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ON $(\mathcal{I},\gamma)$-g^*-CLOSED SETS VIA IDEAL TOPOLOGICAL SPACES

Abstract. In this paper we introduce (I,γ)-g^*-closed sets in topological spaces and also introduce $\gamma g^* - T_I$-spaces and investigate some of their properties.

Key words: (I,γ)-g-closed set, (I,γ)-gs-closed sets, $\gamma\alpha$-T_I -spaces.

AMS Mathematics Subject Classification: 54B05, 54C08, 54D05.

1. Introduction and preliminaries

Recently Julian Dontchev et. al. [3] introduce (I,γ)-generalized closed sets and R. Devi et. al. [1] introduce (I,γ)-generalized semi-closed sets via topological ideals. In this paper we introduce (I,γ)-g^*-closed sets and investigate some of their properties.

An ideal I on a topological space (X,τ) is a non-empty collection of subsets of X satisfying the following two properties:

(i) $A \in I$ and $B \subset A$ implies $B \in I$

(ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$

For a subset $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ. Recall that $A \subseteq (X,\tau,I)$ is called τ^*-closed [5] if $A^* \subseteq A$. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(I)$, finer than τ. An operation γ [6,10] on the topology τ on a given topological space (X,τ) is a function from the topology itself into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V.

The following operators are examples of the operation γ: the closure operator γ_{cl} defined by $\gamma(U) = \text{cl}\ (U)$, the identity operator γ_{id} defined by $\gamma(U) = U$. Another example of the operation γ is the γ_f-operator defined by $U^{\gamma_f} = (FrU)^c = X/FrU$[11]. Two operators γ_1 and γ_2 are called mutually dual [11] if $U^{\gamma_1} \cap U^{\gamma_2} = U$ for each $U \in \tau$. For example the identity operator is mutually dual to any other operator, while the γ_f-operator is mutually dual to the closure operator [11].

Definition 1. *A subset A of a space (X, τ) is called*
(a) *an α-open set [9] if $A \subseteq int(cl(int(A)))$.*
(b) *a semi-open set [8] if $A \subseteq cl(int(A))$.*
(c) *a generalized closed(briefly g-closed) set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).*
(d) *a g^*-closed set [12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).*
(e) *an (I, γ)- generalized closed (briefly (I, γ)- g-closed) set [3] if $A^* \subseteq U^\gamma$ whenever $A \subseteq U$ and U is open in (X, τ).*
(f) *an (I, γ)-generalized semi closed (briefly (I, γ)-gs-closed) set [1] if $A^* \subseteq U^\gamma$, whenever $A \subseteq U$ and U is semi-open in (X, τ).*

Throughout this paper the operator γ is defined as $\gamma : \tau^g \to P(X)$, where τ^g denotes the set of all g-open sets of (X, τ).

2. Properties of $(\mathcal{I}, \gamma)$-g^*-closed sets

Definition 2. *A subset A of a topological space (X, τ) is called $(\mathcal{I}, \gamma)$-g^*-closed if $A^* \subset U^\gamma$, whenever $A \subset U$ and U is g-open in (X, τ).*

Example 1. Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\phi\}$ and $U^\gamma = \text{int}(\text{cl}(U))$. Here $(\mathcal{I}, \gamma)$-g^*-closed sets are X, ϕ, $\{a, c\}$, $\{b, c\}$, $\{c, e\}$, $\{d, e\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, d, e\}$, $\{a, c, e\}$, $\{b, c, d\}$, $\{b, d, e\}$, $\{c, d, e\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$, $\{a, b, d, e\}$, $\{a, b, c, e\}$, $\{b, c, d, e\}$.

Theorem 1. (a) *Every g^*-closed set $\mathcal{I}$-g^*-closed.*
(b) *Every $(\mathcal{I}, \gamma)$-g^*-closed set is $(\mathcal{I}, \gamma)$-g-closed.*
(c) *Every $(\mathcal{I}, \gamma)$-g^*-closed set is $(\mathcal{I}, \gamma)$-gs-closed.*

The converse of the above theorem need not be true by the following example.

Example 2. (a) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Here $A = \{a\}$ is $\mathcal{I}$-g^*-closed but not g^*-closed.
(b) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and $\gamma = id$. Here $A = \{a, b\}$ is $(\mathcal{I}, \gamma)$-g-closed but not $(\mathcal{I}, \gamma)$-g^*-closed.
(c) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and $\gamma = id$. Here $A = \{a, b\}$ is $(\mathcal{I}, \gamma)$-gs-closed but not $(\mathcal{I}, \gamma)$-g^*-closed.

Theorem 2. *If A is $\mathcal{I}$-g^*-closed and g-open, then A is τ^*-closed.*

Proof. Since A is $\mathcal{I}$-g^*-closed, then $A^* \subset U$, U is g-open. It is given that A is g-open implies $A^* \subset A$. Hence A is τ^*-closed. ∎

Lemma 1 ([4], Theorem II3). *Let $(A_i)_{i\in I}$ be a locally finite family of sets in $(X,\tau,\mathcal{I})$. Then $\cup_{i\in I}A_i^*(\mathcal{I}) = (\cup_{i\in I}A_i)^*(\mathcal{I})$.*

Theorem 3. *Let $(X,\tau,\mathcal{I},\gamma)$ be a topological space.*

(*a*) *If $(A_i)_{i\in I}$ is a locally finite family of sets and each $A_i \in IG^*(X)$, then $\cup_{i\in I}A_i \in IG^*(X)$.*

(*b*) *Finite intersection of $(\mathcal{I},\gamma)$-g^*-closed sets need not be $(\mathcal{I},\gamma)$-g^*-closed.*

Proof.

(*a*) Let $\cup_{i\in I}A_i \subset U$, where U is g-open. Since $A_i \in IG^*(X)$ for each $i \in I$, then $A_i^* \subset U^\gamma$. Hence $\cup_{i\in I}A_i^* \subset U^\gamma$. By Lemma 1, $(\cup_{i\in I}A_i)^* = \cup_{i\in I}A_i^*$, then $(\cup_{i\in I}A_i)^* \subset U^\gamma$. Hence $\cup_{i\in I}A_i \in IG^*(x)$.

(*b*) Let $X = \{a,b,c,d,e\}$, $\tau = \{X,\phi,\{c\},\{a,b\},\{a,b,c\}\}$, $\mathcal{I} = \{\phi\}$ and $\gamma = \gamma_{ic}$. Set $A = \{a,c\}$ and $B = \{b,c\}$. Clearly $A, B \in IG^*(X)$ but $A \cap B = \{c\} \notin IG^*(X)$.

■

Lemma 2 ([4]). *If A and B are subsets of $(X,\tau,\mathcal{I})$, then $(A\cap B)^*(\mathcal{I}) \subset A^*(\mathcal{I}) \cap B^*(\mathcal{I})$.*

Theorem 4. *Let $(X,\tau,\mathcal{I},\gamma_{id})$ be a topological space. If $A \subset X$ is $\mathcal{I}$-g^*-closed and B is closed and τ^*-closed, then $A\cap B$ is $\mathcal{I}$-g^*-closed.*

Proof. Let $U \in \tau^g$ be such that $A \cap B \subset U$. Then $A \subset U \cap (X/B)$. Since A is $\mathcal{I}$-g^*-closed, then $A^* \subset U \cap (X/B)$. Hence $B \cap A^* \subset U \cap B \subset U$, but we know that $B^* \subset B$, therefore $(A\cap B)^* \subset A^* \cap B^* \subset A^* \cap B \subset U$, by Lemma 2. Hence $A \cap B$ is $\mathcal{I}$-g^*-closed. ■

Theorem 5. *Let A be a subset of $(X,\tau,\mathcal{I},\gamma_{id})$. Then, A is $\mathcal{I}$-g^*-closed if and only if $A^* - A$ does not contain any non-empty closed subset.*

Proof. *Necessity.* Assume that F is a closed subset of $A^* - A$. Note that clearly $A \subset X - F$, where A is $\mathcal{I}$-g^*-closed and $X - F \in \tau$. Then $A^* \subset X - F$, that is $F \subset X - A^*$. Since due to our assumption $F \subset A^*$, $F \subset (X - A^*) \cap A^* = \phi$.

Sufficiency. Let U be an open subset and hence g-open subset containing A. Since A^* is closed [5, Theorem 2.3(c)] and $A^* \cap (X - U) \subset A^* - A$ holds, then $A^* \cap (X - U)$ is a closed set contained in $A^* - A$. By assumption, $A^* \cap (X - U) = \phi$ and hence $A^* \subset U$. ■

A subset S of a space $(X,\tau,\mathcal{I},\gamma)$ is a topological space with an ideal $\mathcal{I}_s = \{I \in \mathcal{I} : I \subset S\} = \{I \cap S : I \in \mathcal{I}\}$ on S [2].

Lemma 3 ([3]). *Let $(X,\tau,\mathcal{I})$ be a topological space and $A \subset S \subset X$. Then, $A^*(\mathcal{I}_S, \tau/S) = A^*(\mathcal{I},\tau) \cap S$ holds.*

Proof. First we prove the following implication: $A^*(\mathcal{I}_S, \tau/S) \subset A^*(\mathcal{I}, \tau) \cap S$. Let $x \notin A^*(\mathcal{I}, \tau) \cap S$. We consider the following two cases:

Case 1. $x \notin S$. Since $A^*(\mathcal{I}_S, \tau/S) \subset S$, then $x \notin A^*(\mathcal{I}_S, \tau/S)$.

Case 2. $x \in S$. In this case $x \notin A^*(\mathcal{I}, \tau)$. There exists a set $V \in \tau$ such that $x \in V$ and $V \cap A \in \mathcal{I}$. Since $x \in S$, we have a set $S \cap V \in \tau/S$ such that $x \in S \cap V$ and $(S \cap V) \cap A \in \mathcal{I}$ and hence $(S \cap V) \cap A \in \mathcal{I}_S$. Consequently, $x \notin A^*(\mathcal{I}_S, \tau/S)$. Both cases show the implication.

Secondly, we prove the converse implication: $A^*(\mathcal{I}, \tau) \cap S \subset A^*(\mathcal{I}_S, \tau/S)$. Let $x \notin A^*(\mathcal{I}_S, \tau/S)$. Then, for some open subset $U \cap S$ of $(S, \tau/S)$ containing x, we have $(U \cap S) \cap A \in \mathcal{I}_S$. Since $A \subset S$, then $U \cap A \in \mathcal{I}_S \subset \mathcal{I}$, i.e., $U \cap A \in \mathcal{I}$ for some $V \in \tau$ containing x. This shows that $x \notin A^*(\mathcal{I}, \tau)$. ■

Theorem 6. *Let $A \subset S \subset (X, \tau, \mathcal{I}, \gamma_{id})$. If A is $\mathcal{I}_S$-g^*-closed in $(S, \tau/S, \mathcal{I}_S, \gamma_{id})$ and S is closed in (X, τ), then A is $\mathcal{I}$-g^*-closed in $(X, \tau, \mathcal{I}, \gamma_{id})$.*

Proof. Let $A \subset U$, where $U \in \tau^g$. Let $x \notin U$. We consider the following two cases.

Case (i). $x \in S$. By assumption, $A^*(\mathcal{I}_S, \tau/S) \subset U \cap S \subset U$. We show that $A^*(\mathcal{I}) \subset A^*(\mathcal{I}_S, \tau/S)$. Let $x \notin A^*(\mathcal{I}_S, \tau/S)$. Since $x \in S$, then for some open subset V_S of $(S, \tau/S)$ containing x, we have $V_S \cap A \in \mathcal{I}_S$, since $V_S = V \cap S$ for some $V \in \mathcal{I}$, then $(S \cap V) \cap A \in \mathcal{I}_S \subset \mathcal{I}$, that is $V \cap A \in \mathcal{I}$ for some $V \in \tau$ containing x. This shows that $x \notin A^*(\mathcal{I})$. Hence $A^*(\mathcal{I}) \subset U$.

Case (ii). $x \notin S$. Then X/S is an open neighbourhood of x disjoint from A. Hence $x \notin A^*(\mathcal{I})$. Consequently $A^*(\mathcal{I}) \subset U$.

Both cases we show that the local function of A with respect to $\mathcal{I}$ and τ is in U. Hence A is $\mathcal{I}$-g^*-closed in $(X, \tau, \mathcal{I}, \gamma_{id})$. ■

Theorem 7. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space and $A \subset S \subset X$. If A is $\mathcal{I}_S$-g^*-closed in $(S, \tau/S, \mathcal{I}_S, \gamma_{id})$ and S is $\mathcal{I}$-g^*-closed in X, then A is $\mathcal{I}$-g^*-closed in X.*

Proof. Let $A \subset U$ and $U \in \tau^g$. By assumption and Lemma 3, $A^*(\mathcal{I}, \tau) \cap S \subset U \cap S$. Then we have $S \subset U \cup (X/A^*(\mathcal{I}, \tau))$. Since $X/A^*(\mathcal{I}, \tau) \in \tau^g$, then $A^*(\mathcal{I}, \tau) \subset S^*(\mathcal{I}, \tau) \subset U \cup (X/A^*(\mathcal{I}, \tau))$. Therefore, we have that $A^*(\mathcal{I}, \tau) \subset U$ and hence A is $\mathcal{I}$-g^*-closed in X. ■

Corollary 1. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space and A and F subsets of X. If A is $\mathcal{I}$-g^*-closed and F is closed in (X, τ), then $A \cap F$ is $\mathcal{I}$-g^*-closed.*

Proof. Since $A \cap F$ is closed in $A, \tau/A$, then $A \cap F$ is $\mathcal{I}_A$-g^*-closed in $(A, \tau/A, \mathcal{I}_A)$. By Theorem 7, $A \cap F$ is $\mathcal{I}$-g^*-closed. ■

Theorem 8. *Let $A \subset S \subset (X, \tau, \mathcal{I}, \gamma)$. If $A \in IG^*(X)$ and $S \in \tau^g$, then $A \in IG^*(S)$.*

Proof. Let U be a g-open subset of $(S,\tau/S)$ such that $A\subset U$. Since $S\in\tau^g$, then $U\in\tau^g$. Then $A^*(\mathcal{I})\subset U^\gamma$, since $A\in IG^*(X)$. By Lemma 3, we have $A^*(\mathcal{I}_S,\tau/S)\subset U^{\gamma/S}$, where $U^{\gamma/S}$ means the image of the operation $\gamma/S:\tau^g/S\to P(S)$, defined by $(\gamma/S)(U)=\gamma(U)\cap S$ for each $U\in\tau^g/S$. Hence $A\in IG^*(S)$. ∎

Theorem 9. *If the set $A\subset(X,\tau,\mathcal{I})$ is both $(\mathcal{I},\gamma_1)$-g^*-closed and $(\mathcal{I},\gamma_2)$-g^*-closed, then it is $\mathcal{I}$-g^*-closed, granted the operators γ_1 and γ_2 are mutually dual.*

Proof. Let $A\subset U$, where $U\in\tau^g$. Since $A^*\subset U^{\gamma_1}$ and $A^*\subset U^{\gamma_2}$, then $A^*\subset U^{\gamma_1}\cap U^{\gamma_2}=U$, since γ_1 and γ_2 are mutually dual. Hence A is $\mathcal{I}$-g^*-closed. ∎

Theorem 10. *Every set $A\subset(X,\tau,\mathcal{I})$ is $(\mathcal{I},\gamma_{cl})$-$g^*$-closed.*

Proof. Let $A\subset U$, U is g-open. We know that $A\cup A^*=\text{cl}^*(A)\subset\text{cl}(A)\subset\text{cl}(U)$. This implies that $A^*\subset\text{cl}(U)$. Hence A is (I,γ_{cl})-g^*-closed. ∎

Corollary 2. *For a set $A\subset(X,\tau,\mathcal{I},\gamma)$, the following conditions are equivalent.*

(a) *A is $(\mathcal{I},\gamma_f)$-g^*-closed.*

(b) *A is $\mathcal{I}$-g^*-closed.*

Proof. (a) $\Rightarrow$ (b) By Theorem 10, A is $(\mathcal{I},\gamma_{cl})$-$g^*$-closed. Since γ_f and γ_{cl} are mutually dual due to [11], then $\gamma_f(U)\cap\gamma_{cl}(U)=U$. This implies that $A^*\subset U$, that is, A is $\mathcal{I}$-g^*-closed.

(b) $\Rightarrow$ (a) Let $A\subset U$, U is g-open. Since A is $\mathcal{I}$-g^*-closed, $A^*\subset U$. But we know that $U\subset U^{\gamma_f}$, we have $A^*\subset U^{\gamma_f}$, this implies that A is $(\mathcal{I},\gamma_f)$-g^*-closed. ∎

3. γg^*-$T_{\mathcal{I}}$-space

Definition 3. *A space $(X,\tau,\mathcal{I},\gamma)$ is called an γg^*-$T_{\mathcal{I}}$-space if every $(\mathcal{I},\gamma)$-g^*-closed subset of X is an τ^*-closed. We use the simple notation $g^*T_{\mathcal{I}}$-space, in case γ is the identity operator.*

Theorem 11. *For a space $(X,\tau,\mathcal{I})$, the following conditions are equivalent.*

(a) *X is a g^*T_I-space.*

(b) *Each singleton of X is either closed or τ^*-open.*

Proof. (a) $\Rightarrow$ (b) Let $x\in X$. If $\{x\}$ is not closed, then $A=X/\{x\}\notin\tau$ and then A is trivially $\mathcal{I}$-g^*-closed. By (a), A is τ^*-closed. Hence $\{x\}$ is τ^*-open.

$(b) \Rightarrow (a)$ Let A be a $\mathcal{I}$-g^*-closed and let $x \in \mathrm{cl}^*(A)$. We have the following two cases.

Case (i). $\{x\}$ is closed. By Theorem 2.10, $A^* - A$ does not contain any non-empty closed subset. This shows that $x \in A$.

Case (ii). $\{x\}$ is τ^*-open. Then $\{x\} \cap A \neq \phi$. Hence $x \in A$.

Thus in both cases $\{x\}$ is in A and so $A = \mathrm{cl}^* A$, that is A is τ^*-closed, which shows that X is a $g^*T_{\mathcal{I}}$-space. ■

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