

MURAT KIRIŞCI

**INTEGRATED AND DIFFERENTIATED SPACES
OF TRIANGULAR FUZZY NUMBERS**

ABSTRACT. Fuzzy sets are the cornerstone of a non-additive uncertainty theory, namely possibility theory, and of a versatile tool for both linguistic and numerical modeling. Fuzzy sets have become popular in every branch of mathematics such as analysis, topology, algebra, applied mathematics etc. Thus fuzzy sets triggered the creation of a wide range of research topics in all areas of science in a short time. In this paper, we use the triangular fuzzy numbers for matrix domains of sequence spaces with infinite matrices. We construct the new space with triangular fuzzy numbers and investigate to structural, topological and algebraic properties of these spaces.

KEY WORDS: fuzzy numbers, integrated matrix, differentiated matrix, real duals.

AMS Mathematics Subject Classification: 03E72, 46A45, 40C05.

1. Introduction

Fuzziness has revolutionized many areas such as mathematics, science, engineering, medicine. This concept was initiated by Zadeh [17]. Not only Zadeh discovered this concept, but he also developed the infrastructure of today's popular forms of use such as relations of similarity, decision making, and fuzzy programming in a short time.

Just like in the theory of sets, fuzzy sets(FS) also have led to the emergence of new mathematical concepts, research topics, and the design of engineering applications. Therefore, the nature of the classical set theory must be well known and understood. In particular, consider the two fundamental laws of Boolean algebra the law of excluded middle and law of contradiction. In logic, the proposition every proposition is either true or false excludes any third, or middle, possibility, which gave this principle the name of the law of excluded middle. When look at the principles of Boolean algebra, there are two items as prediction: "True" or "False". Whether classical,

Boolean or crisp, set theory can be defined as a characteristic function of the membership of an element x in a set A . For each elements of universal set X , the function that generates the values 0 and 1 is called the characteristic function.

Today, the studies and applications related to the fuzziness are increasing rapidly and this concept is finding in every scientific areas. Zadeh's FS theory succeeded in attracting mathematicians as well as other scientists and was adopted. The most important feature of FS is to generalize the values of "true" and "false" in traditional logic and to produce a logic that can use multi-valued results.

Any change or transition in real life takes place between membership and non-membership. This transition is clear, well-defined. Despite this certainty in classical sets, this is somewhat different in FS. This difference arises because the transition is gradual. The uncertainty of the boundaries of the FS implies that this transition will vary from one membership to another. In that case, membership of an element from the universe in this set is measured by a function that attempts to describe vagueness and ambiguity.

Matloka[7] introduced and discussed the bounded and convergent sequences of fuzzy numbers(FN). Hausdorff metric was used in Matloka's work. Matloka's sequences were used in the classical sequence spaces by Nanda[8]. Following the Nanda, Talo and Basar[12] used the sequences of FN in the space bv_p and defined the space of sequences of p -bounded variation of FN. Later, in [13], dual spaces of the sequences spaces derived by FN was computed and characterized some matrix classes. The quasilinearity of the sequences spaces derived by FN have been studied in [14]. The new classes of sequences of FN defined by Orlicz function and studied their properties in [16]. Dutta and Tripathy [2] have discussed some properties of an important class of sequences of interval numbers with some easy and suitable examples. The new type of open set defined by Dutta and Tripathy [3]. This open set is called fuzzy $b - \theta$ open set, which is a generalization of $b - \theta$ open set. Quite recently, the matrix domains of the sequence spaces derived by FN are studied[9, 10, 15, 18, 19, 20].

We initiate the integrated and differentiated spaces using the triangular fuzzy numbers(TFN) and compute the dual spaces of new spaces. Finally, we characterize some matrix classes.

2. Preliminaries, background and notation

2.1. Definitions and notions

First, we give $\Omega = (a_{nk})$ and $\Gamma = (b_{nk})$ matrices.

The matrix $\Omega = (a_{nk})$ defined by $a_{nk} = k, (n \geq k \geq 1)$ and $a_{nk} = 0,$

($n < k$), and the matrix $\Gamma = (b_{nk})$ defined by $b_{nk} = 1/k, (n \geq k \geq 1)$ and $b_{nk} = 0, (n < k)$, i.e.,

$$a_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 0 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad b_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1/2 & 0 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Further, we can give $\Omega^{-1} = (c_{nk})$ and $\Gamma^{-1} = (d_{nk})$ which are inverses of the above matrices by $c_{nk} = 1/n, (k = n), c_{nk} = -1/n, (k = n - 1), c_{nk} = 0, (otherwise)$ and $d_{nk} = n, (k = n), d_{nk} = -n, (k = n - 1), d_{nk} = 0, (otherwise)$, i.e.,

$$c_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & -1/3 & 1/3 & 0 & \cdots \\ 0 & 0 & -1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad d_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & 2 & 0 & 0 & \cdots \\ 0 & -3 & 3 & 0 & \cdots \\ 0 & 0 & -4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is well known that, the ω denotes the family of all real (or complex)-valued sequences. ω is a linear space and each linear subspace of ω (with the included addition and scalar multiplication) is called a *sequence space* such as the spaces c, c_0 and ℓ_∞ , where c, c_0 and ℓ_∞ denote the set of all convergent sequences in fields \mathbb{R} or \mathbb{C} , the set of all null sequences and the set of all bounded sequences, respectively. It is clear that the sets c, c_0 and ℓ_∞ are the subspaces of the ω . Thus, c, c_0 and ℓ_∞ equipped with a vector space structure, from a sequence space.

Let λ be a sequence spaces. The equations

$$\int \lambda = \{(x_k) \in \omega : (kx_k) \in \lambda\}$$

and

$$d(\lambda) = \{(x_k) \in \omega : (k^{-1}x_k) \in \lambda\},$$

defines the integrated sequence spaces and the differentiated sequence spaces, respectively. Integrated sequence space and differentiated sequence space are obtained from the space ℓ_1 with the matrices Ω and Γ . These spaces were first worked by Goes & Goes [4]. In Goes & Goes [4], some properties related to the integrated space were studied, but the properties related to the differentiated spaces were not given. The integrated rate space is defined and investigated some properties by Subramanian et al. [11]. Kirişci [5] defined new integrated and differentiated spaces by using the new Ω and Γ

matrices. Some of the basic properties of these spaces have been studied in [5].

The definitions, notions and some properties of the FN will be given below.

Let $E_i = \{\delta = [\delta^-, \delta^+] : \delta^- \leq x \leq \delta^+, \delta^- \text{ and } \delta^+ \in \mathbb{R}\}$ denotes the set of all bounded and closed intervals on \mathbb{R} . If we define the metric

$$(1) \quad \mathfrak{D}(\delta, \gamma) = \max \{ |\delta^- - \gamma^-|, |\delta^+ - \gamma^+| \},$$

for $\delta, \gamma \in E_i$, then, the pair (E_i, \mathfrak{D}) is a complete metric space. The fuzzy subset of λ which is nonempty set, is a nonempty subset $\{(a, S(a)) : a \in \lambda\}$ of $\lambda \times [0, 1]$, where $S : \lambda \rightarrow [0, 1]$. If we take $\lambda = \mathbb{R}$, then a subset of nonempty base space \mathbb{R} is defined by $S : \mathbb{R} \rightarrow [0, 1]$.

We give the following properties:

- (i) There exists an $a_0 \in \mathbb{R}$ such that $S(a_0) = 1$.
- (ii) $S[\delta a + (1 - \delta)b] \geq \min\{S(a), S(b)\}$, for any $a, b \in \mathbb{R}$ and $\delta \in [0, 1]$.
- (iii) S is upper semicontinuous.
- (iv) The closure of $\{a \in \mathbb{R} : S(a) > 0\}$ is compact (denoted by S^0).

If the function S provides the conditions (i)-(iv), then S is called the FN. The A -level set of FS S on \mathbb{R} is defined by $S(A) = \{x \in \mathbb{R} : S(x) \geq A\}$ for each $A \in (0, 1]$.

If we choose the membership function of the TFN $\varphi_{(d_1, d_2)}$,

$$(2) \quad \varphi_{(d_1, d_2)} = \begin{cases} (x - (\varphi - d_1))d_1^{-1}, & x \in [\varphi - d_1, \varphi] \\ ((\varphi + d_2) - x)d_2^{-1}, & x \in [\varphi, \varphi + d_2] \\ 0, & \text{otherwise} \end{cases}$$

then, for $d_1 \leq d_2$, $(d_1, d_2 \in \mathbb{R})$, $\varphi_{(d_1, d_2)}$ can be represented with the notation $\varphi_{(sd_1, d_2)} = (\varphi - d_1, \varphi, \varphi + d_2)$. $\varphi_{(d_1, d_2)}$ is a real number, when $d_1 = d_2 = 0$. For convenience, we will understand the triangular (d_1, d_2) -type FN, when we say TFN, through the text. We denote the set F by

$$(3) \quad F = \{(\varphi - d_1, \varphi, \varphi + d_2) : d_1, d_2, \varphi \in \mathbb{R}, d_1 \leq d_2\}.$$

The points $\varphi - d_1, \varphi, \varphi + d_2$ are called first, middle, end points for the TFN $\varphi_{(d_1, d_2)}$, respectively. The notation φ means that the height of the FN $\varphi_{(d_1, d_2)}$ is 1 at the point φ . There is no unique set of FN in the set in form (3). Conversely, there are infinitely-many sets of FN which are different from each other according to structure of their elements.

2.2. $+^F, -^F, \times^F, \div^F$ operators

Errors occur according to algebraic operations, when we choose the FN with A -cut sets. We can explain this with an example. Take any FN

$\delta = [\delta^-(A), \delta^+(A)]$. For $A \in [0, 1]$, $\delta - \delta = [\delta^-(A), \delta^+(A)] - [\delta^-(A), \delta^+(A)] = [\delta^-(A) - \delta^+(A), \delta^+(A) - \delta^-(A)]$. It is clear that this equation is nonzero as expected in the classical mean. Here and throughout the text, θ denotes the fuzzy zero.

Lemma 1. *Let $d_1 \leq d_2$ ($d_1, d_2 \in \mathbb{R}$). All sets of the form F are linear spaces according to algebraic operations*

$$(4) \quad \begin{aligned} \varphi_{(d_1, d_2)} +^F \phi_{(d_1, d_2)} &= (\varphi - d_1, \varphi, \varphi + d_2) +^F (\phi - d_1, \phi, \phi + d_2) \\ &= (\varphi + \phi - d_1, \varphi + \phi, \varphi + \phi + d_2) \end{aligned}$$

and

$$(5) \quad \begin{aligned} \alpha \times^F \varphi_{(d_1, d_2)} &= (\alpha\varphi - d_1, \alpha\varphi, \alpha\varphi + d_2) \\ &= (r - d_1, r, r + d_2) = r_{(d_1, d_2)}. \end{aligned}$$

where, $\varphi_{(d_1, d_2)} \in F$, $\alpha \in \mathbb{R}$ and $\phi_{(d_1, d_2)}$ is nonzero FN.

From the algebraic operations (4) and (5), we can define the following operations:

$$\begin{aligned} \varphi_{(d_1, d_2)} -^F \phi_{(d_1, d_2)} &= (\varphi - d_1, \varphi, \varphi + d_2) -^F (\phi - d_1, \phi, \phi + d_2) \\ &= (\varphi - \phi - d_1, \varphi - \phi, \varphi - \phi + d_2) \\ \varphi_{(d_1, d_2)} \times^F \phi_{(d_1, d_2)} &= (\varphi - d_1, \varphi, \varphi + d_2) \times^F (\phi - d_1, \phi, \phi + d_2) \\ &= (\varphi\phi - d_1, \varphi\phi, \varphi\phi + d_2) \\ \varphi_{(d_1, d_2)} \div^F \phi_{(d_1, d_2)} &= (\varphi - d_1, \varphi, \varphi + d_2) \div^F (\phi - d_1, \phi, \phi + d_2) \\ &= (\varphi \div \phi - d_1, \varphi \div \phi, \varphi \div \phi + d_2) \end{aligned}$$

We find the identity element of F . We choose $\varphi_{(d_1, d_2)} = (\varphi - d_1, \varphi, \varphi + d_2)$ in F . Then, we obtain $(\varphi - d_1, \varphi, \varphi + d_2) + (0 - d_1, 0, 0 + d_2) = (\varphi + 0 - d_1, \varphi + 0, \varphi + 0 + d_2)$. It means that $\theta_{(d_1, d_2)} = (0 - d_1, 0, 0 + d_2)$ is considered as the identity element of F according to operation which is given in (4).

Now, we give the inverse of TFN according to addition. The inverse of FN $\varphi_{(d_1, d_2)} \in F$ is equal to $-\varphi_{(d_1, d_2)} = (-\varphi - d_1, -\varphi, -\varphi + d_2)$ and $-\varphi_{(d_1, d_2)}$ is a FN. Therefore, we can write $x_{(d_1, d_2)} + \varphi_{(d_1, d_2)} = \theta_{(d_1, d_2)}$, where $\theta = \theta_{(d_1, d_2)} = (0 - d_1, 0, 0 + d_2)$ denotes the fuzzy zeros of the sets F . From this idea, we can see that the fuzzy zero of each element of the set F is different. It is clear that the representation $\varphi_{(d_1, d_2)} = (\varphi - d_1, \varphi, \varphi + d_2)$ is unique. Then, for every $-\varphi_{(d_1, d_2)} = (-\varphi - d_1, -\varphi, -\varphi + d_2)$, certainly $\varphi_{(d_1, d_2)}$ is unique.

2.3. Topological structure of the set F

The function $\mathcal{D} : F \times F \rightarrow \mathbb{R}$ defined as a metric in the form

$$\mathcal{D}(\varphi_{(d_1, d_2)}, \phi_{(d_1, d_2)}) := \max\{|\varphi - \phi - d_1|, |\varphi - \phi|, |\varphi - \phi + d_2|\}.$$

It is clear that the (F, \mathcal{D}) is a complete metric space.

Applications have shown that the spread of fuzziness should not be very large. Then, it is necessary to make the value $\max\{|\varphi - \phi - d_1|, |\varphi - \phi|, |\varphi - \phi + d_2|\}$ as small as possible. In fact, we theoretically know that this is not necessary, but this is absolutely necessary in practice. Let us explain this thought with an example: Define the expression "approximately 5" as $5_{(d_1, d_2)} = (-4 - d_1, 5, 15 + d_2)$. In applications, this expression is taken as $5_{(d_1, d_2)} = (5 - d_1, 5, 5 + d_2)$, ($0 \leq d_1 \leq d_2 < 1$) and this choice is more accurate than $5_{(d_1, d_2)} = (-4 - d_1, 5, 15 + d_2)$.

Theorem 1. *The matrices Ω and Γ are regular.*

Proof. Let $\widehat{\varphi} = (\varphi_{(d_1, d_2)}^i)$ be a sequence of TFN. We must show that if for $n \rightarrow \infty$,

$$\mathcal{D}(\varphi_{(d_1, d_2)}^i, \varphi_{(d_1, d_2)}^0) \rightarrow 0,$$

then for $n \rightarrow \infty$,

$$\mathcal{D}\left(\sum_{i=1}^n |i\varphi_{(d_1, d_2)}^i|, \varphi_{(d_1, d_2)}^0\right) \rightarrow 0.$$

Suppose that for $n \rightarrow \infty$,

$$\mathcal{D}(\varphi_{(d_1, d_2)}^i, \varphi_{(d_1, d_2)}^0) \rightarrow 0$$

and choose $\varepsilon > 0$. Then, there exist a positive integer N such that

$$\mathcal{D}(\varphi_{(d_1, d_2)}^i, \varphi_{(d_1, d_2)}^0) < \varepsilon$$

for $n \geq N$. Then, for $n \geq N$ and $N \in \mathbb{N}$,

$$\mathcal{D}(\Omega\varphi_{(d_1, d_2)}^i, \varphi_{(d_1, d_2)}^0) = \mathcal{D}\left(\sum_{i=1}^n |i\varphi_{(d_1, d_2)}^i|, \varphi_{(d_1, d_2)}^0\right) < \varepsilon.$$

Therefore the matrix Ω is regular.

Similarly, we can show that the matrix Γ is regular. ■

2.4. Classical sequence spaces of FN

Define the $f : \mathbb{N} \rightarrow F$, $k \rightarrow f(k) = \varphi_{(d_1, d_2)}^k$. The function f is defined as a sequence of TFN and is represented $\widehat{\varphi} = (\varphi_{(d_1, d_2)}^k)$. The set of all

sequences of TFN denotes with $\omega_F = \left\{ \widehat{\varphi} = \left(\varphi_{(d_1, d_2)}^k \right) : a : \mathbb{N} \rightarrow F, \varphi(k) = \left(\varphi_{(d_1, d_2)}^k \right) = \left(\varphi^k - d_1, \varphi^k, \varphi^k + d_2 \right) \right\}$, where $\varphi^k - d_1 \leq \varphi^k \leq \varphi^k + d_2$, $(d_1, d_2 \in \mathbb{R})$ and $\varphi_{(d_1, d_2)}^k \in F$ for all $k \in \mathbb{N}$. In this place, the elements $\varphi^k - d_1, \varphi^k, \varphi^k + d_2$ is expressed first, middle, end points of general term of a sequences of FN, respectively. If degree of membership at φ^k is equal to 1, then $\widehat{\varphi}$ is a (d_1, d_2) -type FN, if it is not equal to 1, then $\left(\varphi_{(d_1, d_2)}^k \right)$ is a sequence of the FS.

We define the spaces $(\ell_\infty)_F, c_F, (c_0)_F$ and $(\ell_p)_F$ as follows[18, 19]:

$$\begin{aligned} (\ell_\infty)_F &= \left\{ \widehat{\varphi} = \left(\varphi_{(d_1, d_2)}^k \right) \in \omega(F) : \sup_{k \in \mathbb{N}} \mathcal{D} \left(\varphi_{(d_1, d_2)}^k, \theta \right) < \infty \right\}, \\ c_F &= \left\{ \widehat{\varphi} = \left(\varphi_{(d_1, d_2)}^k \right) \in \omega(F) : \lim_k \mathcal{D} \left(\varphi_{(d_1, d_2)}^k, \varphi_{(d_1, d_2)}^0 \right) = 0, \varphi_{(d_1, d_2)}^0 \in F \right\}, \\ (c_0)_F &= \left\{ \widehat{\varphi} = \left(\varphi_{(d_1, d_2)}^k \right) \in \omega(F) : \lim_k \mathcal{D} \left(\varphi_{(d_1, d_2)}^k, \theta \right) = 0 \right\}, \\ (\ell_p)_F &= \left\{ \widehat{\varphi} = \left(\varphi_{(d_1, d_2)}^k \right) \in \omega(F) : \sum_k \mathcal{D} \left(\varphi_{(d_1, d_2)}^k, \theta \right)^p < \infty, 1 \leq p < \infty \right\}. \end{aligned}$$

Let $\lambda_F \subset \omega_F$ and define the function $\|\cdot\| : \lambda_F \rightarrow \mathbb{R}$. Suppose that the function $\|\cdot\|$ is satisfies the conditions $\|\widehat{\varphi}\| = 0 \Leftrightarrow \widehat{\varphi} = \theta, \|\alpha\widehat{\varphi}\| = |\alpha|\|\widehat{\varphi}\|$ ($\alpha \in \mathbb{R}$), $\|\widehat{\varphi} + \widehat{\phi}\| \leq \|\widehat{\varphi}\| + \|\widehat{\phi}\|$.

Then, the function $\|\cdot\|$ is called norm and λ_F is called normed sequence space(NSS) of the (d_1, d_2) -FN. If λ_F is complete with respect to the norm $\|\cdot\|$, then λ_F is called complete NSS of the (d_1, d_2) -FN.

Lemma 2 ([18]). *The bounded, convergent, null sequence spaces are complete NSS with the norm defined by*

$$\|\widehat{\varphi}\| = \sup_k \max \left\{ |\varphi^k - \phi^k - d_1|, |\varphi^k - \phi^k|, |\varphi^k - \phi^k + d_2| \right\},$$

where $\widehat{\varphi}$ is in the any sets of $\{(\ell_\infty)_F, c_F, (c_0)_F\}$.

Let $A = (a_{nk})$ be an infinite matrix and choose two spaces of triangular fuzzy valued sequences λ_F and μ_F . Then, we define the *real* matrix mapping from λ_F to μ_F by $A : \lambda_F \rightarrow \mu_F$. That is, if we take $\widehat{\varphi} = \left(\varphi_{(d_1, d_2)}^k \right) \in \lambda_F$, then we can write $A\widehat{\varphi} = \left[(A\varphi_{(d_1, d_2)})^n \right] \in \mu_F$, where,

$$\begin{aligned} (6) \quad (A\varphi_{(d_1, d_2)})^n &= \sum_k a_{nk} \varphi_{(d_1, d_2)}^k \\ &= \left(\sum_k a_{nk} \varphi^k - d_1, \sum_k a_{nk} \varphi^k, \sum_k a_{nk} \varphi^k + d_2 \right). \end{aligned}$$

In (6), the series $\sum_k a_{nk}\varphi^k - d_1$, $\sum_k a_{nk}\varphi^k$, $\sum_k a_{nk}\varphi^k + d_2$ are convergent for all $n \in \mathbb{N}$. That is, $A : \lambda_F \rightarrow \mu_F$ if and only if for $a \in \lambda_F$ and all $n \in \mathbb{N}$, the series of $\sum_k a_{nk}\varphi^k - d_1$, $\sum_k a_{nk}\varphi^k$, $\sum_k a_{nk}\varphi^k + d_2$ are convergent.

The set

$$[\lambda_F]_A = \left\{ (\varphi_{(d_1, d_2)}^k) \in \omega_F : A\varphi_{(d_1, d_2)}^n \in \lambda_F \right\}.$$

is called the fuzzy domain of infinite matrix A in λ_F

2.5. New spaces with TFN

Let $\varphi_{d_1}^k \leq \varphi^k \leq \varphi_{d_2}^k$ and $\varphi = (\varphi_{(d_1, d_2)}^k) \in F$ for all $k \in \mathbb{N}$.

Now, we introduce the integrated spaces with TFN by

$$[\lambda_F]_\Omega = \left\{ \widehat{\varphi} = (\varphi_{(d_1, d_2)}^k) \in \omega_F : \Omega\varphi_{(d_1, d_2)}^k \in \lambda_F \right\}$$

and the differentiated spaces with TFN by

$$[\lambda_F]_\Gamma = \left\{ \widehat{\varphi} = (\varphi_{(d_1, d_2)}^k) \in \omega_F : \Gamma\varphi_{(d_1, d_2)}^k \in \lambda_F \right\}$$

where $\lambda = \{\ell_\infty, c, c_0\}$.

Let us define the sequence of FN $v = (v_{(d_1, d_2)}^k)$ and $y = (y_{(d_1, d_2)}^k)$, as the Ω -transform and Γ -transform of a sequence of FN $\varphi = (\varphi_{(d_1, d_2)}^k)$, respectively; that is, for $k, n \in \mathbb{N}$,

$$(7) \quad v_{(d_1, d_2)}^n = \sum_{k=1}^n \left| k a \varphi_{(d_1, d_2)}^k \right|$$

$$(8) \quad y_{(sd_1, d_2)}^n = \sum_{k=1}^n \left| \frac{1}{k} \varphi_{(d_1, d_2)}^k \right|$$

where $\varphi_{(d_1, d_2)}^{-1} = \theta$.

Theorem 2. *Integrated and differentiated spaces derived by TFN are norm isomorphic to the spaces $(\ell_\infty)_F$, c_F , $(c_0)_F$ and $(\ell_p)_F$.*

Proof. We show that there is a linear isometry between integrated bounded space with TFN and bounded sequence space of FN. We consider the mapping defined Φ , from $[(\ell_\infty)_F]_\Omega$ to $(\ell_\infty)_F$ by $\varphi \mapsto v = \Phi\varphi = \sum_{k=1}^n \left| k \varphi_{(d_1, d_2)}^k \right|$. Then, it is easy to see that the equality $\Phi(\varphi + \phi) = \Phi(\varphi) + \Phi(\phi)$ holds. Choose $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned} \Phi(\lambda\varphi) &= \Phi(\lambda\varphi_{(d_1, d_2)}^k) = \sum_{k=1}^n \left| \lambda k \varphi_{(d_1, d_2)}^k \right| \\ &= \lambda \sum_{k=1}^n \left| k \varphi_{(d_1, d_2)}^k \right| = \lambda \Phi\varphi. \end{aligned}$$

Therefore, we have Φ is linear map.

We take $\phi \in (\ell_\infty)_F$ and define the sequence φ such that $\varphi = (\varphi_{(d_1, d_2)}^k) = (c_{nk}\phi_{(d_1, d_2)}^k)$, where (c_{nk}) is an inverse of the matrix Ω .

$$\begin{aligned} \|\varphi\|_{[(\ell_\infty)_F]\Omega} &= \sup_k \bar{d} \left(\Omega \varphi_{(d_1, d_2)}^k, \theta \right) \\ &= \sup_k \bar{d} \left(\phi_{(d_1, d_2)}^k, \theta \right) = \|\phi\|_{(\ell_\infty)_F} \end{aligned}$$

Therefore, we can say that Φ is norm preserving.

If we define the transformation $\Psi : [(\ell_\infty)_F]\Gamma \rightarrow (\ell_\infty)_F$ by $\varphi \mapsto y = \Psi\varphi = \sum_{k=1}^n \left| (1/k)\varphi_{(d_1, d_2)}^k \right|$, then we can say that Γ is norm preserving. ■

Theorem 3. *Integrated and differentiated sequence spaces derived by TFN are complete NSS with the norms defined by*

$$(9) \quad \|\widehat{\varphi}\|_\Omega = \sup_k \max \left\{ \Omega \left| \varphi_{(d_1, d_2)}^k - v_{(d_1, d_2)}^k - d_1 \right|, \Omega \left| \varphi_{(d_1, d_2)}^k - v_{(d_1, d_2)}^k \right|, \Omega \left| \varphi_{(d_1, d_2)}^k - v_{(d_1, d_2)}^k + d_2 \right| \right\}$$

$$(10) \quad \|\widehat{\varphi}\|_\Gamma = \sup_k \max \left\{ \Gamma \left| \varphi_{(d_1, d_2)}^k - y_{(d_1, d_2)}^k - d_1 \right|, \Gamma \left| \varphi_{(d_1, d_2)}^k - y_{(d_1, d_2)}^k \right|, \Gamma \left| \varphi_{(d_1, d_2)}^k - y_{(d_1, d_2)}^k + d_2 \right| \right\},$$

respectively.

Proof. Integrated and differentiated spaces derived by TFN are norm isomorphic to the spaces $(\ell_\infty)_F$, c_F , $(c_0)_F$ and $(\ell_p)_F$. Further, matrices Ω and Γ are regular. Hence, integrated and differentiated spaces derived by TFN are complete NSS with the norms (9), (10). ■

We take two spaces of triangular fuzzy valued sequences λ_F and μ_F . We define the set

$$(11) \quad T(\lambda_F, \mu_F) = \left\{ a = (a^k) \in \omega_F : \left(a^k x_{(d_1, d_2)}^k \right) \in \mu_F, \forall x \in \lambda_F \right\}.$$

We denote the duals of the sequence spaces λ_F with $\alpha(r)$, $\beta(r)$, $\gamma(r)$. Using the notation (11), we can define the duals as follows:

$$\begin{aligned} [\lambda_F]^{\alpha(r)} &= T(\lambda_F, (\ell_1)_F), \\ [\lambda_F]^{\beta(r)} &= T(\lambda_F, cS_F), \\ [\lambda_F]^{\gamma(r)} &= T(\lambda_F, bS_F), \end{aligned}$$

where cs_F denote the convergent series of FN and bs_F denotes the spaces bounded series of FN.

Lemma 3 ([13]). *The following statements hold:*

- (i) $A \in ((\ell_\infty)_F : (\ell_\infty)_F)$, $A \in (c_F : (\ell_\infty)_F)$, $A \in ((c_0)_F : (\ell_\infty)_F)$ if and only if $\sup_n \sum_k \mathcal{D}(a_{nk}, \theta) < \infty$ holds.
- (ii) $A \in ((\ell_\infty)_F : (c_0)_F)$ if and only if $\lim_n \sum_k \mathcal{D}(a_{nk}, \theta) = 0$ holds.
- (iii) $A \in ((c_0)_F : c_F)$ if and only if $\sup_n \sum_k \mathcal{D}(a_{nk}, \theta) < \infty$ and $\lim_n \mathcal{D}(a_{nk}, a^k) = 0$ where $(a_{(0,0)}^k) \in \omega_F$ hold.
- (iv) $A \in ((c_0)_F : (c_0)_F)$ if and only if $\sup_n \sum_k \mathcal{D}(a_{nk}, \theta) < \infty$ and $\lim_n \mathcal{D}(a_{nk}, a^k) = 0$ where $(a_{(0,0)}^k) \in \omega_F$ hold with $a^k = \theta$ for all $k \in \mathbb{N}$.

Lemma 4 ([9]). *Let K is the finite subset of \mathbb{N} and A be an infinite matrix of positive numbers a_{nk} .*

- (i) $A \in (c_F : c_F)$ if and only if the conditions $\sup_n \sum_k \mathcal{D}(a_{nk}, \theta) < \infty$ and $\lim_n \mathcal{D}(a_{nk}, a^k) = 0$ where $(a_{(0,0)}^k) \in \omega_F$ hold with $a^k = \theta$ for all $k \in \mathbb{N}$.
- (ii) $A \in ((c_0)_F : (\ell_1)_F)$ if and only if $\sup_K \sum_k \mathcal{D}\left(\sum_{n \in K} a_{n,k}, \theta\right) < \infty$.

Let $U = (u_{nk})$ be an infinite matrix and $V = (v_{nk})$ be an inverse matrix of U . The matrix $F = (f_{nk})$ defined by $f_{nk} = \sum_{j=k}^n a^j v_{jk}$, ($0 \leq k \leq n$) and $f_{nk} = 0$, ($k > n$), for all $k, n \in \mathbb{N}$.

Lemma 5 ([1]). *Let λ be a sequence space. Then,*

$$\begin{aligned} [\lambda_U]^\beta &= \left\{ a = (a^k) \in \omega : F \in (\lambda : c) \right\} \\ [\lambda_U]^\gamma &= \left\{ a = (a^k) \in \omega : F \in (\lambda : \ell_\infty) \right\}. \end{aligned}$$

Theorem 4. *The sets ρ_1 and ρ_2 are defined as follows:*

$$\begin{aligned} \rho_1 &= \left\{ a \in \omega_F : \sup_{n \in \mathbb{N}} \sum_k \mathcal{D}\left(\sum_{n \in \mathbb{N}} \sum_{k \in K} \frac{a_k}{k}, \theta\right) < \infty \right\} \\ \rho_2 &= \left\{ a \in \omega_F : \sup_{n \in \mathbb{N}} \sum_k \mathcal{D}\left(\sum_{n \in \mathbb{N}} \sum_{k \in K} k a_k, \theta\right) < \infty \right\}. \end{aligned}$$

The $\alpha(r)$ -duals of the integrated and differentiated triangular fuzzy sequence spaces are the sets ρ_1 and ρ_2 , respectively.

The proof of this theorem is obtained from (ii) of Lemma 4.

We define the matrices $G = (g_{nk})$ as $g_{nk} = a_n/n$, ($n \geq k \geq 1$) and $g_{nk} = 0$, ($n < k$), and $H = (h_{nk})$ as by $h_{nk} = na_n$, ($n \geq k \geq 1$) and $h_{nk} = 0$, ($n < k$). Also we define the sets $\rho_3, \rho_4, \rho_5, \rho_6$ as follows

$$\begin{aligned} \rho_3 &= \left\{ a \in \omega(F) : \sup_n \sum_k \mathcal{D}(g_{nk}, \theta) < \infty \right\} \\ \rho_4 &= \left\{ a \in \omega(F) : \lim_{n \rightarrow \infty} \mathcal{D}(g_{nk}, \theta) \text{ exists for each } k \in \mathbb{N} \right\} \\ \rho_5 &= \left\{ a \in \omega(F) : \sup_n \sum_k \mathcal{D}(h_{nk}, \theta) < \infty \right\} \\ \rho_6 &= \left\{ a \in \omega(F) : \lim_{n \rightarrow \infty} \mathcal{D}(h_{nk}, \theta) \text{ exists for each } k \in \mathbb{N} \right\} \end{aligned}$$

Theorem 5. $\{[\lambda_F]_\Omega\}^{\beta(r)} = \rho_3 \cap \rho_4$ and $\{[\lambda_F]_\Gamma\}^{\beta(r)} = \rho_5 \cap \rho_6$.

Using Lemma 3 and Lemma 5, the proof of this theorem can be obtained. This proof is similar to proof of Theorem 6. Then, we only prove the Theorem 6.

Theorem 6. *The $\gamma(r)$ -duals of the integrated and differentiated triangular fuzzy sequence spaces are the sets ρ_3 and ρ_5 , respectively.*

Proof. We give only proof of the $\gamma(r)$ -dual of the space $[(c_0)_F]_\Omega$. Let $a \in \omega$ and give the matrix $G = (g_{nk})$. Using the relation (7), we have

$$(12) \quad \sum_{k=0}^n a^k x_{(d_1, t_2)}^k = (Gy_{(d_1, d_2)})^n$$

We obtain that $ax = (a^k x_{(d_1, d_2)}^k) \in bs_F$ whenever $x \in [(c_0)_F]_\Omega$ if and only if $Gy_{(d_1, d_2)} \in (\ell_\infty)_F$ whenever $y_{(d_1, d_2)} \in (c_0)_F$ by 12. From (i) of Lemma 3, we have $\sup_n \sum_k \mathcal{D}(e_{nk}, \theta) < \infty$. Then, we get from the last result that $\{[(c_0)_F]_\Omega\}^\gamma(r) = \rho_3$.

Similarly, it is seen that $\{[(c_0)_F]_\Gamma\}^\gamma(r) = \rho_5$ by using the matrix $H = (h_{nk})$. ■

2. Matrix transformation

In this section, we characterize the matrix mappings. First, we will give some brevity as follows:

$$(13) \quad \widetilde{b}_{nk} = \frac{a_{nk}}{k} - \frac{a_{n,k+1}}{k+1} \quad \text{or} \quad \widetilde{a}_{nk} = \sum_{k=1}^{\infty} \sum_{j=1}^n j b_{nj}$$

and

$$(14) \quad \widehat{b}_{nk} = ka_{nk} - (k+1)a_{n,k+1} \quad \text{or} \quad \widehat{a}_{nk} = \sum_{k=1}^{\infty} \sum_{j=1}^n \frac{1}{j} b_{nj}$$

Theorem 7. *Let $P = (p_{ni})$ and $R = (r_{ni})$ be Ω matrices and λ_F be any given sequence space. Then, $P \in ([c_F]_{\Omega} : \lambda_F)$ if and only if $(p_{ni})_{i \in \mathbb{N}} \in (\ell_1)_F$ and $R \in (c_F : \lambda_F)$.*

Proof. We choose the matrices $P = (p_{ni})$ and $R = (r_{ni})$ which are Ω matrices. Assume that these matrices satisfy the condition (13). Also it is known that the spaces $[c_F]_{\Omega}$ and c_F are linearly isomorphic (Theorem 2).

Suppose that $P \in ([c_F]_{\Omega} : \lambda_F)$. If we take any $z \in c_F$ then, $R\Omega$ is equal to P and $(p_{ni})_{i \in \mathbb{N}} \in \{[c_F]_{\Omega}\}^{\beta(r)}$. Therefore, we obtain that $(r_{ni})_{i \in \mathbb{N}} \in (\ell_1)_F$ for each $n \in \mathbb{N}$. Hence, Rz exists for each $z \in c_F$. Then, we have

$$(15) \quad \sum_i r_{ni} z^i_{(d_1, d_2)} = \sum_i p_{ni} x^i_{(d_1, d_2)}$$

for all $n \in \mathbb{N}$. From the equation 15, $Rz = Px$ and $R \in (c_F : \lambda_F)$.

Conversely, suppose that $R \in (c_F : \lambda_F)$. If we take any $x \in [c_F]_{\Omega}$, then Px exists. Therefore, we obtain from the following equality as $n \rightarrow \infty$ that $Px = Rz$ and this shows that $P \in ([c_F]_{\Omega} : \lambda_F)$ and

$$\sum_{i=0}^n p_{ni} x^i_{(d_1, d_2)} = \sum_{i=0}^n \left\{ \frac{1}{n} \sum_{i=0}^n a_{ni} \right\} z^i_{(d_1, d_2)}$$

for all $n \in \mathbb{N}$, which is the desired result. ■

Theorem 8. *Let $T = (t_{ni})$ and $Q = (q_{ni})$ be Γ matrices and λ_F be any given sequence space. Then, $T \in ([c_F]_{\Gamma} : \lambda_F)$ if and only if $(a_{ni})_{k \in \mathbb{N}} \in (\ell_1)_F$ and $Q \in (c_F : \lambda_F)$.*

Theorem 8 can be proved using (14).

Theorem 9. *Suppose that the elements of the infinite matrices $\Delta = (\delta_{ni})$ and $\Lambda = (\lambda_{ni})$ are connected with relation*

$$(16) \quad \lambda_{nk} = \sum_{k=1}^{\infty} \sum_{j=1}^n j \delta_{jk}$$

for all $n, i \in \mathbb{N}$. Then, $\Delta \in (\lambda_F : [c_F]_{\Omega})$ if and only if $\Lambda \in (\lambda_F : c_F)$.

Proof. Take $x = (x^i_{(d_1, d_2)}) \in \lambda_F$. Then, we have

$$\sum_{k=1}^m \lambda_{nk} x^i_{(d_1, d_2)} = \sum_{k=1}^m \sum_{k=1}^{\infty} \sum_{j=1}^n j \delta_{jk} x^i_{(d_1, d_2)}$$

for all $m, n \in \mathbb{N}$. Then, we obtain $(\Lambda x)_n = [\Gamma(\Delta x)]_n$ as $m \rightarrow \infty$, for all $n \in \mathbb{N}$. Therefore, one can observe from here that $\Delta x \in [c_F]_{\Omega}$ whenever $x \in \lambda_F$ if and only if $\Lambda x \in c_F$ whenever $x \in \lambda_F$ ■

Theorem 10. *Suppose that the elements of the infinite matrices $\Pi = (\pi_{ni})$ and $\Sigma = (\sigma_{ni})$ are connected with relation*

$$(17) \quad \sigma_{nk} = \sum_{k=1}^{\infty} \sum_{j=1}^n \frac{1}{j} \pi_{jk}$$

for all $n, i \in \mathbb{N}$. Then, $\Pi \in (\lambda_F : [c_F]_{\Gamma})$ if and only if $\Sigma \in (\lambda_F : c_F)$.

3. Conclusion

Since FS have emerged, they have made great progress in science and technology. It is seen to have many applications in both theoretical and practical studies from engineering area to arts and humanities, from computer science to health sciences, and from life sciences to physical sciences. In the recent years, ordinary FS have been extended to new types and these extensions have been used in many areas such as energy, medicine, material, economics and pharmacology sciences.

The theory of FS now encompasses a well organized corpus of basic notions including (and not restricted to) aggregation operations, a generalized theory of relations, specific measures of information content, a calculus of FN.

In this paper, using the matrices Ω and Γ , we define the new sequence spaces with TFN. We compute the real-duals of these spaces and characterize the matrix classes of these spaces with well-known sequence spaces.

The properties and results related to the integrated space with TFN and differentiated space with TFN are more general and more extensive than the corresponding consequences of the classical sets consisting of the bounded, convergent and null sequences of FN.

Conflict of Interests. The author declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgements. The author thanks the anonymous referees for their valuable suggestions for the improvement of the manuscript. This

work was supported by Scientific Projects Coordination Unit of Istanbul University. Project number 26287.

References

- [1] ALTAY B., BASAR F., Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence sapces, *J. Math. Anal. Appl.*, 336(1)(2007), 632-645.
- [2] DUTTA A.J., TRIPATHY B.C., On the class of p -absolutely summable sequence $\ell^i(p)$ of interval numbers, *Songklanakarinn Journal of Science and Technology*, 38(2)(2016), 143-146.
- [3] DUTTA A.J., TRIPATHY B.C., On fuzzy $b - \theta$ open sets in fuzzy topological space, *Journal of Intelligent and Fuzzy Systems*, 32(1)(2017), 137-139.
- [4] GOES G., GOES S., Sequences of bounded variation and sequences of Fourier coefficients I, *Math. Z.*, 118(1970), 93-102.
- [5] KIRIŞCI M., Integrated and differentiated sequence spaces, *Journal Nonlinear Analysis and Application*, 2015(1)(2015), 2-16. DOI:10.5899/2015/jnaa-00266.
- [6] KIRIŞCI M., Riesz type integrated and differentiated sequence spaces, *Bull. Math. Anal. Appl.*, 2(2015), 14-27.
- [7] MATLOKA M., Sequences of fuzzy numbers, *BUSEFAL*, 28(1986), 28-37.
- [8] NANDA S., On sequences of fuzzy numbers, *Fuzzy Sets and Systems*, 33(1989), 123-126.
- [9] ŞENGÖNÜL M., On the Zweier sequence spaces of Fuzzy Numbers, *International Journal of Mathematics and Mathematical Sciences*, vol. 2014, Article ID 439169, 9 pages, (2014).
- [10] ŞENGÖNÜL M., The application domain of Cesaro matrix on some sequence spaces of fuzzy numbers, *Int. J. Math. Anal.*, 9(2015), 1-14.
- [11] SUBRAMANIAN N., RAO K.C., GURUMOORTHY N., Integrated rate space $\int \ell_\pi$, *Commun. Korean Math. Soc.*, 22(2007), 527-534.
- [12] TALO Ö., BAŞAR F., On the space $bv_p(F)$ of sequences of p -bounded variation of fuzzy numbers, *Acta Math. Sin. (Engl. Ser.)*, 24(7)(2008), 1205-1212.
- [13] TALO Ö., BAŞAR F., Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, *Comp. Math. Appl.*, 58(2009), 717-733.
- [14] TALO Ö., BAŞAR F., Quasilinearity of the classical sets of sequences of fuzzy numbers and some related results, *Taiwanese J. Math.*, 14(5)(2010), 1799-1819.
- [15] TALO Ö., ÇAKAN C., On the Cesáro convergence of sequences of fuzzy numbers, *Appl. Math. Lett.*, 25(4) (2012), 676-681.
- [16] TRIPATHY B.C., BRAHA N.L., DUTTA A.J., A new class of fuzzy sequences related to the ℓ_p space defined by Orlicz function, *Journal of Intelligent and Fuzzy Systems*, 26(3)(2014), 1273-1278.
- [17] ZADEH L.A., Fuzzy sets, *Inf. Comp.*, 8(1965), 338-353.
- [18] ZARARSIZ Z., Similarity measures of sequences of fuzzy numbers and fuzzy risk analysis, *Advances in Math. Phys.*, vol. 2015, Article ID 724647, 12 pages, (2015).

- [19] ZARARSIZ Z., A contribution to the algebraic structure of fuzzy numbers, *Ann. Fuzzy Math. Inform.*, 12(2016), 205-219.
- [20] ZARARSIZ Z., ŞENGÖNÜL M., On the gravity of center of sequence of fuzzy numbers, *Ann. Fuzzy Math. Inform.*, 6(3)(2013), 479-485.

MURAT KIRIŞCI

DEPARTMENT OF MATHEMATICAL EDUCATION

HASAN ALI YÜCEL EDUCATION FACULTY

ISTANBUL UNIVERSITY

VEFA, 34470, FATİH, ISTANBUL, TURKEY

e-mail: mkirisci@hotmail.com or murat.kirisci@istanbul.edu.tr

Received on 05.04.2017 and, in revised form, on 03.11.2017.