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CONTRA $(\mu g, \lambda)$-CONTINUOUS FUNCTIONS

Abstract. In this paper we introduce and study some properties of contra $(\mu g, \lambda)$-continuous functions. We obtain some characterizations and several properties of such functions.

Key words: contra $(\mu g, \lambda)$-continuous, μg-closed set, generalized topological space

AMS Mathematics Subject Classification: 54C10, 54C08, 54D10.

1. Introduction

In 2002, generalized topological spaces introduced and developed by A. Csaszar [1]. A generalized topology (briefly a GT) μ on a nonempty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. The pair (X, μ) is called a generalized topological space (briefly a GTS). Elements of μ are called μ-open sets and a complement of a μ-open set is called a μ-closed set. The union of all μ-open subsets of a subset S of (X, μ) is called the μ-interior of S [2] and denoted by $i_\mu(S)$. The intersection of μ-closed sets containing S is called the μ-closure of S [2] and denoted by $c_\mu(S)$. A subset S of a space (X, μ) is called μ-regular closed (shortly μr-closed) [3] if $S = c_\mu(i_\mu(S))$. If $X \setminus S$ is μ-regular closed then S is called as μ-regular open (shortly μr-open).

A GTS (X, μ) is called strong if $X \in \mu$ and a quasi-topological space if μ is closed under finite intersections. (X, μ) is said to be extremally disconnected (briefly EDC) if the μ-closure of every μ-open set is μ-open.

Generalized closed sets introduced by N. Levine [6] in 1970. This notion has been studied and developed in many papers and plays a significant role in General Topology. The purpose of this paper is to introduce new types of continuous functions using this concept.

2. Preliminaries

Definition 1. *A subset A of a GTS (X, μ) is said to be μ-semi-open [1] (respectively μ-preopen [1], and μ-δ-open) if $A \subset c_\mu(i_\mu(A))$ (respectively*

$A \subset i_\mu(c_\mu(A))$, and A is the union of μr-open sets). The complements of the above sets are called respective closed ones.

Definition 2. *Let A be a subset of GTS (X, μ). Then, A is called μg-closed [9] if $c_\mu(A) \subset U$ whenever $A \subset U$ and U is μ-open. It is known that every μ-closed set in a GTS (X, μ) is μg-closed,but reverse implication is not true in general.The complement of a μg-closed set is called μg-open. The union of all μg-open subsets of a subset A of (X, μ) is called the μg-interior of A and denoted by $int_{\mu g}(A)$. The intersection of all μg-closed sets containing a subset A is called the μg-closure of A and denoted by $cl_{\mu g}(A)$. If A is μg-closed, then $A = cl_{\mu g}(A)$. The converse does not hold in general.*

The family of all μg-open (respectively μg-closed, μ-closed) sets of (X, μ) is denoted by $GO(\mu)$ (respectively $GC(\mu)$, $C(\mu)$. The family of all μg-open (respectively μg-closed, μ-closed) sets containing a point $x \in X$ is denoted by $GO(\mu, x)$ (respectively $GC(\mu, x)$, $C(\mu, x)$.

Definition 3. *A function $f : (X, \mu) \to (Y, \lambda)$, where (X, μ) and (Y, λ) are two GTS's, is called:*

(*a*) $(\mu g, \lambda)$-continuous *[12] if $f^{-1}(V)$ is μg-closed in (X, μ) for each λ-closed set V in (Y, λ),*
(*b*) $(\mu g, \lambda g)$-irresolute *[12]* ((μ, λ)-irresolute function*) if $f^{-1}(V)$ is μg-closed (μ-closed) in (X, μ) for each λg-closed (λ-closed set) V in (Y, λ),*
(*c*) contra (μ, λ)-continuous *[7] if $f^{-1}(V)$ is μ-closed in (X, μ) for each λ-open set V in (Y, λ),*
(*d*) *contra $(\mu g, \lambda)$-continuous $f^{-1}(V)$ is μg-closed in (X, μ) for each λ-open set V in (Y, λ),*
(*e*) *(μ, λ)-closed if $f(F)$ is λ-closed in (Y, λ) for each μ-closed set F in (X, μ) .*

Remark 1. Assume that $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous. Since $\emptyset \in \lambda$ and f is contra $(\mu g, \lambda)$-continuous, $f^{-1}(\emptyset) = \emptyset$ is μg-closed and this implies that $\emptyset$ is μ-closed, because it is true that $\emptyset \subset \emptyset \in \mu$ and $cl_\mu(\emptyset) \subset \emptyset\,(\subset cl_\mu(\emptyset))$. So, if $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous, then (X, μ) is a strong GTS. Same is true for if $f : (X, \mu) \to (Y, \lambda)$ is contra (μ, λ)-continuous.

The concept of contra $(\mu g, \lambda)$-continuous functions is a generalization of Contra sg-Continuous Maps [8].

Definition 4. *A GTS (X, μ) is called:*

(*a*) *μ-Urysohn if for each pair of distinct points x and y in X, there exist μ-open sets U and V such that $x \in U$, $y \in V$ and $c_\mu(U) \cap c_\mu(V) = \emptyset$.*

(*b*) μ-$T_{\frac{1}{2}}$-space *[9] if each μg-closed subset of (X, μ) is μ-closed,*
(*c*) μg-connected *[12] if X cannot be written as a disjoint union of two nonempty μg-open sets,*
(*d*) weakly μ-Hausdorff (*see [13]*) *if each element of X is an intersection of μr-closed sets.*

Definition 5 ([9]). *Let A be a subset of a GTS (X, μ). The set $\cap\{U \in \mu : A \subset U\}$ is called the* μ-kernel *of A and denoted by* μ-ker(A).

The following Lemma due to D. Jayanthi *stated without proof in [5], we give the proofs for the sake of completeness.*

Lemma 1 ([5]). *Let (X, μ) be a GTS and $A, B \subseteq X$. The following properties hold:*

(*a*) μ-ker$(A) \supset A$ *and if $A \in \mu$ then $A = \mu$-ker(A)*
(*b*) *If $A \subset B$, then μ-ker$(A) \subset \mu$-ker(B).*
(*c*) $x \in \mu$-ker(A) *iff $A \cap F \neq \emptyset$ for any μ-closed set F containing x,*

Proof. (*a*) Let $\mathcal{U}_A=\{O : A \subset O \in \mu\}$ be the family of all μ-open sets containing A. Then we have μ-ker$(A) = \bigcap \mathcal{U}_A \supset A$. If $A \in \mu$, then $A \in \mathcal{U}_A$ and $\bigcap \mathcal{U}_A \subset A$ gives $A = \bigcap \mathcal{U}_A$=$\mu$-ker$(A)$.

(*b*) Let $A \subseteq B$, consider $\mathcal{U}_A = \{U : A \subseteq U \in \mu\}$ and $\mathcal{U}_B = \{U : B \subseteq U \in \mu\}$. Then for $U \in \mathcal{U}_B$, it is true that $A \subseteq B \subseteq U \in \mathcal{U}_B$, that is $U \in \mathcal{U}_A$ and $\mathcal{U}_B \subseteq \mathcal{U}_A$ and this implies $[(\mathcal{U}_A - \mathcal{U}_B) \cup \mathcal{U}_B] = \mathcal{U}_A$, so we have

$$\begin{aligned}\mu\text{-}\ker(A) &= \bigcap \mathcal{U}_A \\ &= \Big(\bigcap (\mathcal{U}_A - \mathcal{U}_B)\Big) \cap \Big(\bigcap \mathcal{U}_B\Big) \\ &\subset \bigcap \mathcal{U}_B \\ &= \mu\text{-}\ker(B).\end{aligned}$$

(*c*) Let $x \in \mu$-ker(A) and suppose that $A \cap F = \emptyset$ for some μ-closed set F containing x. Then $A \subset X - F \in \mu$, and $x \notin X - F$ but we have

$$x \in \mu\text{-}\ker(A) \subset \mu\text{-}\ker(X - F) = X - F$$

which is a contradiction.

Conversely, assume that $A \cap F \neq \varnothing$ for any μ-closed set F containing x, but $x \notin \mu$-ker(A). Then, there exists a μ-open set V such that $A \subset V$ and $x \notin V$. Thus we have $x \in X - V\,(\subset X - A)$ and $X - V$ is μ-closed. But this implies $(X - V) \cap A \subset (X - A) \cap A = \emptyset$ that is $(X - V) \cap A = \emptyset$, but this contradicts with the hypothesis. ■

3. Characterizations of contra $(\mu g, \lambda)$-continuous functions

Remark 2. From the definitions we have stated above, we observe that in a *GTS* (X, μ), every contra (μ, λ)-continuous function is contra $(\mu g, \lambda)$-continuous. However the converse does not hold in general.

Example 1. Let $\mathbb{R}$ be the set of real numbers, $\mu = \{\mathbb{R}, \emptyset, \mathbb{R}\diagdown\{0\}, \mathbb{R}\diagdown\{-1, 1\}\}$ and $\lambda = \{\emptyset, \{1\}, \mathbb{R}\}$. Let $f : (\mathbb{R}, \mu) \to (\mathbb{R}, \lambda)$ be the identity function. Then f is contra $(\mu g, \lambda)$-continuous but not contra (μ, λ)-continuous.

Proposition 1. *Let $f : (X, \mu) \to (Y, \lambda)$ be a function. Suppose that (X, μ) is μ-$T_{\frac{1}{2}}$-space. Then the following properties are equivalent:*
(*i*) *f is contra $(\mu g, \lambda)$-continuous,*
(*ii*) *f is contra (μ, λ)-continuous.*

Proof. This is clear. ■

Theorem 1. *Suppose that $GC(\mu)$ is closed under arbitrary intersections. Then the following are equivalent for a function $f : (X, \mu) \to (Y, \lambda)$:*

(*a*) *f is contra $(\mu g, \lambda)$-continuous,*
(*b*) *The inverse image of each λ-closed set in (Y, λ) is μg-open.*
(*c*) *For each $x \in X$ and each λ-closed set B containing $f(x)$, there exists a μg-open set A in X such that $x \in A$ and $f(A) \subset B$,*
(*d*) *$f(cl_{\mu g}(A)) \subset \lambda\text{-ker}(f(A))$ for every subset A of X,*
(*e*) *$cl_{\mu g}(f^{-1}(B)) \subset f^{-1}(\lambda\text{-ker}(B))$ for every subset B of Y.*

Proof. $(a) \implies (b)$: Let G be a λ-closed set in Y. Then $Y \setminus G$ is λ-open and by (a), $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$ is μg-closed. Thus $f^{-1}(G)$ is μg-open.

$(b) \implies (a)$: Let $U \in \lambda$. Then $Y \setminus U$ is λ-closed and by (b), $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is μg-open, thus $f^{-1}(U)$ is μg-closed. Hence, f is contra $(\mu g, \lambda)$-continuous.

$(a) \implies (c)$: Let $x \in X$ and B be a λ-closed set with $f(x) \in B$. By (a), it follows that $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is μg-closed and so $f^{-1}(B)$ is μg-open. Take $A = f^{-1}(B)$. We obtain that $x \in A$ and $f(A) \subset B$.

$(c) \implies (b)$: Let B be a λ-closed set with $x \in f^{-1}(B)$. Since $f(x) \in B$, by (c) there exists a μg-open set A containing x such that $f(A) \subset B$. It follows that $x \in A \subset f^{-1}(B)$. Hence, $f^{-1}(B)$ is μg-open.

$(b) \implies (d)$: Let A be any subset of X and $y \notin \lambda\text{-ker}(f(A))$. Then by Lemma 1, there exists a λ-closed set F containing y such that $f(A) \cap F = \emptyset$. Hence, we have $A \cap f^{-1}(F) = \emptyset$ and $cl_{\mu g}(A) \cap f^{-1}(F) = \emptyset$. Thus we obtain, $f(cl_{\mu g}(A)) \cap F = \emptyset$ and $y \notin f(cl_{\mu g}(A))$. Therefore, $f(cl_{\mu g}(A)) \subset \lambda\text{-ker}(f(A))$.

$(d) \implies (e)$: Let B be any subset of Y. By (d) we have

$$f(cl_{\mu g}(f^{-1}(B))) \subset \lambda\text{- ker}(f(f^{-1}(B))) \subset \lambda\text{- ker}(B).$$

and this implies

$$cl_{\mu g}(f^{-1}(B)) \subset f^{-1}\left(f(cl_{\mu g}(f^{-1}(B)))\right) \subset f^{-1}\left(\lambda\text{- ker}(B)\right)$$

Then we have the result $cl_{\mu g}(f^{-1}(B)) \subset f^{-1}(\lambda\text{-ker}(B))$.

$(e) \implies (a)$: Let $B \in \lambda$, then by (e), $cl_{\mu g}(f^{-1}(B)) \subset f^{-1}(\lambda\text{-ker}(B)) = f^{-1}(B)$ and $cl_{\mu g}(f^{-1}(B)) = f^{-1}(B)$. Since $GC(\mu)$ is closed under arbitrary intersections, $f^{-1}(B)$ is μg-closed in (X, μ). ■

Notation 1. *Let (X, μ) and (Y, κ) be generalized topological spaces, and let $\mathcal{U} = \{U \times V : U \in \mu, V \in \kappa\}$. It is known that $\mathcal{U}$ generates a generalized topology $\nu = \mu \times \kappa$ on $X \times Y$, called the generalized product topology ([4], [11]) on $X \times Y$, that is, $\nu = \{$ all possible unions of members of $\mathcal{U}\}$*

Theorem 2. *Let $f : (X, \mu) \to (Y, \lambda)$ be a function and $g : (X, \mu) \to (X \times Y, \nu)$ be the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra $(\mu g, \nu)$-continuous, then f is $(\mu g, \lambda)$-continuous.*

Proof. Let U be any λ-open set in (Y, λ). By remark 1, (X, μ) is strong GTS, hence $X \times U$ is a ν-open set in $X \times Y$. It follows that $f^{-1}(U) = g^{-1}(X \times U)$ is μg-closed. Thus, f is contra $(\mu g, \lambda)$-continuous. ■

Definition 6. *For a function $f : (X, \mu) \to (Y, \lambda)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.*

Definition 7. *Let (X, μ) and (Y, λ) are two GTS's, consider ν as generalized product space of the μ and λ on $X \times Y$. The graph $G(f)$ of a function $f : (X, \mu) \to (Y, \lambda)$ is said to be contra νg-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists a μg-open set U in X containing x and a λ-closed set V in Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Proposition 2. *The following properties are equivalent for the graph $G(f)$ of a function $f : (X, \mu) \to (Y, \lambda)$:*

(a) $G(f)$ is contra νg-closed graph,
(b) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exists a μg-open set U in X containing x and a λ-closed set V in Y containing y such that $f(U) \cap V = \emptyset$.

Proof. $(a) \implies (b)$: Let $(x, y) \in (X \times Y) \setminus G(f)$. By (a), there exists a μg-open set U in X containing x and a λ-closed set in Y containing y

such that $(U \times V) \cap G(f) = \emptyset$. Since $(x, y) \notin G(f)$, $x \in U$, $y \in V$ we have $f(x) \neq y$ and therefore $f(U) \cap V = \emptyset$.

$(b) \implies (a)$: Let $(x, y) \in (X \times Y) \setminus G(f)$. By (b), there exists a μg-open set U in X containing x and a λ-closed set V in Y containing y such that $f(U) \cap V = \emptyset$. Hence, $(x, y) \in (U \times V) \subset (X \times Y) \setminus G(f)$. ■

Theorem 3. *If $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous function and (Y, λ) is λ-Urysohn, then $G(f)$ is contra νg-closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. It follows that $f(x) \neq y$. Since (Y, λ) is λ-Urysohn, there exist λ-open sets B and C such that $f(x) \in B$, $y \in C$ and $c_\lambda(B) \cap c_\lambda(C) = \emptyset$. Since f is contra $(\mu g, \lambda)$-continuous, there exists a μg-open set A in X containing x such that $f(A) \subset c_\lambda(B)$. Therefore, $f(A) \cap c_\lambda(C) = \emptyset$ and $G(f)$ is contra νg-closed graph in $X \times Y$. ■

Theorem 4. *Let $\{(X_i, \mu_i) : i \in I\}$ be any family of strong GTS's. If $f : (X, \mu) \to (\Pi X_i, \nu)$ is contra $(\mu g, \nu)$-continuous, then $p_i \circ f : (X, \mu) \to (X_i, \mu_i)$ is contra $(\mu g, \mu_i)$-continuous for each $i \in I$, where p_i is the projection of $(\Pi X_i, \nu)$ onto (X_i, μ_i).*

Proof. We shall consider a fixed $i \in I$. Suppose U_i is an arbitrary μ_i-open set of X_i. Since each (X_i, μ_i) is strong GTS, p_i is (ν, μ_i)-continuous by Proposition 2.7 of [4], that is $p_i^{-1}(U_i)$ is ν-open in $(\Pi X_i, \nu)$. Since f is contra $(\mu g, \nu)$-continuous, we have $f^{-1}(p_i^{-1}(U_i)) = (p_i \circ f)^{-1}(U_i)$ is $\mu_i g$-closed. Therefore, $p_i \circ f$ is contra $(\mu g, \mu_i)$-continuous. ■

Definition 8. *A GTS (X, μ) is said to be locally μg-indiscrete if every μg-open set of (X, μ) is μ-closed.*

Theorem 5. *If $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous with (X, μ) is locally μg-indiscrete, then f is contra (μ, λ)-continuous.*

Proof. This is clear. ■

Theorem 6. *Suppose that (X, μ), (Y, λ) are two GTS's and $GO(\mu)$ is closed under arbitrary unions. If a function $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous and (Y, λ) is λ-regular, then f is $(\mu g, \lambda)$-continuous.*

Proof. Let x be an arbitrary point of (X, μ) and V be a λ-open set of Y containing $f(x)$. Since (Y, λ) is λ-regular, there exists a λ-open set G in Y containing $f(x)$ such that $c_\lambda(G) \subset V$. Because f is contra $(\mu g, \lambda)$-continuous, there exists $U \in GO(\mu)$ containing x such that $f(U) \subset c_\lambda(G)$. Then $f(U) \subset c_\lambda(G) \subset V$. Hence, f is $(\mu g, \lambda)$-continuous. ■

Theorem 7. *Let (X, μ) be a μg-connected GTS and (Y, λ) be any GTS. If there is surjective, contra $(\mu g, \lambda)$-continuous function $f : (X, \mu) \to (Y, \lambda)$, then (Y, λ) is λ-connected.*

Proof. Let $f : (X, \mu) \to (Y, \lambda)$ be a contra $(\mu g, \lambda)$-continuous, surjective function of a μg-connected space (X, μ) to a GTS (Y, λ). Suppose that (Y, λ) is λ-disconnected. Let A and B form a disconnection of (Y, λ). Then A and B are λ-open and $Y = A \cup B$ where $A \cap B = \emptyset$. Since f is contra $(\mu g, \lambda)$-continuous and surjective, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty μg-closed sets in (X, μ). Also $f^{-1}(A) \cap f^{-1}(B) = \emptyset$, so $f^{-1}(A)$ and $f^{-1}(B)$ are μg-open. This contradicts with the fact that (X, μ) is μg-connected. Hence (Y, λ) is λ-connected. ■

Theorem 8. *Let (X, μ) be μg-connected. Then each contra $(\mu g, \lambda)$-continuous function of X into a λ-discrete GTS (Y, λ) with at least two points is a constant function.*

Proof. Let $f : (X, \mu) \to (Y, \lambda)$ be a contra $(\mu g, \lambda)$-continuous function and (X, μ) be a μg-connected GTS. Then (X, μ) is covered by μg-open and μg-closed covering $\{f^{-1}(\{y\}) : y \in Y\}$. By assumption, $f^{-1}(\{y\}) = \emptyset$ or X for each $y \in Y$. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, then it fails to be a function. Then there exists only one point $y \in Y$ such that $f^{-1}(\{y\}) \neq \emptyset$ and hence $f^{-1}(\{y\}) = X$ which shows that f is a constant function. ■

Theorem 9. *If f is a contra $(\mu g, \lambda)$-continuous function from a μg-connected GTS (X, μ) onto a GTS (Y, λ), then Y is not a λ-discrete space.*

Proof. Suppose that (Y, λ) is λ-discrete. Let A be a proper nonempty λ-open and λ-closed subset of (Y, λ). Then $f^{-1}(A)$ is a proper nonempty μg-open subset of (X, μ), which is a contradiction with the fact that (X, μ) is μg-connected. ■

Definition 9. *A GTS (X, μ) is said to be μg-normal (resp. μ-normal [10]) if each pair of nonempty μ-closed sets can be separated by disjoint μg-open (resp. μ-open) sets.*

Theorem 10. *If $f : (X, \mu) \to (Y, \lambda)$ is a contra $(\mu g, \lambda)$-continuous, (μ, λ)-closed, injection and (Y, λ) is λ-normal, then (X, μ) is μg-normal.*

Proof. Let F_1, F_2 be disjoint μ-closed subsets of (X, μ). Since f is (μ, λ)-closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint λ-closed subset of (Y, λ). $f(F_1)$ and $f(F_2)$ are separated by disjoint λ-open sets V_1, V_2, respectively, because (Y, λ) is λ-normal. Hence, $F_i \subset f^{-1}(V_i)$ and $f^{-1}(V_i)$ is μg-open in (X, μ) for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, (X, μ) is μg-normal. ■

4. Composition properties

Remark 3. Let $f:(X,\mu)\to(Y,\lambda)$ be contra $(\mu g,\lambda)$-continuous and $g:(Y,\lambda)\to(Z,\nu)$ be contra $(\lambda g,\nu)$-continuous. Then, the composition $g\circ f:(X,\mu)\to(Z,\nu)$ need not be contra $(\mu g,\nu)$-continuous.

Example 2. Let $\mathbb{R}$ be the set of real numbers, $\mu=\{\emptyset,\mathbb{R}\setminus\{-1\},\mathbb{R}\setminus\{1\},\mathbb{R}\setminus\{-1,1\},\mathbb{R}\}$, $\lambda=\{\emptyset,\mathbb{R}\setminus\{0\},\mathbb{R}\setminus\{0,1\},\mathbb{R}\}$ and $\nu=\{\emptyset,\mathbb{R}\setminus\{-1,1\},\mathbb{R}\}$. Then the identity function $f:(\mathbb{R},\mu)\to(\mathbb{R},\lambda)$ is contra $(\mu g,\lambda)$-continuous and the identity function $g:(\mathbb{R},\lambda)\to(\mathbb{R},\nu)$ is contra $(\lambda g,\nu)$-continuous. But the composition $g\circ f:(\mathbb{R},\mu)\to(\mathbb{R},\nu)$ is not contra $(\mu g,\nu)$-continuous.

Theorem 11. *Let $(X,\mu),(Z,\nu)$ be two GTS's and (Y,λ) be a λ-$T_{\frac{1}{2}}$-space. Let $f:(X,\mu)\to(Y,\lambda)$ be (μ,λ)-irresolute function and $g:(Y,\lambda)\to(Z,\nu)$ be contra $(\lambda g,\nu)$-continuous. Then $g\circ f:(X,\mu)\to(Z,\nu)$ is contra $(\mu g,\nu)$-continuous.*

Proof. Let F be any ν-open subset of (Z,ν). Since g is contra $(\lambda g,\nu)$-continuous, $f^{-1}(F)$ is λg-closed in (Y,λ). But (Y,λ) is λ-$T_{\frac{1}{2}}$-space, so $f^{-1}(F)$ is λ-closed. Since f is (μ,λ)-irresolute, $f^{-1}(g^{-1}(F))=(g\circ f)^{-1}(F)$ is μ-closed. Since every μ-closed set in a GTS (X,μ) is μg-closed, $g\circ f:(X,\mu)\to(Z,\nu)$ is contra $(\mu g,\nu)$-continuous. ■

Theorem 12. *Let $f:(X,\mu)\to(Y,\lambda)$ be $(\mu g,\lambda g)$-irresolute function and $g:(Y,\lambda)\to(Z,\nu)$ be contra $(\lambda g,\nu)$-continuous function. Then $g\circ f:(X,\mu)\to(Z,\nu)$ is contra $(\mu g,\nu)$-continuous.*

Proof. Let F be a ν-open set in (Z,ν). Then $g^{-1}(F)$ is λg-closed in (Y,λ), because g is contra $(\lambda g,\nu)$-continuous. Since f is $(\mu g,\lambda g)$-irresolute, $f^{-1}(g^{-1}(F))=(g\circ f)^{-1}(F)$ is μg-closed. Thus, $g\circ f:(X,\mu)\to(Z,\nu)$ is contra $(\mu g,\nu)$-continuous. ■

Corollary 1. *Let $f:(X,\mu)\to(Y,\lambda)$ be $(\mu g,\lambda g)$-irresolute and $g:(Y,\lambda)\to(Z,\nu)$ be contra (λ,ν)-continuous function. Then $g\circ f:(X,\mu)\to(Z,\nu)$ is contra $(\mu g,\nu)$-continuous.*

Definition 10. *A function $f:(X,\mu)\to(Y,\lambda)$ is said to be pre-$(\mu g,\lambda g)$-open if the image of every μg-open set is λg-open.*

Theorem 13. *Let $f:(X,\mu)\to(Y,\lambda)$ be surjective, $(\mu g,\lambda g)$-irresolute, pre-$(\mu g,\lambda g)$-open function and $g:(Y,\lambda)\to(Z,\nu)$ be any function. Then $g\circ f:(X,\mu)\to(Z,\nu)$ is contra $(\mu g,\nu)$-continuous if g is contra $(\lambda g,\nu)$-continuous.*

Proof. Let $g:(Y,\lambda)\to(Z,\nu)$ be a contra $(\lambda g,\nu)$-continuous function and F be a ν-open subset of (Z,ν). Since g is contra $(\lambda g,\nu)$-continuous,

$g^{-1}(F)$ is λg-closed. But $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is μg-closed because f is $(\mu g, \lambda g)$-irresolute. Thus, $g \circ f : (X, \mu) \to (Z, \nu)$ is contra $(\mu g, \nu)$-continuous.

Conversely, let $g \circ f : (X, \mu) \to (Z, \nu)$ be contra $(\mu g, \nu)$-continuous and let F be a ν-closed subset of (Z, ν). Then, $(g \circ f)^{-1}(F)$ is a μg-open. Since f is pre-$(\mu g, \lambda g)$-open and surjective, $f(f^{-1}(g^{-1}(F))) = g^{-1}(F)$ is λg-open. Hence, $g : (Y, \lambda) \to (Z, \nu)$ is contra $(\lambda g, \nu)$-continuous. ■

Theorem 14. *If $f : (X, \mu) \to (Y, \lambda)$ is $(\mu g, \lambda g)$-irresolute function with (Y, λ) as locally λg-indiscrete space and $g : (Y, \lambda) \to (Z, \nu)$ is contra $(\lambda g, \nu)$-continuous function, then $g \circ f : (X, \mu) \to (Z, \nu)$ is $(\mu g, \nu)$-continuous.*

Proof. Let F be a ν-closed subset of (Z, ν). Since, g is contra $(\lambda g, \nu)$-continuous, $g^{-1}(F)$ is λg-open in (Y, λ). But (Y, λ) is locally λg-indiscrete, so $g^{-1}(F)$ is λg-closed. Since f is $(\mu g, \lambda g)$-irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is μg-closed. Therefore, $g \circ f$ is $(\mu g, \nu)$-continuous. ■

5. Some covering and separation properties

Definition 11. *A GTS (X, μ) is said to be*

(a) μg-compact *if every μg-open cover of (X, μ) has a finite subcover,*
(b) strongly μ-S-closed *if every μ-closed cover of (X, μ) has a finite subcover,*
(c) countably μg-compact *if every countable cover of (X, μ) by μg-open sets has a finite subcover,*
(d) strongly countably μ-S-closed *if every countable cover of (X, μ) by μ-closed sets has a finite subcover,*
(e) μg-Lindelöf *if every μg-open cover of (X, μ) has a countable subcover,*
(f) strongly μ-S-Lindelöf *if every μ-closed cover of (X, μ) has a countable subcover.*

Theorem 15. *The surjective contra $(\mu g, \lambda)$-continuous image of a μg-compact (resp. μg-Lindelöf, countably μg-compact) space is strongly λ-S-closed (resp. strongly λ-S-Lindelöf, strongly countable λ-S-closed).*

Proof. Suppose that $f : (X, \mu) \to (Y, \lambda)$ is a contra $(\mu g, \lambda)$-continuous surjection. Let $\{V_\alpha : \alpha \in \nabla\}$ be any λ-closed cover of (Y, λ). Since f is contra $(\mu g, \lambda)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a μg-open cover of X and hence there exists a finite subset ∇_0 of ∇ such that $X = \cup_{\alpha \in \nabla_0} f^{-1}(V_\alpha)$. Therefore we have, $Y = \cup_{\alpha \in \nabla_0} V_\alpha$ and (Y, λ) is strongly λ-S-closed.

The other proofs can be obtained similarly. ■

Definition 12. *A GTS (X, μ) is said to be*

(a) μg-closed-compact *if every* μg*-closed cover of* (X, μ) *has a finite subcover,*
(b) countably μg-closed compact *if every countable cover of* (X, μ) *by* μg*-closed sets has a finite subcover,*
(c) μg-closed-Lindelöf *if every* μg*-closed cover of* (X, μ) *has a countable subcover.*

Theorem 16. *Surjective, contra* $(\mu g, \lambda)$*-continuous image of a* μg*-closed compact (resp.* μg*-closed Lindelöf, countably* μg*-closed compact) space is* λ*-compact (resp.* λ*-Lindelöf, countably* λ*-compact).*

Proof. Suppose that $f : (X, \mu) \to (Y, \lambda)$ is a contra $(\mu g, \lambda)$-continuous surjection. Let $\{V_\alpha : \alpha \in \nabla\}$ be any λ-open cover of (Y, λ). Since f is contra $(\mu g, \lambda)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a μg-closed cover of (X, μ), hence there exists a finite subset ∇_0 of ∇ such that $X = \cup_{\alpha \in \nabla_0} f^{-1}(V_\alpha)$. Therefore we have $Y = \cup_{\alpha \in \nabla_0} V_\alpha$ and Y is λ-compact.The other proofs can be obtained similarly. ■

Definition 13. *A GTS* (X, μ) *is said to be* μg*-*T_1 *if for each pair of distinct points* x *and* y *in* (X, μ)*, there exist* μg*-open sets* U *and* V *containing* x *and* y *respectively, such that* $y \notin U$ *and* $x \notin V$*.*

Definition 14. *A GTS* (X, μ) *is said to be* μg*-*T_2 *if for each pair of distinct points* x *and* y *in* (X, μ)*, there exist disjoint* μg*-open sets* U *and* V *containing* x *and* y *respectively.*

Theorem 17. *Let* (X, μ)*,* (Y, λ) *be two GTS's. If*

(a) *for each pair of distinct points* x *and* y *in* (X, μ)*, there exists a function* f *of* X *on to* Y *such that* $f(x) \neq f(y)$*,*
(b) (Y, λ) *is* λ*-Urysohn space, and*
(c) f *is contra* $(\mu g, \lambda)$*-continuous at* x *and* y*.*

Then (X, μ) *is* μg*-*T_2*.*

Proof. Let x and y be distinct points in (X, μ), from the hypothesis by (b) there exists a λ-Urysohn space (Y, λ), by (a) there exists a function $f : (X, \mu) \to (Y, \lambda)$ such that $f(x) \neq f(y)$ and by (c). f is contra $(\mu g, \lambda)$-continuous at x and y. Let $v = f(x)$ and $w = f(y)$, then $v \neq w$. Since (Y, λ) is λ-Urysohn, there exists λ-open sets V and W containing v and w respectively, such that $c_\lambda(V) \cap c_\lambda(W) = \emptyset$. Since f is contra $(\mu g, \lambda)$-continuous at x and y, there exist μg-open sets A and B containing x and y respectively, such that $f(A) \subset c_\lambda(V)$ and $f(B) \subset c_\lambda(W)$. We have $A \cap B = \emptyset$ since $c_\lambda(V) \cap c_\lambda(W) = \emptyset$. Hence, (X, μ) is μg-T_2. ■

Theorem 18. *If $f : (X, \mu) \to (Y, \lambda)$ is a contra $(\mu g, \lambda)$-continuous injection and (Y, λ) is weakly λ-Hausdorff, then (X, μ) is μg-T_1.*

Proof. Suppose that (Y, λ) is weakly λ-Hausdorff, then for any pair of distinct points x and y in (X, μ), there exist λr-closed sets A, B in (Y, λ) such that $f(x) \in A$, $f(x) \notin B$ and $f(y) \in B$, $f(y) \notin A$. Since f is contra $(\mu g, \lambda)$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are μg-open subsets of (X, μ) such that $x \in f^{-1}(A)$, $x \notin f^{-1}(B)$ and $y \in f^{-1}(B)$, $y \notin f^{-1}(A)$. Hence, (X, μ) is μg-T_1. ■

Theorem 19. *Let $f : (X, \mu) \to (Y, \lambda)$ have a contra $(\mu g, \lambda)$-closed graph. If f is injective, then (X, μ) is μg-T_1.*

Proof. Let x and y be distinct points in (X, μ). Then we have $(x, f(y)) \in (X \times Y) \setminus G(f)$. Then, there exists a μg-open set U in (X, μ) containing x and a λ-closed set F containing $f(y)$ such that $f(U) \cap F = \emptyset$. Hence, $U \cap f^{-1}(F) = \emptyset$. Therefore, we have $y \notin U$. This implies (X, μ) is μg-T_1. ■

Theorem 20. *Let $f : (X, \mu) \to (Y, \lambda)$ be a contra $(\mu g, \lambda)$-continuous injection. If (Y, λ) is ultra λ-Hausdorff, then (X, μ) is μg-T_2.*

Proof. Let x and y be two distinct points in (X, μ). Then $f(x) \neq f(y)$ and there exist λ-clopen sets A, B containing $f(x)$, $f(y)$ respectively, such that $A \cap B = \emptyset$. Since f is contra $(\mu g, \lambda)$-continuous, then $f^{-1}(A)$, $f^{-1}(B)$ are μg-open sets such that $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence, (X, μ) is μg-T_2. ■

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