on a b-metric space $(\mathcal{E}, d_b, s)$.

Following terminologies are used in the paper:
$\mathcal{P}_b(\mathcal{E})$: the class of non-empty and bounded subsets of $\mathcal{E}$,
$\mathcal{P}_{cb}(\mathcal{E})$: the class of non-empty, closed and bounded subsets of $\mathcal{E}$.
For $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{P}_b(\mathcal{E})$, we define:
$$\mathcal{D}_b(\mathcal{U}, \mathcal{V}) = \inf\{d_b(u,v) : u \in \mathcal{U},\ v \in \mathcal{V}\} \text{ and}$$
$$\delta_b(\mathcal{U}, \mathcal{V}) = \sup\{d_b(u,v) : u \in \mathcal{U},\ v \in \mathcal{V}\}$$
with $\mathcal{D}_b(w, \mathcal{W}) = \mathcal{D}_b(\{w\}, \mathcal{W}) = \inf\{d_b(w,x) : x \in \mathcal{W}\}$.

The following are some easy properties of $\mathcal{D}_b$ and δ_b (see, e.g., [8, 9, 10]).
(i) if $\mathcal{U} = \{u\}$ and $\mathcal{V} = \{v\}$ then $\mathcal{D}_b(\mathcal{U}, \mathcal{V}) = \delta_b(\mathcal{U}, \mathcal{V}) = d_b(u,v)$,

(*ii*) $\mathcal{D}_b(\mathcal{U},\mathcal{V}) \leq \delta_b(\mathcal{U},\mathcal{V})$,
(*iii*) $\mathcal{D}_b(x,\mathcal{V}) \leq d_b(x,b)$ for any $b \in \mathcal{V}$,
(*iv*) $\delta_b(\mathcal{U},\mathcal{V}) \leq s[\delta_b(\mathcal{U},\mathcal{W}) + \delta_b(\mathcal{W},\mathcal{V})]$,
(*v*) $\delta_b(\mathcal{U},\mathcal{V}) = 0$ iff $\mathcal{U} = \mathcal{V} = \{v\}$.

It is obvious that $\delta_b(\mathcal{U},\mathcal{U})$ is need not to be zero. Indeed $\delta_b(\mathcal{U},\mathcal{U}) = \text{dia}(\mathcal{U})$, where $\text{dia}(\mathcal{U}) = \sup\{d_b(u,v) : u,v \in \mathcal{U}\}$ is called the diameter of the set $\mathcal{U}$.

Recall that $z \in \mathcal{E}$ is called a fixed point of a multi-valued mapping $\mathcal{T} : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ if $z \in \mathcal{T}z$.

The concepts of an orbit, the orbital completeness and the orbital continuous mappings given in [6, 13, 22] for metric spaces, can be extended to the case of b-metric spaces, as follows:

Definition 1. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space and $\mathcal{T} : \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be a mapping.*

1. *The orbit of the mapping $\mathcal{T}$ at point $x_0 \in \mathcal{E}$ is the set $\mathcal{O}(x_0;\mathcal{T}) = \{x_0\} \cup \{x_n : x_n \in \mathcal{T}x_{n-1}, n = 1, 2, \ldots\}$.*
2. *The space $(\mathcal{E}, d_b, s)$ is said to be $\mathcal{T}$-orbitally complete at $x_0 \in \mathcal{E}$ if every Cauchy subsequence $\{x_{n_i}\}$ in $\mathcal{O}(x_0;\mathcal{T})$ converges in $\mathcal{E}$.*
3. *The mapping $\mathcal{T}$ is said to be orbitally continuous at a point $x_0 \in \mathcal{E}$ if for any sequence $\{x_n\}_{n\geq 0} \subset \mathcal{O}(x_0;\mathcal{T})$ and $z \in \mathcal{E}$, $d(x_n, z) \to 0$ as $n \to \infty$ implies $\delta_b(\mathcal{T}x_n, \mathcal{T}z) \to 0$ as $n \to \infty$. $\mathcal{T}$ is called orbitally continuous in $\mathcal{E}$ if it is orbitally continuous at every point of $\mathcal{E}$.*
4. *The graph $G(\mathcal{T})$ of $\mathcal{T}$ is defined as $G(\mathcal{T}) = \{(x, y) : x \in \mathcal{E},\, y \in \mathcal{T}x\}$. The graph $G(\mathcal{T})$ of $\mathcal{T}$ is called $\mathcal{T}$-orbitally closed if, for any sequence $\{x_n\}$, we have $(x, x) \in G(\mathcal{T})$ whenever $(x_n, x_{n+1}) \in G(\mathcal{T})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$.*

Wardowski [25] introduced a new type of contractions which he called F-contractions. Several authors proved various variants of fixed point results using such contractions. In particular, Acar and Altun proved in [1] a fixed point theorem for multivalued mappings under δ-distances.

Acclimatizing Wardowski's approach to b-metric space, Cosentino et al. used in [7] the set of functions $\mathfrak{F}_s$ defined as follows:

Definition 2. *Let $s \geq 1$ be a real number. We denote by $\mathfrak{F}_s$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:*

($F1$) *F is strictly increasing,*
($F2$) *for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,*
($F3$) *for each sequence $\{\alpha_n\}$ of positive numbers with $\lim_{n\to\infty} \alpha_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0$,*

$(F4)$ *there exists* $\tau \in \mathbb{R}^+$ *such that for each sequence* $\{\alpha_n\}$ *of positive numbers, if* $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ *for all* $n \in \mathbb{N}$, *then* $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$ *for all* $n \in \mathbb{N}$.

Example 1. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln\alpha$ or $F(\alpha) = \alpha + \ln\alpha$. It can be easily checked [7, Example 3.2] that F satisfies the properties (F1)-(F4).

Cosentino et al. [7] proved the following theorem (note that $\mathcal{H}_b$ here denotes the b-Hausdorff-Pompeiu metric):

Theorem 1 ([7], Theorem 3.4)**.** *Let* $(\mathcal{E}, d_b, s)$ *be a complete b-metric space and let* $\mathcal{T}\colon \mathcal{E} \to \mathcal{P}_{cb}(\mathcal{E})$. *Assume that there exists a continuous from the right function* $F \in \mathfrak{F}_s$ *and* $\tau \in \mathbb{R}^+$ *such that*

$$2\tau + F(s\mathcal{H}_b(\mathcal{T}x, \mathcal{T}y)) \leq F(d_b(x, y)), \tag{1}$$

for all $x, y \in \mathcal{E}$, $\mathcal{T}x \neq \mathcal{T}y$. *Then* $\mathcal{T}$ *has a fixed point.*

3. Result-1

In this section, we prove results on multivalued almost (GF, δ_b)-contraction in a b-metric space. For this, we first introduce the notion of multivalued almost (GF, δ_b)-contraction in a b-metric space and then derive fixed point results.

To define multivalued almost (GF, δ_b)-contraction, we need following family of new functions (see also, [11]).

Let Δ_G denotes the set of all functions $G : \left(\mathbb{R}_0^+\right)^4 \to \mathbb{R}_0^+$ satisfying:

(G) there exists $\tau > 0$ such that $\lim_{n\to\infty} G(a, b, c, \varepsilon_n) = \tau$ for all $a, b, c \in \mathbb{R}_0^+$ and for every sequence $\{\varepsilon_n\} \subset \mathbb{R}_0^+$ with $\lim_{n\to\infty} \varepsilon_n = 0$.

Example 2. If $G(a, b, c, d) = L\min\{a, b, c, d\} + \tau$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Example 3. If $G(a, b, c, d) = \tau e^{L\min\{a,b,c,d\}}$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Delta_G$.

Example 4. If $G(a, b, c, d) = L\ln(\min\{a, b, c, d\} + 1) + \tau$ where $L \in \mathbb{R}^+$ $\tau > 0$, then $G \in \Delta_G$.

Example 5. If $G(a, b, c, d) = \tau - \dfrac{\tau d}{L + d}$ where $L \in \mathbb{R}^+$ $\tau > 0$, then $G \in \Delta_G$.

Definition 3. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$. We say that a multi-valued mapping $\mathcal{T}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ is a multi-valued almost (GF, δ_b)-contraction if $F \in \mathfrak{F}_s$, $G \in \Delta_G$ and there exists $\lambda \geq 0$ such that*

$$\begin{aligned}(2)\qquad & G(\mathcal{D}_b(x,\mathcal{T}x),\mathcal{D}_b(y,\mathcal{T}y),\mathcal{D}_b(x,\mathcal{T}y),\mathcal{D}_b(y,\mathcal{T}x)) + F(s\delta_b(\mathcal{T}x,\mathcal{T}y)) \\ & \qquad \leq F(\Theta_1(x,y) + \lambda\Theta_2(x,y)),\end{aligned}$$

for all $x, y \in \mathcal{E}$ with $\min\{\delta_b(\mathcal{T}x,\mathcal{T}y), d_b(x,y)\} > 0$, where

$$(3)\quad \Theta_1(x,y) = \max\left\{\begin{array}{c} d_b(x,y), \mathcal{D}_b(x,\mathcal{T}x), \mathcal{D}_b(y,\mathcal{T}y), \frac{\mathcal{D}_b(x,\mathcal{T}y)+\mathcal{D}_b(y,\mathcal{T}x)}{2s}, \\ \frac{\mathcal{D}_b(y,\mathcal{T}y)[1+\mathcal{D}_b(x,\mathcal{T}x)]}{s[1+d_b(x,y)]}, \frac{\mathcal{D}_b(y,\mathcal{T}x)[1+\mathcal{D}_b(x,\mathcal{T}x)]}{s[1+d_b(x,y)]} \end{array}\right\}$$

and

$$\Theta_2(x,y) = \min\{\mathcal{D}_b(x,\mathcal{T}x), \mathcal{D}_b(y,\mathcal{T}y), \mathcal{D}_b(x,\mathcal{T}y), \mathcal{D}_b(y,\mathcal{T}x)\}.$$

If (2) is satisfied just for $x, y \in \overline{\mathcal{O}(x_0;\mathcal{T})}$ (for some $x_0 \in \mathcal{E}$), we say that $\mathcal{T}$ is a multi-valued almost orbitally (GF, δ_b)-contraction at x_0.

Now we are position to state our first main result.

Theorem 2. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and let $\mathcal{T}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be a multi-valued almost orbitally (GF, δ_b)-contraction at $x_0 \in \mathcal{E}$. Suppose that $(\mathcal{E}, d_b, s)$ is $\mathcal{T}$-orbitally complete at x_0. If F is continuous and $\mathcal{T}x$ is closed for all $x \in \overline{\mathcal{O}(x_0;\mathcal{T})}$; or $\mathcal{T}$ has $\mathcal{T}$-orbitally closed graph, then $\mathcal{T}$ has a fixed point in $\mathcal{E}$.*

Proof. Since $\mathcal{T}: \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$, let $x_1 \in \mathcal{T}x_0$. Continuing in this manner, we can choose a sequence $\{x_n\}$ in $\mathcal{E}$ such that $x_{n+1} \in \mathcal{T}x_n$, for all $n \in \mathbb{N}\cup\{0\}$. Now, if $x_{n_0} \in \mathcal{T}x_{n_0}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of $\mathcal{T}$ and the proof is finished. Therefore, we assume that $x_n \notin \mathcal{T}x_n$, i.e., $x_n \neq x_{n+1}$ for all $n \geq 0$. So $d_b(x_n,\ x_{n+1}) > 0$ and $\delta_b(\mathcal{T}x_n, \mathcal{T}x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Using the condition (2) for the elements $x = x_n$, $y = x_{n+1}$, arbitrary $n \in \mathbb{N} \cup \{0\}$ we obtain

$$\begin{aligned}(4)\qquad & G(\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(x_{n+1},\mathcal{T}x_{n+1}),\mathcal{D}_b(x_n,\mathcal{T}x_{n+1}),\mathcal{D}_b(x_{n+1},\mathcal{T}x_n)) \\ & \quad + F(s\delta_b(\mathcal{T}x_n,\mathcal{T}x_{n+1})) \\ & \leq F(\Theta_1(x_n,x_{n+1}) + \lambda\Theta_2(x_n,x_{n+1})).\end{aligned}$$

Since $x_{n+1} \in \mathcal{T}x_n$ for all $n \geq 0$, by definition we have

$$\mathcal{D}_b(x_{n+1},\mathcal{T}x_n) = \mathcal{D}_b(\{x_{n+1}\},\mathcal{T}x_n) = \inf\{d_b(x_{n+1},z) \colon z \in \mathcal{T}x_n\} = 0.$$

Again, since $x_{n+1} \in \mathcal{T}x_n$ for all $n \geq 0$, by definition we have

$$\begin{aligned} d_b(x_{n+1},x_{n+2}) &\leq \sup\{d_b(u,v) : u \in \mathcal{T}x_n,\ v \in \mathcal{T}x_{n+1}\} \\ &= \delta_b(\mathcal{T}x_n,\mathcal{T}x_{n+1}).\end{aligned}$$

Therefore, it follows from the inequality (4) and (F1) that

$$
\begin{aligned}
(5)\qquad & G(\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(x_{n+1},\mathcal{T}x_{n+1}),\mathcal{D}_b(x_n,\mathcal{T}x_{n+1}),0)\\
&\quad + F(sd_b(x_{n+1},\ x_{n+2}))\\
&\leq G(\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(x_{n+1},\mathcal{T}x_{n+1}),\mathcal{D}_b(x_n,\mathcal{T}x_{n+1}),0)\\
&\quad + F(s\delta_b(\mathcal{T}x_n,\mathcal{T}x_{n+1}))\\
&\leq F(\Theta_1(x_n,x_{n+1})+\lambda\Theta_2(x_n,x_{n+1}))
\end{aligned}
$$

where

$$
\begin{aligned}
&\Theta_1(x_n,x_{n+1})\\
&\quad=\max\left\{\begin{array}{c} d_b(x_n,x_{n+1}),\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(x_{n+1},\mathcal{T}x_{n+1}),\\ \frac{1}{2s}[\mathcal{D}_b(x_n,\mathcal{T}x_{n+1})+\mathcal{D}_b(x_{n+1},\mathcal{T}x_n)]\\ \frac{\mathcal{D}_b(x_{n+1},\mathcal{T}x_{n+1})[1+\mathcal{D}_b(x_n,\mathcal{T}x_n)]}{s[1+d_b(x_n,x_{n+1})]},\frac{\mathcal{D}_b(x_{n+1},\mathcal{T}x_n)[1+\mathcal{D}_b(x_n,\mathcal{T}x_n)]}{s[1+d_b(x_n,x_{n+1})]}\end{array}\right\}\\
&\quad\leq\max\left\{\begin{array}{c} d_b(x_n,x_{n+1}),d_b(x_n,x_{n+1}),d_b(x_{n+1},x_{n+2}),\\ \frac{1}{2s}d_b(x_n,x_{n+2})),\frac{1}{s}d_b(x_{n+1},x_{n+2}))\end{array}\right\}\\
&\quad=\max\left\{d_b(x_n,x_{n+1}),d_b(x_{n+1},x_{n+2}),\frac{1}{2s}d_b(x_n,x_{n+2})\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Theta_2(x_n,x_{n+1})\\
&\quad=\min\{\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(x_{n+1},\mathcal{T}x_{n+1}),\mathcal{D}_b(x_n,\mathcal{T}x_{n+1}),\mathcal{D}_b(x_{n+1},\mathcal{T}x_n)\}\\
&\quad=0.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{1}{2s}d_b(x_n,x_{n+2}) &\leq \frac{1}{2}[d_b(x_n,x_{n+1})+d_b(x_{n+1},x_{n+2})]\\
&\leq \max\{d_b(x_n,x_{n+1}),d_b(x_{n+1},x_{n+2})\}.
\end{aligned}
$$

Also, since $G\in\Delta_G$, there exist $\tau>0$, such that

$$G(d_b(x_n,x_{n+1}),d_b(x_{n+1},x_{n+2}),d_b(x_n,x_{n+2}),0)=\tau.$$

Therefore the above inequalities with (5) yields

$$(6)\qquad \tau+F(sd_b(x_{n+1},x_{n+2}))\leq F(\max\{d_b(x_n,x_{n+1}),d_b(x_{n+1},x_{n+2})\}).$$

Suppose that $d_b(x_n,x_{n+1})\leq d_b(x_{n+1},x_{n+2})$, for some $n\in\mathbb{N}$. Then from (6), we have

$$\tau+F(sd_b(x_{n+1},x_{n+2}))\leq F(d_b(x_{n+1},x_{n+2})),$$

which with (F1) yields a contradiction (since $\tau > 0$). Hence, we must have

$$\max\{d_b(x_n, x_{n+1}), d_b(x_{n+1}, x_{n+2})\} = d_b(x_n, x_{n+1}),$$

and consequently from (6) we have

$$\tau + F(sd_b(x_{n+1}, x_{n+2})) \leq F(d_b(x_n, x_{n+1})) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{7}$$

It follows by (7) and the property (F4) that

$$\tau + F(s^n d_b(x_n, x_{n+1})) \leq F(s^{n-1} d_b(x_{n-1}, x_n)) \text{ for all } n \in \mathbb{N}. \tag{8}$$

Denote $\varrho_n = d_b(x_n, x_{n+1})$ for $n = 0, 1, 2, \ldots$. Then, $\varrho_n > 0$ for all n and, using (8), the following holds:

$$F(s^n \varrho_n) \leq F(s^{n-1} \varrho_{n-1}) - \tau \leq F(s^{n-2} \varrho_{n-2}) - 2\tau \leq \cdots \leq F(\varrho_0) - n\tau \tag{9}$$

for all $n \in \mathbb{N}$. From (9), we get $F(s^n \varrho_n) \to -\infty$ as $n \to \infty$. Thus, from (F2), we have

$$s^n \varrho_n \to 0 \quad \text{as } n \to \infty. \tag{10}$$

Now, by the property (F3) there exists $k \in (0,1)$ such that

$$\lim_{n \to \infty} (s^n \varrho_n)^k F(s^n \varrho_n) = 0. \tag{11}$$

By (9), the following holds for all $n \in \mathbb{N}$:

$$(s^n \varrho_n)^k F(s^n \varrho_n) - (s^n \varrho_n)^k F(\varrho_0) \leq (s^n \varrho_n)^k (-n\tau) \leq 0. \tag{12}$$

Passing to the limit as $n \to \infty$ in (12) and using (10) and (11), we obtain

$$\lim_{n \to \infty} n (s^n \varrho_n)^k = 0$$

and hence $\lim_{n \to \infty} n^{1/k} s^n \varrho_n = 0$. Now, the last limit implies that the series $\Sigma_{n=1}^{\infty} s^n \varrho_n$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{T})$. Since $\mathcal{E}$ is $\mathcal{T}$-orbitally complete, there exists a $z \in \mathcal{E}$ such that

$$x_n \to z \text{ as } n \to \infty.$$

We shall show that $z \in \mathcal{T}z$, i.e., z is a fixed point of $\mathcal{T}$.

First, suppose that F is continuous and $\mathcal{T}z$ is closed.

We observe that, if there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $x_{n_k} \in \mathcal{T}z$ for all $k \in \mathbb{N}$, since $\mathcal{T}z$ is closed and $\lim_{k \to \infty} x_{n_k} = z$, we deduce that $z \in \mathcal{T}z$ and hence the proof is completed. Therefore, we assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin \mathcal{T}z$ for all $n \in \mathbb{N}$ with $n > n_0$. It follows that $\delta_b(\mathcal{T}x_n, \mathcal{T}z) > 0$ for all $n > n_0$. Using the condition (2) for $x = x_n$, $y = z$, we have

$$\begin{aligned}(13)\qquad & G(\mathcal{D}_b(x_n, \mathcal{T}x_n), \mathcal{D}_b(z, \mathcal{T}z), \mathcal{D}_b(x_n, \mathcal{T}z), \mathcal{D}_b(z, \mathcal{T}x_n)) \\ & \quad + F(s\delta_b(\mathcal{T}x_n, \mathcal{T}z)) \\ & \leq F(\Theta_1(x_n, z) + \lambda\Theta_2(x_n, z))).\end{aligned}$$

Since $x_{n+1} \in \mathcal{T}x_n$ we have

$$(14)\qquad \mathcal{D}_b(x_{n+1}, \mathcal{T}z) = \inf\{d_b(x_{n+1}, v) : v \in \mathcal{T}z\} \leq \delta_b(\mathcal{T}x_n, \mathcal{T}z),$$

$$(15)\qquad \mathcal{D}_b(z, \mathcal{T}x_n) = \inf\{d_b(z, u) : u \in \mathcal{T}x_n, \} \leq d_b(z, x_{n+1}).$$

So, it follows from the inequality (14) and (F1) that $F(s\mathcal{D}_b(x_{n+1}, \mathcal{T}z)) \leq F(s\delta_b(\mathcal{T}x_n, \mathcal{T}z))$ which with (13) yields

$$\begin{aligned}(16)\qquad & G(\mathcal{D}_b(x_n, \mathcal{T}x_n), \mathcal{D}_b(z, \mathcal{T}z), \mathcal{D}_b(x_n, \mathcal{T}z), \mathcal{D}_b(z, \mathcal{T}x_n)) \\ & \quad + F(s\mathcal{D}_b(x_{n+1}, \mathcal{T}z)) \\ & \leq F(\Theta_1(x_n, z) + \lambda\Theta_2(x_n, z))\end{aligned}$$

where

$$\begin{aligned}\Theta_1(x_n, z) &= \max\left\{\begin{array}{c} d_b(x_n, z), \mathcal{D}_b(x_n, \mathcal{T}x_n), \mathcal{D}_b(z, \mathcal{T}z), \frac{\mathcal{D}_b(x_n, \mathcal{T}z) + \mathcal{D}_b(z, \mathcal{T}x_n)}{2s} \\ \frac{\mathcal{D}_b(z, \mathcal{T}z)[1 + \mathcal{D}_b(x_n, \mathcal{T}x_n)]}{s[1 + d_b(x_n, z)]}, \frac{\mathcal{D}_b(z, \mathcal{T}x_n)[1 + \mathcal{D}_b(x_n, \mathcal{T}x_n)]}{s[1 + d_b(x_n, z)]} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d_b(x_n, z), d_b(x_n, x_{n+1}), \mathcal{D}_b(z, \mathcal{T}z), \frac{\mathcal{D}_b(x_n, \mathcal{T}z) + d_b(z, x_{n+1})}{2s} \\ \frac{\mathcal{D}_b(z, \mathcal{T}z)[1 + \mathcal{D}_b(x_n, x_{n+1})]}{s[1 + d_b(x_n, z)]}, \frac{\mathcal{D}_b(z, x_{n+1})[1 + \mathcal{D}_b(x_n, x_{n+1})]}{s[1 + d_b(x_n, z)]} \end{array}\right\} \\ &\to \mathcal{D}_b(z, \mathcal{T}z), \quad \text{as } n \to \infty,\end{aligned}$$

and

$$\begin{aligned}\Theta_2(x_n, z) &= \min\left\{ \mathcal{D}_b(x_n, \mathcal{T}x_n), \mathcal{D}_b(z, \mathcal{T}z), \mathcal{D}_b(x_n, \mathcal{T}z), \mathcal{D}_b(z, \mathcal{T}x_n) \right\} \\ &\leq \min\left\{ d_b(x_n, x_{n+1}), \mathcal{D}_b(z, \mathcal{T}z), \mathcal{D}_b(x_n, \mathcal{T}z), d_b(z, x_{n+1}) \right\} \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Since $\lim_{n\to\infty} d_b(z, x_n) = 0$, from inequality (15) and the property (G), there exist $\tau > 0$ such that

$$\lim_{n\to\infty} G(\mathcal{D}_b(x_n, \mathcal{T}x_n), \mathcal{D}_b(z, \mathcal{T}z), \mathcal{D}_b(x_n, \mathcal{T}z), \mathcal{D}_b(z, \mathcal{T}x_n)) = \tau.$$

Therefore, it follows from the continuity of F and the inequality (16) that

$$F\left(s \lim_{n\to\infty} \mathcal{D}_b(x_{n+1}, \mathcal{T}z)\right) \leq F(\mathcal{D}_b(z, \mathcal{T}z)) - \tau.$$

Also, since F is strictly increasing and $\tau > 0$, it follows from the above inequality that

$$s \lim_{n\to\infty} \mathcal{D}_b(x_{n+1}, \mathcal{T}z) < \mathcal{D}_b(z, \mathcal{T}z). \tag{17}$$

On the other hand,

$$\mathcal{D}_b(z, \mathcal{T}z) \leq s[d_b(z, x_{n+1}) + \mathcal{D}_b(x_{n+1}, \mathcal{T}z)].$$

Letting $n \to \infty$ and using (25) we have

$$\mathcal{D}_b(z, \mathcal{T}z) \leq s \lim_{n\to\infty} \mathcal{D}_b(x_{n+1}, \mathcal{T}z) < \mathcal{D}_b(z, \mathcal{T}z).$$

This contradiction shows that $\mathcal{D}_b(z, \mathcal{T}z) = 0$, and, since $\mathcal{T}z$ is closed, we have $z \in \mathcal{T}z$. Thus, z is a fixed point of $\mathcal{T}$.

Now, suppose that $G(\mathcal{T})$ is $\mathcal{T}$-orbitally closed.

Since $(x_n, x_{n+1}) \in G(\mathcal{T})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = z$, we have $(z, z) \in G(\mathcal{T})$ by the $\mathcal{T}$-orbitally closedness. Hence, $z \in \mathcal{T}z$. Thus, z is a fixed point of $\mathcal{T}$. ■

Example 6. Let $\mathcal{E} = [0, 1]$ and define a function $d_b : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_0^+$ by

$$d_b(x, x) = 0, d_b(x, y) = [\max\{x, y\}]^2 + (x - y)^2 \quad \text{for} \quad x \neq y.$$

Then $(\mathcal{E}, d_b, s)$ is a b-metric space with $s = 2$. Define a mapping $\mathcal{T} \colon \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ by

$$\mathcal{T}x = \begin{cases} \left\{\frac{x}{8}, \frac{x}{4}\right\}, & \text{if} \quad x \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ \{1\}, & \text{otherwise}. \end{cases}$$

Then it is easy to see that $\mathcal{E}$ is $\mathcal{T}$-orbitally complete at $x_0 \in \mathcal{E}$ for arbitrary $x_0 \in \mathcal{E}$. Let $\tau > 0$, $G(a, b, c, d) = \tau$ for all $a, b, c, d \in \mathbb{R}^+$ and $F(t) = \ln(t)$ for all $t \in \mathbb{R}^+$, then F is continuous and $F \in \mathfrak{F}_s$. One can see that $\mathcal{T}$ is a multi-valued almost orbitally (GF, δ_b)-contraction at $x_0 \in \mathcal{E}$ with $\tau = \ln\left(\frac{5}{4}\right)$ and $\lambda = 5(2)^{10}$. Thus, all the conditions of Theorem 2 are satisfied and we can conclude the existence of fixed point of $\mathcal{T}$. Indeed, $1 \in \mathcal{T}1$. On the other hand, Theorem 1 is not applicable here. Indeed, the contractive condition (1) is not satisfied, e.g., if $y = \frac{1}{2}$ and $x \in \left(\frac{1}{4}, \frac{1}{2}\right)$ and $F \in \mathfrak{F}_s$, then we have (see, [7]):

$$\begin{aligned} F(s\mathcal{H}_b(\mathcal{T}x, \mathcal{T}y)) &= F\left(2\mathcal{H}_b\left(\left\{\frac{x}{8}, \frac{x}{4}, \right\}, \{1\}\right)\right) \\ &= F\left(2\max\left\{\sup\left\{1 + \left(1 - \frac{x}{8}\right)^2, 1 + \left(1 - \frac{x}{4}\right)^2\right\},\right.\right. \\ &\qquad \left.\left.\inf\left\{1 + \left(1 - \frac{x}{8}\right)^2, 1 + \left(1 - \frac{x}{4}\right)^2\right\}\right\}\right) \\ &= F\left(2 + 2\left(1 - \frac{x}{8}\right)^2\right). \end{aligned}$$

And $F(d_b(x,y)) = F\left(d_b\left(x,\frac{1}{2}\right)\right) = F\left(\frac{1}{4}+\left(\frac{1}{2}-x\right)^2\right)$. Therefore, by the property (F1) we have $F(s\mathcal{H}_b(\mathcal{T}x,\mathcal{T}y)) > F(d_b(x,y))$. Thus, the contractive condition (1) is not satisfied.

Combining Theorem 2 and Example 2 with $F(\alpha)=\ln\alpha$ in (2), we get the following corollary:

Corollary 1. *Let $(\mathcal{E},d_b,s)$ be a b-metric space with $s>1$ and let $\mathcal{T}\colon\mathcal{E}\to\mathcal{P}_b(\mathcal{E})$ be a multivalued mapping satisfying, for some $\tau>0$, $x_0\in\mathcal{E}$, $\lambda\geq 0$, the condition*

$$\begin{aligned} s\delta_b(\mathcal{T}x,\mathcal{T}y) \ \leq\ & e^{-L\min\{\mathcal{D}_b(x,\mathcal{T}x),\mathcal{D}_b(y,\mathcal{T}y),\mathcal{D}_b(x,\mathcal{T}y),\mathcal{D}_b(y,\mathcal{T}x)\}-\tau} \\ & \times\{\Theta_1(x,y)+\lambda\Theta_2(x,y)\} \end{aligned}$$

for all $x,y\in\overline{\mathcal{O}(x_0;\mathcal{T})}$ with $\min\{\delta_b(\mathcal{T}x,\mathcal{T}y),d_b(x,y)\}>0$, where Θ_1,Θ_2 are given by (3). Suppose that $(\mathcal{E},d_b,s)$ is $\mathcal{T}$-orbitally complete at x_0. If $\mathcal{T}x$ is closed for all $x\in\overline{\mathcal{O}(x_0;\mathcal{T})}$; or $\mathcal{T}$ has $\mathcal{T}$-orbitally closed graph, then $\mathcal{T}$ has a fixed point in $\mathcal{E}$.

Combining Theorem 2 and Example 3 with $F(\alpha)=\alpha+\ln\alpha$ in (2), we get following corollary:

Corollary 2. *Let $(\mathcal{E},d_b,s)$ be a b-metric space with $s>1$ and let $\mathcal{T}\colon\mathcal{E}\to\mathcal{P}_b(\mathcal{E})$ be a multivalued mapping satisfying, for some $\tau>0$, $x_0\in\mathcal{E}$, $L,\lambda\geq 0$, the condition*

$$\begin{aligned} & \frac{s\delta_b(\mathcal{T}x,\mathcal{T}y)}{\Theta_1(x,y)+\lambda\Theta_2(x,y)}\, e^{s\delta_b(\mathcal{J}x,\mathcal{T}y)-(\Theta_1(x,y)+\lambda\Theta_2(x,y))} \\ & \qquad\leq e^{-\tau L\min\{\mathcal{D}_b(x,\mathcal{T}x),\mathcal{D}_b(y,\mathcal{T}y),\mathcal{D}_b(x,\mathcal{T}y),\mathcal{D}_b(y,\mathcal{T}x)\}} \end{aligned}$$

for all $x,y\in\overline{\mathcal{O}(x_0;\mathcal{T})}$ with $\min\{\delta_b(\mathcal{T}x,\mathcal{T}y),d_b(x,y)\}>0$, where Θ_1,Θ_2 are given by (3). Suppose that $(\mathcal{E},d_b,s)$ is $\mathcal{T}$-orbitally complete at x_0. If $\mathcal{T}x$ is closed for all $x\in\overline{\mathcal{O}(x_0;\mathcal{T})}$; or $\mathcal{T}$ has $\mathcal{T}$-orbitally closed graph, then $\mathcal{T}$ has a fixed point in $\mathcal{E}$.

The following corollary is a special case of Theorem 2 when $\mathcal{T}$ is a single-valued mapping.

Corollary 3. *Let $(\mathcal{E},d_b,s)$ be a b-metric space with $s>1$ and let $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ be a self-mapping such that $\mathcal{E}$ is $\mathcal{T}$-orbitally complete at some $x_0\in\mathcal{E}$. Suppose that $F\in\mathfrak{F}_s$, $G\in\Delta_G$, there exist $\lambda\geq 0$ such that*

$$\begin{aligned} G(d_b(x,y),d_b(x,\mathcal{T}x),d_b(y,\mathcal{T}y),d_b(x,\mathcal{T}y),d_b(y,\mathcal{T}x)) & \\ +F(sd_b(\mathcal{T}x,\mathcal{T}y)) & \\ \leq F(\Theta_1'(x,y)+\lambda\Theta_2'(x,y)), & \end{aligned} \tag{18}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{T})}$ *with* $\min\{d_b(\mathcal{T}x, \mathcal{T}y), d_b(x, y)\} > 0$*, where*

$$\Theta_1'(x, y) = \max \left\{ \begin{array}{c} d_b(x, y), d_b(x, \mathcal{T}x), d_b(y, \mathcal{T}y), \frac{d_b(x,\mathcal{T}y)+d_b(y,\mathcal{T}x)}{2s} \\ \frac{d_b(y,\mathcal{T}y)[1+d_b(x,\mathcal{T}x)]}{s[1+d_b(x,y)]}, \frac{d_b(y,\mathcal{T}x)[1+d_b(x,\mathcal{T}x)]}{s[1+d_b(x,y)]} \end{array} \right\}$$

and

$$\Theta_2'(x, y) = \min\{d_b(x, \mathcal{T}x), d_b(y, \mathcal{T}y), d_b(x, \mathcal{T}y), d_b(y, \mathcal{T}x)\}.$$

If $\mathcal{T}$ *is continuous, then* $\mathcal{T}$ *has a fixed point in* $\mathcal{E}$*.*

4. Result-2

In this section, we prove a fixed point result on an ordered b-metric space with some relaxed contractive conditions on the self-mappings of the space. For this purpose, we apply some additional conditions on the space.

Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$ and $\preceq$ be a partial order on $\mathcal{E}$. We say that a multi-valued mapping $\mathcal{T} \colon \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ is a multi-valued ordered almost (GF, δ_b)-contraction if $F \in \mathfrak{F}_s$, $G \in \Delta_G$ and there exists $\lambda \geq 0$ such that

$$\begin{aligned} & G(\mathcal{D}_b(x, \mathcal{T}x), \mathcal{D}_b(y, \mathcal{T}y), \mathcal{D}_b(x, \mathcal{T}y), \mathcal{D}_b(y, \mathcal{T}x)) \\ & \quad + F(s\delta_b(\mathcal{T}x, \mathcal{T}y)) \\ & \quad \leq F(\Theta_1(x, y) + \lambda\Theta_2(x, y)), \end{aligned} \tag{19}$$

for all $x, y \in \mathcal{E}$ with $\min\{\delta_b(\mathcal{T}x, \mathcal{T}y), d_b(x, y)\} > 0$ and $x \preceq y$, where

$$\Theta_1(x, y) = \max \left\{ \begin{array}{c} d_b(x, y), \mathcal{D}_b(x, \mathcal{T}x), \mathcal{D}_b(y, \mathcal{T}y), \frac{\mathcal{D}_b(x,\mathcal{T}y)+\mathcal{D}_b(y,\mathcal{T}x)}{2s}, \\ \frac{\mathcal{D}_b(y,\mathcal{T}y)[1+\mathcal{D}_b(x,\mathcal{T}x)]}{s[1+d_b(x,y)]}, \frac{\mathcal{D}_b(y,\mathcal{T}x)[1+\mathcal{D}_b(x,\mathcal{T}x)]}{s[1+d_b(x,y)]} \end{array} \right\} \tag{20}$$

and

$$\Theta_2(x, y) = \min\{\mathcal{D}_b(x, \mathcal{T}x), \mathcal{D}_b(y, \mathcal{T}y), \mathcal{D}_b(x, \mathcal{T}y), \mathcal{D}_b(y, \mathcal{T}x)\}.$$

If (19) is satisfied just for $x, y \in \overline{\mathcal{O}(x_0; \mathcal{T})}$ (for some $x_0 \in \mathcal{E}$) with $x \preceq y$, we say that $\mathcal{T}$ is a multi-valued ordered almost orbitally (GF, δ_b)-contraction at x_0.

The following example shows that the condition (19) on multi-valued ordered almost (GF, δ_b)-contraction, is actually a relaxed contractive condition than the condition (2) on multi-valued almost (GF, δ_b)-contraction .

Example 7. Let $\mathcal{E} = \{0, 1, 2, 3\}$ and define a function $d_b : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_0^+$ by

$$d_b(x, y) = (x - y)^2 \text{ for all } x, y \in \mathcal{E}.$$

Then $(\mathcal{E}, d_b, s)$ is a b-metric space with $s = 2$. Define a mapping $\mathcal{T}\colon \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ by

$$\mathcal{T}0 = \{1,3\}, \quad \mathcal{T}1 = \{3\}, \quad \mathcal{T}2 = \{1,2\}, \quad \mathcal{T}3 = \{1\}.$$

Define a partial order $\preceq$ on $\mathcal{E}$ by

$$\preceq := \{(0,0),(1,1),(2,2),(3,3),(0,3),(2,3)\}.$$

Then $\overline{\mathcal{O}(0;\mathcal{T})} = \{0,1,3\}$ and it is easy to see that $\mathcal{T}$ is a multi-valued ordered almost (GF, δ_b)-contraction at $0 \in \mathcal{E}$ with $G(a,b,c,d) = \tau$ for all $a,b,c,d \in \mathbb{R}^+$, $0 < \tau < \max\left\{\ln\left(\frac{9}{8}\right), \ln\left(\frac{1+\lambda}{8}\right)\right\}$, where $\lambda > 7$; and $F(t) = \ln(t)$ for all $t \in \mathbb{R}^+$. Indeed, we have to check the validity of (19) only at points $x = y = 0$, $x = y = 1$, $x = y = 3$ and $x = 0$, $y = 3$. Then:

(i) $s\delta_b(\mathcal{T}0, \mathcal{T}0) = 8$ and $\Theta_1(0,0) = 1$, $\Theta_2(x,y) = 1$. Therefore,

$$\tau + \ln(\delta_b(\mathcal{T}0, \mathcal{T}0)) \le \ln(\Theta_1(0,0) + \lambda\Theta_2(0,0)).$$

(ii) $s\delta_b(\mathcal{T}1, \mathcal{T}1) = 0$. Therefore,

$$\tau + \ln(\delta_b(\mathcal{T}1, \mathcal{T}1)) \le \ln(\Theta_1(1,1) + \lambda\Theta_2(1,1)).$$

(iii) $s\delta_b(\mathcal{T}3, \mathcal{T}3) = 0$. Therefore,

$$\tau + \ln(\delta_b(\mathcal{T}3, \mathcal{T}3)) \le \ln(\Theta_1(3,3) + \lambda\Theta_2(3,3)).$$

(iv) $s\delta_b(\mathcal{T}0, \mathcal{T}3) = 8$ and $\Theta_1(0,3) = 9$, $\Theta_2(0,3) = 0$. Therefore,

$$\tau + \ln(\delta_b(\mathcal{T}0, \mathcal{T}0)) \le \ln(\Theta_1(0,0) + \lambda\Theta_2(0,0)).$$

On the other hand, there exists no $x_0 \in \mathcal{E}$ such that $\mathcal{T}$ is a multi-valued almost (GF, δ_b)-contraction at $x_0 \in \mathcal{E}$.

Definition 4. *The mapping $\mathcal{T}$ is called d_b-nondecreasing if*

$$x \preceq y,\ u \in \mathcal{T}x,\ v \in \mathcal{T}y,\ d_b(u,v) < d_b(x,y) \implies u \preceq v.$$

In addition, we consider the following hypothesis on X:

(UC) For every nondecreasing sequence $\{x_n\}$ (with respect to $\preceq$) with $x_n \to x \in \mathcal{E}$ we have $x_n \preceq x$ for all $n \in \mathbb{N}$.

(LC) For every nonincreasing sequence $\{x_n\}$ (with respect to $\preceq$) with $x_n \to x \in \mathcal{E}$ we have $x \preceq x_n$ for all $n \in \mathbb{N}$.

Theorem 3. *Let $(\mathcal{E}, d_b, s)$ be a b-metric space with $s > 1$, $\preceq$ a partial order on $\mathcal{E}$ and let $\mathcal{T}\colon \mathcal{E} \to \mathcal{P}_b(\mathcal{E})$ be a multi-valued ordered almost orbitally (GF, δ_b)-contraction at $x_0 \in \mathcal{E}$. Suppose that $(\mathcal{E}, d_b, s)$ is $\mathcal{T}$-orbitally complete at x_0 and the following conditions are satisfied:*

(a) $\mathcal{T}$ is d_b-nondecreasing;

(b) there exists $x_1 \in \mathcal{E}$ such that $x_1 \in \mathcal{T}x_0$ and $x_0 \preceq x_1$.

If F is continuous, $\mathcal{T}x$ is closed for all $x \in \overline{\mathcal{O}(x_0;\mathcal{T})}$ and the property (UC) holds on $\mathcal{E}$; or $\mathcal{T}$ has $\mathcal{T}$-orbitally closed graph, then $\mathcal{T}$ has a fixed point in $\mathcal{E}$.

Proof. Let $x_1 \in \mathcal{E}$ be such that $x_1 \in \mathcal{T}x_0$ and $x_0 \preceq x_1$. If $x_0 = x_1$, then x_0 is itself a fixed point of $\mathcal{T}$. Assume that $x_0 \neq x_1$, i.e., $d_b(x_0, x_1) > 0$. Choose $x_2 \in \mathcal{T}x_1$. Again, we can assume that $x_1 \neq x_2$, otherwise, x_1 is a fixed point of $\mathcal{T}$, and so, $\delta_b(\mathcal{T}x_0, \mathcal{T}x_1) > 0$. Then using (19) for $x = x_0, y = x_1$ we have

$$
\begin{aligned}
&G(\mathcal{D}_b(x_0,\mathcal{T}x_0),\mathcal{D}_b(x_1,\mathcal{T}x_1),\mathcal{D}_b(x_0,\mathcal{T}x_1),\mathcal{D}_b(x_1,\mathcal{T}x_0))\\
&\qquad + F(s\delta_b(\mathcal{T}x_0,\mathcal{T}x_1))\\
&\quad \leq F(\Theta_1(x_0,x_1)+\lambda\Theta_2(x_0,x_1)).
\end{aligned} \tag{21}
$$

Since $x_1 \in \mathcal{T}x_0$, by definition we have

$$\mathcal{D}_b(x_1,\mathcal{T}x_0) = \mathcal{D}_b(\{x_1\},\mathcal{T}x_0) = \inf\{d_b(x_1,z)\colon z \in \mathcal{T}x_0\} = 0.$$

Since $x_1 \in \mathcal{T}x_0$, $x_2 \in Tx_1$, by definition we have

$$d_b(x_1,x_2) \leq \sup\{d_b(u,v) : u \in \mathcal{T}x_0,\ v \in \mathcal{T}x_1\} = \delta_b(\mathcal{T}x_0,\mathcal{T}x_1).$$

Therefore, it follows from the inequality (21) and (F1) that

$$
\begin{aligned}
&G(\mathcal{D}_b(x_0,\mathcal{T}x_0),\mathcal{D}_b(x_1,\mathcal{T}x_1),\mathcal{D}_b(x_0,\mathcal{T}x_1),0) + F(sd_b(x_1,\ x_2))\\
&\quad \leq G(\mathcal{D}_b(x_0,\mathcal{T}x_0),\mathcal{D}_b(x_1,\mathcal{T}x_1),\mathcal{D}_b(x_0,\mathcal{T}x_1),0)\\
&\qquad + F(s\delta_b(\mathcal{T}x_0,\mathcal{T}x_1))\\
&\quad \leq F(\Theta_1(x_0,x_1)+\lambda\Theta_2(x_0,x_1))
\end{aligned} \tag{22}
$$

where

$$
\begin{aligned}
\Theta_1(x_0,x_1) &= \max\left\{\begin{array}{c} d_b(x_0,x_1),\mathcal{D}_b(x_0,\mathcal{T}x_0),\mathcal{D}_b(x_1,\mathcal{T}x_1),\\ \frac{1}{2s}[\mathcal{D}_b(x_0,\mathcal{T}x_1)+\mathcal{D}_b(x_1,\mathcal{T}x_0)]\\ \frac{\mathcal{D}_b(x_1,\mathcal{T}x_1)[1+\mathcal{D}_b(x_0,\mathcal{T}x_0)]}{s[1+d_b(x_0,x_1)]},\frac{\mathcal{D}_b(x_1,\mathcal{T}x_0)[1+\mathcal{D}_b(x_0,\mathcal{T}x_0)]}{s[1+d_b(x_0,x_1)]}\end{array}\right\}\\
&\leq \max\left\{d_b(x_0,x_1),d_b(x_0,x_1),d_b(x_1,x_2),\frac{1}{2s}d_b(x_0,x_2),\frac{1}{s}d_b(x_1,x_2),0\right\}\\
&= \max\left\{d_b(x_0,x_1),d_b(x_1,x_2),\frac{1}{2s}d_b(x_0,x_2)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta_2(x_0,x_1) &= \min\{\mathcal{D}_b(x_0,\mathcal{T}x_0),\mathcal{D}_b(x_1,\mathcal{T}x_1),\mathcal{D}_b(x_0,\mathcal{T}x_1),\mathcal{D}_b(x_1,\mathcal{T}x_0)\}\\
&= 0.
\end{aligned}
$$

Now,

$$\begin{aligned}\frac{1}{2s}d_b(x_0,x_2) &\le \frac{1}{2}[d_b(x_0,x_1)+d_b(x_1,x_2)]\\ &\le \max\{d_b(x_0,x_1),d_b(x_1,x_2)\}.\end{aligned}$$

Therefore, following similar arguments to those given in the proof of Theorem 2 we obtain

$$\tau + F(sd_b(x_1,x_2)) \le F(d_b(x_0,x_1)).$$

Since $s \ge 1$ by (F1) we obtain

$$\tau + F(d_b(x_1,x_2)) \le \tau + F(sd_b(x_1,x_2)) \le F(d_b(x_0,x_1)).$$

As $\tau > 0$, again by (F1) we obtain $d_b(x_1,x_2) < d_b(x_0,x_1)$, which with condition (a) implies that $x_1 \preceq x_2$. Repeating this process one can obtain that: for all $n \in \mathbb{N}$

$$\tau + F(sd_b(x_n,x_{n+1})) \le F(d_b(x_{n-1},x_n)) \tag{23}$$

and $x_{n-1} \preceq x_n \in \mathcal{T}x_{n-1}$, for all $n \in \mathbb{N}$.

Now, repeating similar arguments to those given in the proof of Theorem 2 we obtain: there exists a $z \in \mathcal{E}$ such that

$$x_n \to z \quad\text{as}\quad n \to \infty.$$

We shall show that $z \in \mathcal{T}z$, i.e., z is a fixed point of $\mathcal{T}$.

First, suppose that F is continuous, $\mathcal{T}z$ is closed and the property (UC) holds.

Then, we have $x_n \preceq z$ for all $n \in \mathbb{N}$. Without loss of generality, we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin \mathcal{T}z$ for all $n \in \mathbb{N}$ with $n > n_0$. It follows that $\delta_b(\mathcal{T}x_n,\mathcal{T}z) > 0$ for all $n > n_0$. Since $x_n \preceq z$ for all $n \in \mathbb{N}$, following the proof of Theorem 2 we obtain

$$\begin{aligned}&G(\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(z,\mathcal{T}z),\mathcal{D}_b(x_n,\mathcal{T}z),\mathcal{D}_b(z,\mathcal{T}x_n))\\ &\quad + F(s\mathcal{D}_b(x_{n+1},\mathcal{T}z))\\ &\le F(\Theta_1(x_n,z)+\lambda\Theta_2(x_n,z))\end{aligned} \tag{24}$$

where $\Theta_1(x_n,z) \to \mathcal{D}_b(z,\mathcal{T}z)$ and $\Theta_2(x_n,z) \to 0$, as $n \to \infty$.

Since $\lim_{n\to\infty} d_b(z,x_{n+1}) = 0$, we have $\lim_{n\to\infty} \mathcal{D}_b(z,\mathcal{T}x_n) = 0$, and so, by the property (G), there exist $\tau > 0$ such that

$$\lim_{n\to\infty} G(\mathcal{D}_b(x_n,\mathcal{T}x_n),\mathcal{D}_b(z,\mathcal{T}z),\mathcal{D}_b(x_n,\mathcal{T}z),\mathcal{D}_b(z,\mathcal{T}x_n)) = \tau.$$

Therefore, it follows from the continuity of F and the inequality (24) that

$$F\left(s\lim_{n\to\infty}\mathcal{D}_b(x_{n+1},\mathcal{T}z)\right)\le F(\mathcal{D}_b(z,\mathcal{T}z))-\tau.$$

which with (F1) implies that

$$s\lim_{n\to\infty}\mathcal{D}_b(x_{n+1},\mathcal{T}z)<\mathcal{D}_b(z,\mathcal{T}z). \tag{25}$$

On the other hand,

$$\mathcal{D}_b(z,\mathcal{T}z)\le s[d_b(z,x_{n+1})+\mathcal{D}_b(x_{n+1},\mathcal{T}z)].$$

Letting $n\to\infty$ and using (25) we have

$$\mathcal{D}_b(z,\mathcal{T}z)\le s\lim_{n\to\infty}\mathcal{D}_b(x_{n+1},\mathcal{T}z)<\mathcal{D}_b(z,\mathcal{T}z).$$

This contradiction shows that $\mathcal{D}_b(z,\mathcal{T}z)=0$, and, since $\mathcal{T}z$ is closed, we have $z\in\mathcal{T}z$. Thus, z is a fixed point of $\mathcal{T}$.

Now, suppose that $G(\mathcal{T})$ is $\mathcal{T}$-orbitally closed.

Since $(x_n,x_{n+1})\in G(\mathcal{T})$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}x_n=z$, we have $(z,z)\in G(\mathcal{T})$ by the $\mathcal{T}$-orbitally closedness. Hence, $z\in\mathcal{T}z$. Thus, z is a fixed point of $\mathcal{T}$. ∎

Let $\Delta_{G'}$ denotes the set of all functions $G':\left(\mathbb{R}_0^+\right)^4\to\mathbb{R}_0^+$ satisfying:

(G') there exists $\tau>0$ such that $\lim_{n\to\infty}G(a,b,\varepsilon_n,c)=\tau$ for all $a,b,c\in\mathbb{R}_0^+$ and for every sequence $\{\varepsilon_n\}\subset\mathbb{R}_0^+$ with $\lim_{n\to\infty}\varepsilon_n=0$.

If we replace the class Δ_G by $\Delta_{G'}$ in Theorem 2, then due to the symmetry of the functions $d_b,\mathcal{D}_d,\ \delta_b$, the conclusion of this theorem remains true. While, if we replace the class Δ_G by $\Delta_{G'}$ in the ordered version of Theorem 2, we obtain the following theorem (the proof of this theorem is similar to that of the proof of Theorem 3, therefore we omit it):

Theorem 4. *Let $(\mathcal{E},d_b,s)$ be a b-metric space with $s>1$, $\preceq$ be a partial order on $\mathcal{E}$ and let $\mathcal{T}\colon\mathcal{E}\to\mathcal{P}_b(\mathcal{E})$ be a multi-valued ordered almost orbitally (GF,δ_b)-contraction at $x_0\in\mathcal{E}$. Suppose that $(\mathcal{E},d_b,s)$ is $\mathcal{T}$-orbitally complete at x_0 and the following conditions are satisfied:*

(a) $\mathcal{T}$ is d_b-nondecreasing;

(b) there exists $x_1\in\mathcal{E}$ such that $x_1\in\mathcal{T}x_0$ and $x_1\preceq x_0$.

If F is continuous, $\mathcal{T}x$ is closed for all $x\in\overline{\mathcal{O}(x_0;\mathcal{T})}$ and the property (LC) holds on X or $\mathcal{T}$ has $\mathcal{T}$-orbitally closed graph, then $\mathcal{T}$ has a fixed point in $\mathcal{E}$.

5. Solution of nonlinear integral equation

In this section, we prove an existence theorem for a solution of the following nonlinear integral equation :

$$x(t) = g(t) + \int_a^b \mathcal{M}(t, r, x(r))dr, \tag{26}$$

where $a, b \in \mathbb{R}$ such that $a < b, x \in C[a, b]$ (the set of all continuous functions from $[a, b]$ into $\mathbb{R}$), $g : [a, b] \to \mathbb{R}$ and $\mathcal{M} : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

Now, for $p \geq 1$, consider the b-metric d_b on $C[a, b]$ defined by

$$d_b(x, y) = (\max_{t\in[a,b]} |x(t) - y(t)|)^p = \max_{t\in[a,b]} |x(t) - y(t)|^p \tag{27}$$

for all $x, y \in C[a, b]$. Then $(C[a, b], d_b, 2^{p-1})$ is a complete b-metric space.

Let $\mathcal{E} = C[a, b]$ and let $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ be defined by

$$(\mathcal{T}x)(t) = g(t) + \int_a^b \mathcal{M}(t, r, x(r))dr,$$

for all $x \in \mathcal{E}$ and $t \in [a, b]$.

Theorem 5. *Suppose that the following conditions hold:*

(i) $\mathcal{M} : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing in the third order,*

(ii) there exists a continuous function $\varrho : [a, b] \times [a, b] \to [0, \infty)$ *such that*

$$|\mathcal{M}(t, r, u(r)) - \mathcal{M}(t, r, v(r))|^p \leq \varrho(t, r)$$

for all $t, r \in [a, b]$ *and for all* $u, v \in \mathcal{E}$ *where* $p > 1$*,*

(iii) there exists $G \in \Delta_G$ *such that*

$$\begin{aligned} \sup_{t\in[a,b]} \left(\int_a^b \varrho(t, r)dr\right) &< \frac{1}{2^{p-1}} e^{-G(d_b(u(r),\mathcal{T}u(r)),d_b(v(r),\mathcal{T}v(r)),d_b(u(r),\mathcal{T}v(r)),d_b(v(r),\mathcal{T}u(r)))} \\ &\times \left(\max \left\{ \begin{array}{c} d_b(u(r), v(r)), d_b(u(r), \mathcal{T}u(r)), d_b(v(r), \mathcal{T}v(r)), \\ \frac{d_b(u(r),\mathcal{T}v(r))+d_b(v(r),\mathcal{T}u(r))}{2^p} \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}v(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]}, \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}u(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]} \end{array} \right\} \right. \\ &\left. + \lambda \min \left\{ \begin{array}{c} d_b(u(r), \mathcal{T}u(r)), d_b(v(r), \mathcal{T}v(r)), \\ d_b(u(r), \mathcal{T}v(r))), d_b(v(r), \mathcal{T}u(r))) \end{array} \right\} \right) \end{aligned} \tag{28}$$

for all $u, v \in \mathcal{E}$ *where* $\lambda \geq 0$.

Then the nonlinear integral equation (26) has a solution.

Proof. For all $u, v \in \mathcal{E}$, it follows from (ii)-(iii) that

$$
\begin{aligned}
(29)\quad d_b(\mathcal{T}u,\mathcal{T}v) &\leq \max_{t\in[a,b]} \int_a^b |\mathcal{M}(t,r,u(r)) - \mathcal{M}(t,r,v(r))|^p \, dr \\
&\leq \frac{e^{-G(d_b(u(r),\mathcal{T}u(r)),d_b(v(r),\mathcal{T}v(r)),d_b(u(r),\mathcal{T}v(r)),d_b(v(r),\mathcal{T}u(r)))}}{2^{p-1}} \\
&\quad \times \left(\begin{array}{c} \max \left\{ \begin{array}{c} d_b(u(r),v(r)), d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), \\ \frac{d_b(u(r),\mathcal{T}v(r))+d_b(v(r),\mathcal{T}u(r))}{2^p} \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}v(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]}, \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}u(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]} \end{array} \right\} \\ + \lambda \min \left\{ \begin{array}{c} d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), \\ d_b(u(r),\mathcal{T}v(r))), d_b(v(r),\mathcal{T}u(r))) \end{array} \right\} \end{array} \right).
\end{aligned}
$$

Taking logarithms to (29), we have

$$
\begin{aligned}
&\ln(2^{p-1} d_b(\mathcal{T}u,\mathcal{T}v)) \\
&\leq \ln \left(\begin{array}{c} e^{-G(d_b(u(r),\mathcal{T}u(r)),d_b(v(r),\mathcal{T}v(r)),d_b(u(r),\mathcal{T}v(r)),d_b(v(r),\mathcal{T}u(r)))} \times \\ \left(\begin{array}{c} \max \left\{ \begin{array}{c} d_b(u(r),v(r)), d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), \\ \frac{d_b(u(r),\mathcal{T}v(r))+d_b(v(r),\mathcal{T}u(r))}{2^p} \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}v(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]}, \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}u(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]} \end{array} \right\} \\ + \lambda \min \left\{ \begin{array}{c} d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), \\ d_b(u(r),\mathcal{T}v(r))), d_b(v(r),\mathcal{T}u(r))) \end{array} \right\} \end{array} \right) \end{array} \right).
\end{aligned}
$$

Consider the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(\alpha) = \ln \alpha$, belonging to $\mathfrak{F}_s$ and on routine calculations for $u, v \in \mathcal{E}$, we obtain

$$
\begin{aligned}
&G(d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), d_b(u(r),\mathcal{T}v(r)), d_b(v(r),\mathcal{T}u(r))) \\
&\quad + F(2^{p-1} d_b(\mathcal{T}u,\mathcal{T}v)) \\
&\quad \leq F \left(\begin{array}{c} \max \left\{ \begin{array}{c} d_b(u(r),v(r)), d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), \\ \frac{d_b(u(r),\mathcal{T}v(r))+d_b(v(r),\mathcal{T}u(r))}{2^p} \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}v(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]}, \\ \frac{\mathcal{D}_b(v(r),\mathcal{T}u(r))[1+\mathcal{D}_b(u(r),\mathcal{T}u(r))]}{2^{p-1}[1+d_b(u(r),v(r))]} \end{array} \right\} \\ + \lambda \min \left\{ \begin{array}{c} d_b(u(r),\mathcal{T}u(r)), d_b(v(r),\mathcal{T}v(r)), \\ d_b(u(r),\mathcal{T}v(r))), d_b(v(r),\mathcal{T}u(r))) \end{array} \right\} \end{array} \right),
\end{aligned}
$$

for all $r \in [a, b]$.

Since the above inequality is true for any $r \in [a, b]$, we deduce easily that for all $u, v \in \mathcal{E}$

$$G(d_b(u,\mathcal{T}u), d_b(v,\mathcal{T}v), d_b(u,\mathcal{T}v), d_b(v,\mathcal{T}u)) + F(2^{p-1} d_b(\mathcal{T}u,\mathcal{T}v))$$
$$\leq F\left(\begin{array}{c} \max\left\{\begin{array}{c} d_b(u,v), d_b(u,\mathcal{T}u), d_b(v,\mathcal{T}v), \frac{d_b(u,\mathcal{T}v)+d_b(v,\mathcal{T}u)}{2^p} \\ \frac{\mathcal{D}_b(v,\mathcal{T}v)[1+\mathcal{D}_b(u,\mathcal{T}u)]}{2^{p-1}[1+d_b(u,v)]}, \frac{\mathcal{D}_b(v,\mathcal{T}u)[1+\mathcal{D}_b(u,\mathcal{T}u)]}{2^{p-1}[1+d_b(u,v)]} \end{array}\right\} \\ +\lambda \min\left\{ d_b(u,\mathcal{T}u), d_b(v,\mathcal{T}v), d_b(u,\mathcal{T}v)), d_b(v,\mathcal{T}u)) \right\} \end{array}\right).$$

Thus all the conditions of Corollary 3 are satisfied $s = 2^{p-1}$ and hence f has a fixed point in $\mathcal{E}$ (namely, x^*). It follows that x^* is a solution of the nonlinear integral equation (26). ■

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