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A FIXED POINT THEOREM FOR ψ_F-GERAGHTY CONTRACTION ON METRIC-LIKE SPACES

Abstract. In this paper, we define a new type Geraghty type contraction, ψ_F-Geraghty contraction, and prove a fixed point theorem for this type contraction and give an illustrative example.

Key words: fixed point, ψ_F-Geraghty contraction, metric-like space.

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1. Introduction

It is well known that functional analysis is made up of two main methods which are variational methods and fixed point methods. Variational methods are used to prove the existence of solutions for differential equations. However, fixed point methods are studied by many scholars in different spaces. Especially, in 2012, Amini-Harandi [1] introduced the notion of metric-like spaces which are considered to be interesting generalizations of metric spaces, partial metric spaces and quasi-metric spaces. At the same time, the author proved some fixed point theorems in complete metric-like spaces. Based on the work of Amini-Harandi, Aydi et al. [2] and Karapınar et al. [10] proved some interesting fixed point theorems for Geraghty type contraction in complete metric-like spaces. (See [3, 4, 5, 6, 7, 8] for more results on fixed point theory on metric-like space).

Now we recall some definitions and lemmas about metric-like space.

Definition 1 ([1])**.** *Let X be any nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a metric-like on X if for any $x, y, z \in X$ the following conditions are satisfied:*

(σ_1) $\sigma(x, y) = 0 \Rightarrow x = y,$

(σ_2) $\sigma(x, y) = \sigma(y, x),$

(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y).$

The pair (X, σ) is called a metric-like space.

Each metric-like σ on X generates a T_0 topology τ_p on X which has as a base the family open σ-balls

$$\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 2 ([1])**.** (*i*) *A sequence* $\{x_n\}$ *in a metric-like space* (X,σ) *converges to a point* $x \in X$ *if and only if* $\sigma(x,x) = \lim_{n\to\infty} \sigma(x,x_n)$.

(*ii*) *A sequence* $\{x_n\}$ *in a metric-like space* (X,σ) *is called a Cauchy sequence if* $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ *exists* (*and is finite*).

(*iii*) *A metric-like space* (X,σ) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_p, *to a point* $x \in X$ *such that*

$$\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m).$$

Lemma 1 ([9])**.** *Let* (X,σ) *be a metric-like space. Let* $\{x_n\}$ *be a sequence in* X *such that* $x_n \to x$, *where* $x \in X$ *and* $\sigma(x,x) = 0$. *Then for all* $y \in X$, *we have*

$$\lim_{n\to\infty} \sigma(x_n,y) = \sigma(x,y).$$

Now let $\mathcal{B}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition $\lim_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Also, let Ψ denote to the class of the functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:
- ψ is nondecreasing,
- ψ is continuous,
- $\psi(t) = 0$ if and ony if $t = 0$.

In [10], the authors proved the following particular result .

Theorem 1. *Let* (X,σ) *be a complete metric-like space and* $T : X \to X$ *be a mapping. Suppose that there exists* $\beta \in \mathcal{B}$ *such that*

$$\sigma(Tx,Ty) \leq \beta((\sigma(x,y))\sigma(x,y)$$

for all $x,y \in X$. *Then* T *has a unique fixed point* $u \in X$ *with* $\sigma(u,u) = 0$.

Recently, in [2], the authors consider new type of Geraghty contractions in the class of metric-like spaces and proved a fixed point theorem for this type of contractive mappings such that:

Theorem 2 ([2]). *Let (X, σ) be a complete metric-like space and $T : X \to X$ be a mapping. Suppose that there exists $\beta \in \mathcal{B}$ such that*

$$(1) \qquad \sigma(Tx, Ty) \leq \beta((F(x, y))F(x, y)$$

for all $x, y \in X$ where

$$F(x, y) = \sigma(x, y) + |\sigma(x, Tx) - \sigma(y, Ty)|.$$

Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

In this paper, in the light of the above information, we define a new type Geraghty type contraction, ψ_F-Geraghty contraction, and prove a fixed point theorem for this type contraction. At the end we give an illustrative example.

2. Main results

Definition 3. *Let (X, σ) be a metric-like space. A mapping $T : X \to X$ is said to be a ψ_F-Geraghty contraction on (X, σ) if there exists $\psi \in \Psi$ and $\beta \in \mathcal{B}$ such that*

$$(2) \qquad \psi(\sigma(Tx, Ty)) \leq \beta(\psi(F(x, y)))\psi(F(x, y))$$

for all $x, y \in X$ where

$$F(x, y) = \sigma(x, y) + |\sigma(x, Tx) - \sigma(y, Ty)|.$$

Theorem 3. *Let (X, σ) be a complete metric-like space and $T : X \to X$ be a ψ_F-Geraghty contraction. Then T has a unique fixed point $z \in X$ with $\sigma(z, z) = 0$.*

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. Suppose that $\sigma(x_n, x_{n+1}) = 0$ for some n_0, so the proof is completed. Consequently, we assume that

$$\sigma(x_n, x_{n+1}) \neq 0$$

for all n. From (2), we have

$$(3) \qquad \begin{aligned} \psi(\sigma(x_n, x_{n+1})) &= \psi(\sigma(Tx_{n-1}, Tx_n)) \\ &\leq \beta(\psi(F(x_{n-1}, x_n)))\psi(F(x_{n-1}, x_n)), \quad n \geq 1 \end{aligned}$$

where

$$\begin{aligned} F(x_{n-1}, x_n) &= \sigma(x_{n-1}, x_n) + |\sigma(x_{n-1}, Tx_{n-1}) - \sigma(x_n, Tx_n)| \\ &= \sigma(x_{n-1}, x_n) + |\sigma(x_{n-1}, x_n) - \sigma(x_n, x_{n+1})|. \end{aligned}$$

Take $\sigma_n = \sigma(x_{n-1}, x_n)$ and (3) becomes

(4) $\psi(\sigma_{n+1}) \leq \beta(\psi(\sigma_n + |\sigma_n - \sigma_{n+1}|))\psi((\sigma_n + |\sigma_n - \sigma_{n+1}|)).$

Assume that there exists $n > 0$ such that $\sigma_n \leq \sigma_{n+1}$. From (4), we get

$$\psi(\sigma_{n+1}) \leq \beta(\psi(\sigma_{n+1}))\psi(\sigma_{n+1}) < \psi(\sigma_{n+1})$$

which is a contradiction. Thus for all $n > 0$, $\sigma_n > \sigma_{n+1}$. Hence we deduce that the sequence $\{\sigma_n\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} \sigma_n = r.$$

Now, we shall prove that $r = 0$. Suppose that $r > 0$. From (2), we have

$$\psi(\sigma(x_n, x_{n+1})) \leq \beta(\psi(F(x_{n-1}, x_n)))\psi(F(x_{n-1}, x_n))$$

which implies

$$\psi(\sigma_{n+1}) \leq \beta(\psi(2\sigma_n - \sigma_{n+1}))\psi(2\sigma_n - \sigma_{n+1}).$$

Hence

$$\frac{\psi(\sigma_{n+1})}{\psi(2\sigma_n - \sigma_{n+1})} \leq \beta(\psi(2\sigma_n - \sigma_{n+1})) < 1.$$

This implies that $\lim_{n\to\infty} \beta(\psi(2\sigma_n - \sigma_{n+1})) = 1$. Since $\beta \in \mathcal{B}$ we have

$$\lim_{n \to \infty} \psi(2\sigma_n - \sigma_{n+1}) = 0,$$

which yields

(5) $$r = \lim_{n \to \infty} \sigma(x_n, x_{n+1}) = 0$$

which is a contradiction. So $r = 0$. Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. We will prove that

(6) $$\lim_{n,m \to \infty} \sigma(x_n, x_m) = 0.$$

We argue by contradiction. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

(7) $$\sigma(x_{m(k)}, x_{n(k)}) \geq \varepsilon.$$

Moreover corresponding to $n(k)$ we can choose $m(k)$ in such a way that is the smallest integer with $m(k) > n(k)$ and satisfying (7). Then

(8) $$\sigma(x_{m(k)-1}, x_{n(k)}) < \varepsilon.$$

Using (7) and (8)

$$
\begin{aligned}
\varepsilon &\le \sigma(x_{m(k)}, x_{n(k)}) \\
&\le \sigma(x_{m(k)}, x_{m(k)-1}) + \sigma(x_{m(k)-1}, x_{n(k)}) \\
&< \sigma(x_{m(k)}, x_{m(k)-1}) + \varepsilon.
\end{aligned}
$$

By (5), we get

(9)
$$
\lim_{k\to\infty} \sigma(x_{m(k)}, x_{n(k)}) = \varepsilon.
$$

On the other hand, it is easy to see that

$$
\begin{aligned}
\big|\sigma(x_{m(k)-1}, x_{n(k)-1}) &- \sigma(x_{m(k)}, x_{n(k)})\big| \\
&\le \big|\sigma(x_{m(k)-1}, x_{m(k)}) + \sigma(x_{n(k)}, x_{n(k)-1})\big|.
\end{aligned}
$$

Again by (5) and (9)

(10)
$$
\lim_{k\to\infty} \sigma(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon.
$$

We go back to (2) to have

$$
\begin{aligned}
\psi(\varepsilon) &\le \psi(\sigma(x_{m(k)}, x_{n(k)})) \\
&= \psi(\sigma(Tx_{m(k)-1}, Tx_{n(k)-1})) \\
&\le \beta(\psi(F(x_{m(k)-1}, x_{n(k)-1})))\psi(F(x_{m(k)-1}, x_{n(k)-1}))
\end{aligned}
$$

where

$$
\begin{aligned}
F(x_{m(k)-1}, x_{n(k)-1}) &= \sigma(x_{m(k)-1}, x_{n(k)-1}) \\
&+ \big|\sigma(x_{m(k)-1}, Tx_{m(k)-1}) - \sigma(x_{n(k)-1}, Tx_{n(k)-1})\big|.
\end{aligned}
$$

By (5) and (10)

(11)
$$
\lim_{k\to\infty} F(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon.
$$

We deduce

$$
\lim_{k\to\infty} \beta\left(\psi\left(F(x_{m(k)-1}, x_{n(k)-1})\right)\right) = 1.
$$

Since $\beta \in \mathcal{B}$ we have

$$
\lim_{k\to\infty} F(x_{m(k)-1}, x_{n(k)-1}) = 0
$$

which is a contradiction with respect to (11). Thus $\{x_n\}$ is a Cauchy sequence in the complete metric-like space (X, σ). So there exists $z \in X$ such that

$$
\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{n,m\to\infty} \sigma(x_n, x_m).
$$

By (6), we write

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = 0.$$

We shall prove that z is a fixed point of T. Assume that $z \neq Tz$, so $\sigma(z, Tz) > 0$. From (2), we have

$$(12) \qquad \begin{aligned} \psi(\sigma(x_{n+1}, Tz)) &= \psi(\sigma(Tx_n, Tz)) \\ &\leq \beta(\psi(F(x_n, z)))\psi(F(x_n, z)) \\ &\leq \psi(F(x_n, z)) \end{aligned}$$

where

$$F(x_n, z) = \sigma(x_n, z) + |\sigma(x_n, Tx_n) - \sigma(z, Tz)| \to \sigma(z, Tz)$$

as $n \to \infty$. Letting again $n \to \infty$ in (12) and using Lemma 1

$$\psi(\sigma(z, Tz)) \leq \lim_{n\to\infty} \beta(\psi(F(x_n, z)))\psi(\sigma(z, Tz)) \leq \psi(\sigma(z, Tz)).$$

So $\lim_{n\to\infty} \beta(\psi(F(x_n, z))) = 1$ which implies that

$$\lim_{n\to\infty} F(x_n, z) = 0,$$

which is a contradiction. Thus $\sigma(z, Tz) = 0$ and so $z = Tz$, that is, z is a fixed point of T with $\sigma(z, z) = 0$.

We shall prove that such z verifying $\sigma(z, z) = 0$ is the unique fixed point of T. We argue by contradiction. Assume that there exists $z \neq w$ so $\sigma(z, w) > 0$ such that

$$z = Tz, \quad w = Tw, \quad \sigma(z, z) = \sigma(w, w) = 0.$$

We have

$$\begin{aligned} F(z, w) &= \sigma(z, w) + |\sigma(z, Tz) - \sigma(w, Tw)| \\ &= \sigma(z, w) + |\sigma(z, z) - \sigma(w, w)| \\ &= \sigma(z, w). \end{aligned}$$

By (2), we get

$$\begin{aligned} \psi(\sigma(z, w)) &= \psi(\sigma(Tz, Tw)) \\ &\leq \beta(\psi(F(z, w)))\psi(F(z, w)) \\ &= \beta(\psi(\sigma(z, w)))\psi(\sigma(z, w)) \\ &< \psi(\sigma(z, w)) \end{aligned}$$

which is a contradiction. Thus there exists a unique $z \in X$ such that $z = Tz$ with $\sigma(z, z) = 0$. ∎

Example 1. Let $X = [0,1]$ and $\sigma(x,y) = x+y$. Then (X, σ) is a complete metric-like space. Define $T : X \to X$ by

$$Tx = \left\{ \begin{array}{ll} 0, & x = 1 \\ \frac{x}{2} & x \neq 1 \end{array} \right. .$$

Take $\psi(t) = \frac{t}{2}$ and $\beta(\alpha) = \frac{1}{2}$, then it's clear that T is a ψ_F-Geraghty contraction and T has a fixed point $z = 0$ with $\sigma(z,z) = 0$. But the metric version of (2) dosen't satisfy for $x = 0$, $y = \frac{3}{4}$ and $\psi(t) = \frac{t}{2}, \beta(\alpha) = \frac{1}{2}$. Indeed, let $d(x,y) = |x-y|$,then

$$d(T0, T\frac{3}{4}) = \frac{3}{8} \Rightarrow \psi(d(T0, T\frac{3}{4})) = \frac{3}{16}$$

and

$$
\begin{aligned}
F(0, \frac{3}{4}) &= d(0, \frac{3}{4}) + \left| d(0, T0) - d(\frac{3}{4}, T\frac{3}{4}) \right| \\
&= \frac{3}{4} + \frac{3}{8} = \frac{9}{8}.
\end{aligned}
$$

So, we get

$$\psi(d(T0, T\frac{3}{4})) = \frac{3}{16} \lneqq \frac{1}{2}\frac{9}{16} = \beta(\psi(F(0, \frac{3}{4})))\psi(F(0, \frac{3}{4})).$$

References

[1] AMINI-HARANDI A., Metric-like spaces, partial metric spaces and fixed points, *Fixed Point Theory Appl.,* 2012(2012), 10 pages.

[2] AYDI H., FELHI A., AFSHARI H., New Geraghty type contractions on metric-like spaces, *J. Nonlinear Sci. Appl.,* 10(2017), 780-788.

[3] AYDI H., FELHI A., SAHMIM S., A Suzuki fixed point theorem for generalized multivalued mappings on metric-like spaces, *Glasnik Mathematicki,* 52(72)(2017), 147-161.

[4] AYDI H., FELHI A., On best proximity points for various α-proximal contractions on metric-like spaces, *J. Nonlinear Sci. Appl.,* 9(2016), 5202-5218.

[5] AYDI H., FELHI A., Best proximity points for cyclic Kannan-Chatterjea-Ciric type contractions on metric-like spaces, *J. Nonlinear Sci. Appl.,* 9(2016), 2458-2466.

[6] AYDI H., KARAPINAR E., SALIMI P., ERHAN I.M., Best proximity points of generalized almost ψ-Geraghty contractive non-self mappings, *Fixed Point Theory and Applications,* 2014, 2014:32.

[7] AYDI H., FELHI A., SAHMIM S., Common fixed points via implicit contractions on b-metric-like spaces, *J. Nonlinear Sci. Appl.,* 10(4)(2017), 1524-1537.

[8] AYDI H., KARAPINAR E., Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications, *Electronic Journal of Differential Equations,* 133 (2015), 1-15.

[9] Karapinar E., Salimi P., *Dislocated metric space to metric spaces with some fixed point theorems*, Fixed Point Theory Appl., 2013(2013), 19 pages.

[10] Karapinar E., Alsulami H.H., Noorwali M., *Some extensions for Geragthy type contractive mappings*, Fixed Point Theory Appl., 2015(2015), 22 pages.

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