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FASCICULI MATHEMATICI

Nr 59 2017

DOI:10.1515/fascmath-2017-0023

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SOME NEW FRACTIONAL FEJÉR TYPE INEQUALITIES FOR CONVEX FUNCTIONS

Abstract. In this paper, firstly, a new identity for conformable fractional integrals is established. Then by making use of the established identity, some new fractional Fejér type inequalities are established. The results presented here have some relationships with the results of Set *et al.* (2015), proved in [6].

Key words: Convex function, Fejér inequality, Riemann-Liouville fractional integral, conformable fractional integral.

AMS Mathematics Subject Classification: 26A33, 26D10, 26D15, 33B15.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v)$$

holds for all $u, v \in I$ and $\lambda \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known as the Hermite-Hadamard inequality.

In [2], Fejér established the following inequality which is the weighted generalization of Hermite-Hadamard inequality:

Theorem 1. *If $f : [a,b] \to \mathbb{R}$ is a convex function, $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &\leq \frac{1}{b-a}\int_a^b f(x)\,g(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \end{aligned} \tag{2}$$

We give some necessary definitions related to fractional calculus which are used throughout this paper.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Lioville integrals $J^{\mu}_{a+}f$ and $J^{\mu}_{b-}f$ of order $\mu > 0$ are defined by*

$$J^{\mu}_{a+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt, \quad x > a$$

and

$$J^{\mu}_{b-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\mu) = \int_0^{\infty} e^{-t}u^{\mu-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral reduces to classical integral. In [5] , Khalil et al. gave a new definition that is called "Conformable fractional derivative". They not only proved further properties of this definitons but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. In [3], Abdeljawad gave the following definitions of Right-Left fractional integrals:

Definition 2. *Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$ then the left conformable fractional integral starting at a if order α is defined by*

$$I^a_{\alpha} f(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

Analogously, the right conformable fractional integral is defined by

$${}^{b}I_{\alpha} f(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n + 1 - n = 1$ and hence $(I^a_{\alpha} f)(t) = (J^a_{n+1} f)(t)$. We define the Beta function [1, p18]:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0,$$

where Γ is Gamma function. The incomplete Beta function is defined by

$$B_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0, \quad 0 \le x \le 1.$$

The incomplete beta function satisfies following identity

$$B_t(a,b) + B_{1-t}(b,a) = B(a,b). \tag{3}$$

The Newton-Leibnitz integral formula is as the following:

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\delta f}{\delta t}dx + f(b(t),t)b^{'}(t) + f(a(t),t)a^{'}(t) \tag{4}$$

where $f(x,t)$ be a function such that the partial derivative of f with respect to t exists, and is continuous.

In [6], Set *et al.* gave following theorems:

Theorem 2. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f^{'}|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{(\frac{a+b}{2})^-}g(a) + J^{\alpha}_{(\frac{a+b}{2})^+}g(b)\right] \right. \\ &\left. - \left[J^{\alpha}_{(\frac{a+b}{2})^-}(fg)(a) + J^{\alpha}_{(\frac{a+b}{2})^+}(fg)(b)\right] \right| \\ &\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)}(|f^{'}(a)| + |f^{'}(b)|) \end{aligned} \tag{5}$$

where $\|g\|_{[a,b],\infty} = \sup_{x\in[a,b]} |g(x)|$ and $\alpha > 0$.

Theorem 3. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f^{'}|^q$ is convex on $[a,b]$,$q > 1$ then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{(\frac{a+b}{2})^-}g(a) + J^{\alpha}_{(\frac{a+b}{2})^+}g(b)\right] \right. \\ &\left. - \left[J^{\alpha}_{(\frac{a+b}{2})^-}(fg)(a) + J^{\alpha}_{(\frac{a+b}{2})^+}(fg)(b)\right] \right| \\ &\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{1/q}\Gamma(\alpha+1)}\Big[((\alpha+3)|f^{'}(a)|^q \\ &+ (\alpha+1)|f^{'}(b)|^q)^{1/q} + ((\alpha+1)|f^{'}(a)|^q + (\alpha+3)|f^{'}(b)|^q)^{1/q}\Big] \end{aligned} \tag{6}$$

where $\|g\|_{[a,b],\infty} = \sup_{x\in[a,b]} |g(x)|$ and $\alpha > 0$.

Theorem 4. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f^{'}|^q$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{(\frac{a+b}{2})^-}g(a) + J^{\alpha}_{(\frac{a+b}{2})^+}g(b)\right] \right. \tag{7}$$

$$- \left[J^{\alpha}_{(\frac{a+b}{2})^-}(fg)(a) + J^{\alpha}_{(\frac{a+b}{2})^+}(fg)(b) \right]$$
$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha+1+\frac{2}{q}}(\alpha p+1)^{1/p}\Gamma(\alpha+1)}$$
$$\times \left[(3|f^{'}(a)|^q + |f^{'}(b)|^q)^{1/q} + (|f^{'}(a)|^q + 3|f^{'}(b)|^q)^{1/q} \right]$$

where $1/p + 1/q = 1$, $\|g\|_{[a,b],\infty} = \sup_{x\in[a,b]}|g(x)|$ *and* $\alpha > 0$.

Motivated by the recent results given in [3, 4, 5, 6, 7, 8, 9], in the paper, we obtain here new Hermite-Hadamard-Fejer type inequalities for convex functions via conformable fractional integral. An interesting feature of the results given in this paper is that they provide new estimates on Fejer type inequalities for conformable fractional integrals.

2. Main results

Lemma 1. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$ *and* $f^{'} \in L[a,b]$. *If* $g : [a,b] \to \mathbb{R}$ *is integrable and symmetric to* $(a+b)/2$, $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ *and then the following equality for conformable fractional integral holds:*

$$f\left(\frac{a+b}{2}\right)[I_{\alpha}^{\frac{a+b}{2}}g(b) + {}^{\frac{a+b}{2}}I_{\alpha}g(a)] - [I_{\alpha}^{\frac{a+b}{2}}(fg)(b) + {}^{\frac{a+b}{2}}I_{\alpha}(fg)(a)] \tag{8}$$
$$= \frac{(b-a)^{\alpha+1}}{n!}\int_0^1 k(t)f^{'}((1-t)a+tb)dt$$

where

$$k(t) = \begin{cases} \int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}g((1-s)a+sb)ds, & t\in[0,\frac{1}{2}] \\ \int_1^t (1-s)^n(s-\frac{1}{2})^{\alpha-n-1}g((1-s)a+sb)ds, & t\in[\frac{1}{2},1]. \end{cases}$$

Proof. It is clear that

$$\begin{aligned} I &= \int_0^1 k(t)f^{'}((1-t)a+tb)dt \\ &= \int_0^{1/2}\left(\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}g((1-s)a+sb)ds\right)f^{'}((1-t)a+tb)dt \\ &\quad + \int_{1/2}^1\left(\int_1^t (1-s)^n(s-\frac{1}{2})^{\alpha-n-1}g((1-s)a+sb)ds\right) \\ &\qquad \times f^{'}((1-t)a+tb)dt \\ &= I_1 + I_2. \end{aligned}$$

By integrating by parts we get

$$\begin{aligned}
I_1 &= \left(\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}g((1-s)a+sb)ds\right)\frac{f((1-t)a+tb)}{b-a}\Big|_0^{1/2} \\
&\quad -\frac{1}{b-a}\int_0^{1/2} t^n(\frac{1}{2}-t)^{\alpha-n-1}g((1-t)a+tb)f((1-t)a+tb)dt \\
&= \left(\int_0^{1/2} s^n(\frac{1}{2}-s)^{\alpha-n-1}g((1-s)a+sb)ds\right)\frac{f(\frac{a+b}{2})}{b-a} \\
&\quad -\frac{1}{b-a}\int_0^{1/2} t^n(\frac{1}{2}-t)^{\alpha-n-1}(fg)((1-t)a+tb)dt \\
&= \left(\int_a^{(a+b)/2}(x-a)^n(\frac{a+b}{2}-x)^{\alpha-n-1}g(x)dx\right)\frac{f(\frac{a+b}{2})}{(b-a)^{\alpha+1}} \\
&\quad -\frac{1}{(b-a)^{\alpha+1}}\int_a^{(a+b)/2}(x-a)^n(\frac{a+b}{2}-x)^{\alpha-n-1}(fg)(x)dx \\
&= \frac{n!}{(b-a)^{\alpha+1}}\ {}^{\frac{a+b}{2}}I_\alpha g(a)f(\frac{a+b}{2})-\frac{n!}{(b-a)^{\alpha+1}}\ {}^{\frac{a+b}{2}}I_\alpha(fg)(a)
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \left(\int_1^t (1-s)^n(s-\frac{1}{2})^{\alpha-n-1}g((1-s)a+sb)ds\right)\frac{f((1-t)a+tb)}{b-a}\Big|_{\frac{1}{2}}^{1} \\
&\quad -\frac{1}{b-a}\int_{\frac{1}{2}}^{1}(1-t)^n(t-\frac{1}{2})^{\alpha-n-1}g((1-t)a+tb)f((1-t)a+tb)dt \\
&= \left(\int_{\frac{1}{2}}^{1}(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}g((1-s)a+sb)ds\right)\frac{f(\frac{a+b}{2})}{b-a} \\
&\quad -\frac{1}{b-a}\int_{\frac{1}{2}}^{1}(1-t)^n(t-\frac{1}{2})^{\alpha-n-1}(fg)((1-t)a+tb)dt \\
&= \left(\int_a^{(a+b)/2}(b-x)^n(x-\frac{a+b}{2})^{\alpha-n-1}g(x)dx\right)\frac{f(\frac{a+b}{2})}{(b-a)^{\alpha+1}} \\
&\quad -\frac{1}{(b-a)^{\alpha+1}}\int_a^{(a+b)/2}(b-x)^n(x-\frac{a+b}{2})^{\alpha-n-1}(fg)(x)dx \\
&= \frac{n!}{(b-a)^{\alpha+1}}I_\alpha^{\frac{a+b}{2}}g(b)f(\frac{a+b}{2})-\frac{n!}{(b-a)^{\alpha+1}}I_\alpha^{\frac{a+b}{2}}(fg)(b).
\end{aligned}$$

So, we can write

$$\begin{aligned}
I &= I_1+I_2 \\
&= \frac{n!}{(b-a)^{\alpha+1}}\left\{f\left(\frac{a+b}{2}\right)[I_\alpha^{\frac{a+b}{2}}g(b)+\ {}^{\frac{a+b}{2}}I_\alpha g(a)]\right.
\end{aligned}$$

$$- \left[I_{\alpha}^{\frac{a+b}{2}}(fg)(b) + {}^{\frac{a+b}{2}}I_{\alpha}(fg)(a) \right] \Big\} .$$

Multiplying the both sides by $\frac{(b-a)^{\alpha+1}}{n!}$, we obtain the desired result which completes the proof. ■

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$ with $a < b$. If $|f^{'}|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continious and symmetric to $(a+b)/2$, then the following inequality for comformable fractional integrals holds*

$$\tag{9} \begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \left[I_{\alpha}^{\frac{a+b}{2}} g(b) + {}^{\frac{a+b}{2}}I_{\alpha} g(a) \right] - \left[I_{\alpha}^{\frac{a+b}{2}}(fg)(b) + {}^{\frac{a+b}{2}}I_{\alpha}(fg)(a) \right| \\ &\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{n!} \left(\frac{|f^{'}(a)| + |f^{'}(b)|}{2^{\alpha+1}} \right) \\ &\times \left[B(n+1,\alpha-n) - B(n+2,\alpha-n) \right] \end{aligned}$$

where $\|g\|_{[a,b],\infty} = \sup_{x\in[a,b]} |g(x)|$ *and* $\alpha \in (n, n+1]$, $n = 0,1,2,....$

Proof. From Lemma 1 and $|f^{'}|$ is convex, we have

$$\tag{10} \begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \left[I_{\alpha}^{\frac{a+b}{2}} g(b) + {}^{\frac{a+b}{2}}I_{\alpha} g(a) \right] - \left[I_{\alpha}^{\frac{a+b}{2}}(fg)(b) + {}^{\frac{a+b}{2}}I_{\alpha}(fg)(a) \right| \\ &\leq \frac{(b-a)^{\alpha+1}}{n!} \int_0^1 |k(t)||f^{'}((1-t)a+tb)|dt \\ &\leq \frac{(b-a)^{\alpha+1}}{n!} \int_0^1 |k(t)| \left[(1-t)|f^{'}(a)| + t|f^{'}(b)| \right] dt \\ &= \frac{(b-a)^{\alpha+1}}{n!} \Bigg\{ \int_0^{1/2} \left| \int_0^t s^n (\frac{1}{2}-s)^{\alpha-n-1} g((1-s)a+sb) ds \right| \\ &\times \left[(1-t)|f^{'}(a)| + t|f^{'}(b)| \right] dt \\ &+ \int_{1/2}^1 \left| \int_1^t (1-s)^n (s-\frac{1}{2})^{\alpha-n-1} g((1-s)a+sb) ds \right| \\ &\times \left[(1-t)|f^{'}(a)| + t|f^{'}(b)| \right] dt \Bigg\} \\ &\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,\frac{a+b}{2}],\infty}}{n!} \\ &\times \int_0^{1/2} \left(\int_0^t s^n (\frac{1}{2}-s)^{\alpha-n-1} ds \right) \left[(1-t)|f^{'}(a)| + t|f^{'}(b)| \right] dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-a)^{\alpha+1}\|g\|_{[\frac{a+b}{2},b],\infty}}{n!} \\
&\times \int_{1/2}^{1}\left(\int_{1}^{t}(1-s)^{n}(s-\frac{1}{2})^{\alpha-n-1}ds\right)\left[(1-t)|f^{'}(a)|+t|f^{'}(b)|\right]dt \\
\leq{}& \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{n!}\Big\{\int_{0}^{1/2}\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)(1-t)|f^{'}(a)|dt \\
&+\int_{0}^{1/2}\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)t|f^{'}(b)|dt \\
&+\int_{1/2}^{1}\left(\int_{1}^{t}(1-s)^{n}(s-\frac{1}{2})^{\alpha-n-1}ds\right)(1-t)|f^{'}(a)|dt \\
&+\int_{1/2}^{1}\left(\int_{1}^{t}(1-s)^{n}(s-\frac{1}{2})^{\alpha-n-1}ds\right)t|f^{'}(b)|dt \\
={}& \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{n!}\{\Phi_1+\Phi_2+\Phi_3+\Phi_4\}.
\end{aligned}
$$

By integration by parts and using Newton-Leibniz formula, we get

$$
\begin{aligned}
\Phi_1 &= \int_{0}^{1/2}\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)(1-t)|f^{'}(a)|dt \\
&= |f^{'}(a)|\Big\{\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)(t-\frac{t^2}{2})\Big|_{0}^{1/2} \\
&\quad -\int_{0}^{1/2}t^{n}(\frac{1}{2}-t)^{\alpha-n-1}(t-\frac{t^2}{2})dt\Big\} \\
&= |f^{'}(a)|\Big\{\frac{3}{2^{\alpha+3}}\int_{0}^{1}u^{n}(1-u)^{\alpha-n-1}du \\
&\quad -\frac{1}{2^{\alpha+1}}\int_{0}^{1}u^{n}(1-u)^{\alpha-n-1}(u-\frac{u^2}{4})du\Big\} \\
&= \frac{|f^{'}(a)|}{2^{\alpha+1}}\big[\frac{3}{4}B(n+1,\alpha-n)-B(n+2,\alpha-n) \\
&\quad +\frac{1}{4}B(n+3,\alpha-n)\big],
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\Phi_2 &= \int_{0}^{1/2}\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)t|f^{'}(b)|dt \\
&= |f^{'}(b)|\Big\{\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)\frac{t^2}{2}\Big|_{0}^{1/2} \\
&\quad -\int_{0}^{1/2}t^{n}(\frac{1}{2}-t)^{\alpha-n-1}\frac{t^2}{2}dt\Big\}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&= |f^{'}(b)|\Big\{\frac{1}{2^{\alpha+3}}\int_0^1 u^n(1-u)^{\alpha-n-1}du \\
&\quad - \frac{1}{2^{\alpha+3}}\int_0^1 u^n(1-u)^{\alpha-n-1}u^2du\Big\} \\
&= \frac{|f^{'}(b)|}{2^{\alpha+3}}\big[B(n+1,\alpha-n)-B(n+3,\alpha-n)\big],
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 &= \int_{1/2}^1\Big(\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}ds\Big)(1-t)|f^{'}(a)|dt \\
&= |f^{'}(a)|\Big\{\Big(\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}ds\Big)\Big(t-\frac{t^2}{2}\Big)\Big|_{1/2}^1 \\
&\quad + \int_{1/2}^1(1-t)^n(t-\frac{1}{2})^{\alpha-n-1}(t-\frac{t^2}{2})dt\Big\} \\
&= |f^{'}(a)|\Big\{-\frac{3}{2^{\alpha+3}}\int_0^1 u^n(1-u)^{\alpha-n-1}du \\
&\quad + \frac{1}{2^{\alpha+1}}\int_0^1 u^n(1-u)^{\alpha-n-1}(1-\frac{u^2}{4})du\Big\} \\
&= \frac{|f^{'}(a)|}{2^{\alpha+1}}\big[-\frac{3}{4}B(n+1,\alpha-n) \\
&\quad + B(n+1,\alpha-n)-\frac{1}{4}B(n+3,\alpha-n)\big]
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\Phi_4 &= \int_{1/2}^1\Big(\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}ds\Big)t|f^{'}(b)|dt \\
&= |f^{'}(b)|\Big\{\Big(\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}ds\Big)\frac{t^2}{2}\Big|_{1/2}^1 \\
&\quad + \int_{1/2}^1(1-t)^n(t-\frac{1}{2})^{\alpha-n-1}\frac{t^2}{2}dt\Big\} \\
&= |f^{'}(b)|\Big\{-\frac{1}{2^{\alpha+3}}\int_0^1 u^n(1-u)^{\alpha-n-1}du \\
&\quad + \frac{1}{2^{\alpha+1}}\int_0^1 u^n(1-u)^{\alpha-n-1}(1-u+\frac{u^2}{4})du\Big\} \\
&= \frac{|f^{'}(b)|}{2^{\alpha+1}}\big[-\frac{1}{4}B(n+1,\alpha-n)+B(n+1,\alpha-n) \\
&\quad - B(n+2,\alpha-n)+\frac{1}{4}B(n+3,\alpha-n)\big].
\end{aligned} \tag{14}
$$

Using (11), (12), (13) and (14) in (10), we get desired result. ■

Remark 1. In Theorem 5, If we take $\alpha = n+1$, then inequality (9) becomes ineqaulity (5).

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$ with $a < b$. If $|f^{'}|^q$ is convex on $[a,b]$ for $q > 1$ and $g : [a,b] \to \mathbb{R}$ is continious and symmetric to $(a+b)/2$, then the following inequality for comformable fractional integrals holds*

$$
\begin{aligned}
(15)\quad & \left| f\left(\frac{a+b}{2}\right)[I_{\alpha}^{\frac{a+b}{2}} g(b) + {}^{\frac{a+b}{2}}I_{\alpha} g(a)] - [I_{\alpha}^{\frac{a+b}{2}}(fg)(b) + {}^{\frac{a+b}{2}}I_{\alpha}(fg)(a) \right| \\
& \leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha} n!} \\
& \times \left[\left(\int_0^{1/2} [B_{2t}(n+1,\alpha-n)]^p dt \right)^{1/p} \left(\frac{3|f^{'}(a)|^q + |f^{'}(b)|^q}{8} \right)^{1/q} \right. \\
& + \left. \left(\int_{1/2}^{1} [B_{2-2t}(n+1,\alpha-n)]^p dt \right)^{1/p} \left(\frac{|f^{'}(a)|^q + 3|f^{'}(b)|^q}{8} \right)^{1/q} \right]
\end{aligned}
$$

where $\|g\|_{[a,b],\infty} = \sup_{x\in[a,b]} |g(x)|$, $\alpha \in (n, n+1]$, $n = 0,1,2,...$ and $1/p + 1/q = 1$.

Proof. Using Lemma 1, Hölder inequality and the convexity $|f^{'}|^q$, it follows that

$$
\begin{aligned}
(16)\quad & \left| f\left(\frac{a+b}{2}\right)[I_{\alpha}^{\frac{a+b}{2}} g(b) + {}^{\frac{a+b}{2}}I_{\alpha} g(a)] - [I_{\alpha}^{\frac{a+b}{2}}(fg)(b) + {}^{\frac{a+b}{2}}I_{\alpha}(fg)(a) \right| \\
& \leq \frac{(b-a)^{\alpha+1}}{n!} \int_0^1 |k(t)||f^{'}((1-t)a+tb)|dt \leq \frac{(b-a)^{\alpha+1}}{n!} \\
& \times \left(\int_0^{1/2} \left(\int_0^t |s^n (\frac{1}{2}-s)^{\alpha-n-1} g((1-s)a+sb)|ds \right)^p dt \right)^{1/p} \\
& \times \left(\int_0^{1/2} |f^{'}(1-t)a+tb|^q dt \right)^{1/q} + \frac{(b-a)^{\alpha+1}}{n!} \\
& \times \left(\int_{1/2}^{1} \left(\int_1^t |(1-s)^n (s-\frac{1}{2})^{\alpha-n-1} g((1-s)a+sb)|ds \right)^p dt \right)^{1/p} \\
& \times \left(\int_{1/2}^{1} |f^{'}(1-t)a+tb|^q dt \right)^{1/q}
\end{aligned}
$$

$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,\frac{a+b}{2}],\infty}}{n!}\left(\int_0^{1/2}\left(\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1})ds\right)^p dt\right)^{1/p}$$
$$\times\left(\frac{3|f^{'}(a)|^q+|f^{'}(b)|^q}{8}\right)^{1/q}+\frac{(b-a)^{\alpha+1}\|g\|_{[\frac{a+b}{2},b],\infty}}{n!}$$
$$\times\left(\int_0^{1/2}\left(\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1})ds\right)^p dt\right)^{1/p}$$
$$\times\left(\frac{|f^{'}(a)|^q+3|f^{'}(b)|^q}{8}\right)^{1/q}$$
$$=\frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^\alpha n!}\left[\left(\int_0^{1/2}[B_{2t}(n+1,\alpha-n)]^p dt\right)^{1/p}\right.$$
$$\times\left(\frac{3|f^{'}(a)|^q+|f^{'}(b)|^q}{8}\right)^{1/q}$$
$$\left.+\left(\int_{1/2}^1[B_{2-2t}(n+1,\alpha-n)]^p dt\right)^{1/p}\left(\frac{|f^{'}(a)|^q+3|f^{'}(b)|^q}{8}\right)^{1/q}\right].$$

So, the proof is completed. ■

Remark 2. In Theorem 6, if we take $\alpha=n+1$, inequality (15) becomes ineqaulity (7).

Theorem 7. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o and $f^{'}\in L[a,b]$ with $a<b$. If $|f^{'}|^q$ is convex on $[a,b]$ for $q>1$ and $g:[a,b]\to\mathbb{R}$ is continious and symmetric to $(a+b)/2$, then the following inequality for comformable fractional integrals holds:*

$$\left|f\left(\frac{a+b}{2}\right)[I_\alpha^{\frac{a+b}{2}}g(b)+{}^{\frac{a+b}{2}}I_\alpha g(a)]-[I_\alpha^{\frac{a+b}{2}}(fg)(b)+{}^{\frac{a+b}{2}}I_\alpha(fg)(a)\right| \tag{17}$$
$$\leq\frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha-\frac{\alpha}{q}}n!}\left[\left(\int_0^{1/2}B_{2t}(n+1,\alpha-n)dt\right)^{1-1/q}\right.$$
$$\times\left(\frac{|f^{'}(a)|}{2^{\alpha+1}}[\frac{3}{4}B(n+1,\alpha-n)\right.$$
$$-B(n+2,\alpha-n)+\frac{1}{4}B(n+3,\alpha-n)]$$
$$\left.+\frac{|f^{'}(b)|}{2^{\alpha+3}}[B(n+1,\alpha-n)-B(n+3,\alpha-n)]\right)^{1/q}$$

$$+\Big(\int_{1/2}^{1} B_{2-2t}(n+1,\alpha-n)dt\Big)^{1-1/q}$$
$$\times\Big(\frac{|f^{'}(a)|}{2^{\alpha+1}}\big[-\frac{3}{4}B(n+1,\alpha-n)+B(n+1,\alpha-n)-\frac{1}{4}B(n+3,\alpha-n)\big]$$
$$+\frac{|f^{'}(b)|}{2^{\alpha+1}}\big[-\frac{1}{4}B(n+1,\alpha-n)+B(n+1,\alpha-n)$$
$$-B(n+2,\alpha-n)+\frac{1}{4}B(n+3,\alpha-n)\big]\Big)^{1/q}\Bigg]$$

where $\|g\|_{[a,b],\infty}=\sup_{x\in[a,b]}|g(x)|$ *and* $\alpha\in(n,n+1]$, $n=0,1,2,....$

Proof. Using Lemma 1, Power-Mean inequality and the convexity $|f^{'}|^q$, it follows that

$$\left|f\left(\frac{a+b}{2}\right)[I_{\alpha}^{\frac{a+b}{2}}g(b)+{}^{\frac{a+b}{2}}I_{\alpha}g(a)]-[I_{\alpha}^{\frac{a+b}{2}}(fg)(b)+{}^{\frac{a+b}{2}}I_{\alpha}(fg)(a)\right| \tag{18}$$
$$\le\frac{(b-a)^{\alpha+1}}{n!}\int_0^1|k(t)||f^{'}((1-t)a+tb)|dt\le\frac{(b-a)^{\alpha+1}}{n!}$$
$$\times\Bigg[\Big(\int_0^{1/2}\Big|\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}g((1-s)a+sb)ds\Big|dt\Big)^{1-1/q}$$
$$\times\Bigg(\int_0^{1/2}\Big|\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}g((1-s)a+sb)ds\Big|$$
$$\times\ |f^{'}((1-t)a+tb)|^q dt\Bigg)^{1/q}$$
$$+\Bigg(\int_{1/2}^1\Big|\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}g((1-s)a+sb)ds\Big|dt\Bigg)^{1-1/q}$$
$$\times\Bigg(\int_{1/2}^1\Big(\Big|\int_1^t(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}g((1-s)a+sb)ds\Big|\Big)$$
$$\times|f^{'}((1-t)a+tb)|^q dt\Bigg)^{1/q}\Bigg]$$
$$\le\frac{(b-a)^{\alpha+1}}{n!}\Bigg[\|g\|_{[a,\frac{a+b}{2}],\infty}\Bigg(\int_0^{1/2}\Big(\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}ds\Big)dt\Bigg)^{1-1/q}$$
$$\times\ \Bigg(\int_0^{1/2}\Big(\int_0^t s^n(\frac{1}{2}-s)^{\alpha-n-1}ds\Big)$$
$$\times\ [(1-t)|f^{'}(a)|^q+t|f^{'}(b)|^q]dt\Bigg)^{1/q}\Bigg]$$
$$+\|g\|_{[\frac{a+b}{2},b],\infty}\Bigg(\int_{1/2}^1\Big(\int_t^1(1-s)^n(s-\frac{1}{2})^{\alpha-n-1}ds\Big)dt\Bigg)^{1-1/q}$$

$$\times \left(\int_{1/2}^{1}\left(\int_{t}^{1}(1-s)^{n}(s-\frac{1}{2})^{\alpha-n-1}ds\right)\right.$$
$$\left.\left.\times\,[(1-t)|f^{'}(a)|^{q}+t|f^{'}(b)|^{q}]dt\right)^{1/q}\right]$$
$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{n!}\left[\left(\int_{0}^{1/2}B_{2t}(n+1,\alpha-n)dt\right)^{1-1/q}\right.$$
$$\times\left(\int_{0}^{1/2}\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)[(1-t)|f^{'}(a)|^{q}+t|f^{'}(b)|^{q}]dt\right)^{1/q}$$
$$+\left(\int_{1/2}^{1}B_{2-2t}(n+1,\alpha-n)dt\right)^{1-1/q}$$
$$\times \left(\int_{1/2}^{1}\left(\int_{t}^{1}(1-s)^{n}(s-\frac{1}{2})^{\alpha-n-1}ds\right)\right.$$
$$\left.\left.\times\,[(1-t)|f^{'}(a)|^{q}+t|f^{'}(b)|^{q}]dt\right)^{1/q}\right].$$

From (11) and (12) , we get

$$\int_{0}^{1/2}\left(\int_{0}^{t}s^{n}(\frac{1}{2}-s)^{\alpha-n-1}ds\right)[(1-t)|f^{'}(a)|^{q}+t|f^{'}(b)|^{q}]dt \tag{19}$$
$$\leq \frac{|f^{'}(a)|}{2^{\alpha+1}}\Big[\frac{3}{4}B(n+1,\alpha-n)-B(n+2,\alpha-n)+\frac{1}{4}B(n+3,\alpha-n)\Big]$$
$$+\frac{|f^{'}(b)|}{2^{\alpha+3}}\Big[B(n+1,\alpha-n)-B(n+3,\alpha-n)\Big]$$

and from (13) and (14), we get

$$\int_{1/2}^{1}\left(\int_{t}^{1}(1-s)^{n}(s-\frac{1}{2})^{\alpha-n-1}ds\right)[(1-t)|f^{'}(a)|^{q}+t|f^{'}(b)|^{q}]dt \tag{20}$$
$$\leq \frac{|f^{'}(a)|}{2^{\alpha+1}}\Big[-\frac{3}{4}B(n+1,\alpha-n)$$
$$+B(n+1,\alpha-n)-\frac{1}{4}B(n+3,\alpha-n)\Big]$$
$$+\frac{|f^{'}(b)|}{2^{\alpha+1}}\Big[-\frac{1}{4}B(n+1,\alpha-n)+B(n+1,\alpha-n)$$
$$-B(n+2,\alpha-n)+\frac{1}{4}B(n+3,\alpha-n)\Big].$$

Using (19) and (20) in (18), we get desired result. ■

Remark 3. In Theorem 7, if we take $\alpha = n+1$, then inequality (17) becomes ineqaulity (6).

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Received on 15.03.2017 and, in revised form, on 02.11.2017.