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RADIUS OF CONVEXITY OF SECTIONS OF A CLASS OF CLOSE-TO-CONVEX FUNCTIONS OF ORDER α

Abstract. In this paper we study radius of convexity of sections of a class of univalent close-to-convex functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For functions in this class, coefficient bounds, an integral representation and radius of convexity of n^{th} sections have been obtained.

Key words: univalent, close-to-convex, starlike and convex functions.

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1. Introduction

Let $\mathcal{A}$ be the set of all analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the unit disc $\mathbb{D}$ and let $\mathcal{S}$ denote the class of all univalent (one-to-one and analytic) functions $f \in \mathcal{A}$. Let $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{S}$ that are convex, starlike with respect to origin and close-to-convex functions respectively. It is well known that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. Let $S_n(f) = z + \sum_{k=2}^{n} a_k z^k$ be the n^{th} section of f.

In [8], Szego proved that every section $S_n(f)$ of $f \in \mathcal{S}$ is univalent in the disk $|z| < \frac{1}{4}$ and $\frac{1}{4}$ is the best possible as $S_2(k(z)) = z + 2z^2$ is univalent in $|z| < \frac{1}{4}$, where $k(z) = \dfrac{z}{(1-z)^2}$ is Koebe function, extremal function of the class $\mathcal{S}$.

Various problems about sections have been solved for subclasses $\mathcal{C}$, $\mathcal{S}^*$, $\mathcal{K}$ in [1], [3], [4], [6]. In [2], MacGregor considered the class $\mathcal{R} = \{f \in \mathcal{A} : \mathrm{Re}\,(f'(z)) > 0,\ z \in \mathbb{D}\}$ and proved that every section $S_n(f)$ of $f \in \mathcal{R}$ is univalent in $|z| < \dfrac{1}{2}$ and $\dfrac{1}{2}$ is the best constant. In [5], Ponnusamy et.al. considered the class $\mathcal{F} = \left\{ f \in \mathcal{A} / \mathrm{Re}\left(1 + \dfrac{zf''(z)}{f'(z)}\right) > -\dfrac{1}{2},\ z \in \mathbb{D} \setminus \{0\} \right\}$

and proved that every section of a function $f \in \mathcal{F}$ is convex in the disc $|z| < \frac{1}{6}$. The radius $\frac{1}{6}$ cannot be replaced by a larger one. In this paper, we consider the class

$$\mathcal{F}_\alpha = \left\{ f \in \mathcal{A} / \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha}{2},\ 0 \le \alpha \le 1,\ z \in \mathbb{D} \setminus \{0\} \right\}$$

and find coefficient bound, distortion theorem and radius of convexity of sections of functions in this class.

In Theorem 1, though the result in (a) below had been proved in [7] for a wider range, we prove it here for the sake of completeness of this paper.

2. Main theorems

Coefficient bound and distortion theorem

Theorem 1. *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{F}_\alpha$, $0 \le \alpha \le 1$, then we have*

(a) $|a_n| \le \dfrac{\Gamma(n+1+\alpha)}{n!\Gamma(2+\alpha)}$ for $n \ge 2$, equality holds for the extremal function of the class $\mathcal{F}_\alpha$, which is given by $f_\alpha = \dfrac{1}{1+\alpha}\left(\dfrac{1}{(1-z)^{1+\alpha}} - 1\right)$;

(b) $\left|\dfrac{zf''(z)}{f'(z)}\right| < \dfrac{(2+\alpha)r}{1-r}$ for $|z| = r$;

(c) $\dfrac{1}{(1+r)^{2+\alpha}} < |f'(z)| < \dfrac{1}{(1-r)^{2+\alpha}}$.

Proof. As $f \in \mathcal{F}_\alpha$, we can write $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{\alpha}{2} = (1 + \dfrac{\alpha}{2})P(z)$, where $P(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ is such that $\operatorname{Re}(P(z)) > 0$ having $|p_n| \le 2, n \ge 1$. Also,

$$\begin{aligned} \frac{2+\alpha}{2} + \frac{zf''(z)}{f'(z)} &= \frac{2+\alpha}{2}(P(z)) \\ \frac{\sum\limits_{2}^{\infty} n(n-1)a_n z^{n-1}}{1 + \sum\limits_{2}^{\infty} n a_n z^{n-1}} &= \frac{2+\alpha}{2}\left(\sum_{n=1}^{\infty} p_n z^n\right) \\ \sum_{1}^{\infty} n(n+1)a_{n+1}z^n &= \frac{2+\alpha}{2}\left(1 + \sum_{1}^{\infty}(n+1)a_{n+1}z^n\right)\left(\sum_{n=1}^{\infty} p_n z^n\right). \end{aligned}$$

Equating coefficient of z we have as $|p_1| < 2$,

$$1.2|a_2| = \frac{2+\alpha}{2}|p_1| < (2+\alpha);\ \text{(i.e)}\ |a_2| < \frac{2+\alpha}{2}.$$

For $n \geq 2$ we have

$$\begin{aligned} n(n+1)|a_{n+1}| &\leq \frac{2+\alpha}{2}\left(|p_n| + \sum_{k=1}^{n-1}(k+1)|a_{k+1}||p_{n-k}|\right) \\ &< (2+\alpha)\sum_{k=1}^{n} k|a_k|. \end{aligned}$$

We claim that for $n \geq 2$,

$$n(n+1)|a_{n+1}| \leq (2+\alpha)\sum_{k=1}^{n} k|a_k| \leq \frac{(2+\alpha)(3+\alpha)...(n+1+\alpha)}{(n-1)!}.$$

For $n = 2$,

$$\begin{aligned} 2.3|a_3| &\leq (2+\alpha)\sum_{k=1}^{2} k|a_k| < (2+\alpha)(3+\alpha) \\ |a_3| &\leq \frac{(2+\alpha)(3+\alpha)}{3!}. \end{aligned}$$

Now assume that for $n = m$, the following is true.

$$m(m+1)|a_{m+1}| \leq (2+\alpha)\sum_{k=1}^{m} k|a_k| \leq \frac{(2+\alpha)(3+\alpha)...(m+1+\alpha)}{(m-1)!}.$$

Next we consider $n = m+1$,

$$\begin{aligned} (m+1)(m+2)|a_{m+2}| &\leq (2+\alpha)\sum_{k=1}^{m+1} k|a_k| \\ &= (2+\alpha)\sum_{k=1}^{m} k|a_k| + (2+\alpha)(m+1)|a_{m+1}| \\ &= \frac{(2+\alpha)(3+\alpha)...(m+2+\alpha)}{m!}. \end{aligned}$$

Therefore, $|a_{m+2}| \leq \dfrac{(2+\alpha)(3+\alpha)...(m+2+\alpha)}{(m+2)!}$. This bound is sharp as $f_\alpha = \dfrac{1}{1+\alpha}\left(\dfrac{1}{(1-z)^{1+\alpha}} - 1\right) \in \mathcal{F}_\alpha$. To Prove (b), we know by definition of $\mathcal{F}_\alpha$, $1+\dfrac{zf''(z)}{f'(z)} \prec \dfrac{1+(1+\alpha)z}{1-z}$ for $f \in \mathcal{F}_\alpha$. Therefore, $\dfrac{zf''(z)}{f'(z)} \prec \dfrac{(2+\alpha)z}{1-z} = h_\alpha(z)$, where $\prec$ denotes subordination. Hence $\left|\dfrac{zf''(z)}{f'(z)}\right| < \dfrac{(2+\alpha)r}{1-r}$. From the above, it is clear that $f'(z) \prec e^{\int_0^z \frac{h_\alpha(t)}{t}dt} = \frac{1}{(1-z)^{2+\alpha}}$ and hence (c) holds. ∎

Theorem 2. *For $f \in \mathcal{F}_\alpha$, if $f(z) = s_n(z) + \sigma_n(z)$, where $s_n(z) = z + \sum_{k=2}^{n} a_k z^k$ and $\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$, then for $|z| = r$ we have*

$$|\sigma_n'(z)| < \frac{\Gamma(n+3+\alpha)}{\Gamma(2+\alpha)(n+1)!}\frac{r^{n+1}}{(1-r)^{n+3+\alpha}},$$

$$|z\sigma_n''(z)| < \frac{\Gamma(n+4+\alpha)}{\Gamma(2+\alpha)(n+1)!}\frac{r^{n+2}}{(1-r)^{n+4+\alpha}}.$$

Proof. Consider $|\sigma_n(z)| \le \sum_{n+1}^{\infty} |a_k||z|^k < \sum_{n+1}^{\infty} \frac{\Gamma(k+1+\alpha)}{\Gamma(2+\alpha)k!} r^k = \sigma_{n0}(r)$, where $\sigma_{n0}(r)$ is the remainder, after n terms, of the extremal function

$$f_0(r) = \frac{1}{1+\alpha}\Big(\frac{1}{(1-r)^{1+\alpha}} - 1\Big).$$

By Integral form of remainder of a Taylor series, we get

$$\begin{aligned}
|\sigma_{n0}(r)| &= \Big|\int_0^r \frac{f_0^{(n+1)}(t)(r-t)^n}{n!}dt\Big| \\
&< \int_0^r \Big|\frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)n!}\frac{(r-t)^n}{(1-t)^{n+2+\alpha}}\Big|dt \\
&< \frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)n!(1-r)^{n+2+\alpha}}\int_0^r (r-t)^n dt \qquad \text{as} \qquad 0<t<r \\
&= \frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)(n+1)!}\frac{r^{n+1}}{(1-r)^{n+2+\alpha}}.
\end{aligned}$$

Hence

$$|\sigma_n(z)| \le \frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)(n+1)!}\frac{r^{n+1}}{(1-r)^{n+2+\alpha}} \quad \forall z \in \mathbb{D}.$$

Similarly for all $z \in \mathbb{D}$ we can obtain,

$$|\sigma_n'(z)| \le \frac{\Gamma(n+3+\alpha)}{\Gamma(2+\alpha)(n+1)!}\frac{r^{n+1}}{(1-r)^{n+3+\alpha}}$$

$$|z\sigma_n''(z)| \le \frac{\Gamma(n+4+\alpha)}{\Gamma(2+\alpha)(n+1)!}\frac{r^{n+2}}{(1-r)^{n+4+\alpha}}.$$

∎

Theorem 3 (Integral representation of the class)**.** *If $f \in \mathcal{F}_\alpha$, then the integral representation of $f(z)$ is of the form*

$$f'(z) = \frac{e^{[(2+\alpha)/2]\int_0^z [P(t)/t]dt}}{z^{(2+\alpha)/2}},$$

for some $P(z)$ with $\mathrm{Re}\,(P(z)) > 0$ *and vice versa.*

Proof. As $f \in \mathcal{F}_\alpha$, we have $1 + \frac{zf''(z)}{f'(z)} + \frac{\alpha}{2} = (1+\frac{\alpha}{2})P(z)$, where $P(z)$ is such that $\mathrm{Re}\,(P(z)) > 0$. Therefore, we have

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= (1+\frac{\alpha}{2})P(z) - \frac{\alpha}{2}; \\ \text{(i.e)} \quad z\frac{d}{dz}\log(zf'(z)) &= \frac{2+\alpha}{2}P(z) - \frac{\alpha}{2}; \\ \frac{d}{dz}\log(zf'(z)) &= \frac{2+\alpha}{2}\frac{P(z)}{z} - \frac{\alpha}{2z}. \end{aligned}$$

Integrating we get,

$$\begin{aligned} \log(zf'(z)) &= \frac{2+\alpha}{2}\int_0^z \frac{P(t)}{t}dt - \int_0^z \frac{\alpha dt}{2t} \\ &= \frac{2+\alpha}{2}\int_0^z \frac{P(t)}{t}dt - \frac{\alpha}{2}\log(z). \end{aligned}$$

Taking exponentiation we obtain,

$$\begin{aligned} zf'(z) &= z^{-\alpha/2}e^{\frac{2+\alpha}{2}\int_0^z \frac{P(t)}{t}dt}, \\ f'(z) &= \frac{e^{[(2+\alpha)/2]\int_0^z [P(t)/t]dt}}{z^{(2+\alpha)/2}}. \end{aligned}$$

Hence the proof and converse can be proved by retracing the above steps. ■

Theorem 4. *For $f \in \mathcal{F}_\alpha$, if $f(z) = s_n(z) + \sigma_n(z)$, where $s_n(z) = z + \sum_{k=2}^{n} a_k z^k$ and $\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$, then every section $s_n(z)$ of functions in this class is convex in the disk $|z| < \frac{1}{2(2+\alpha)}$ and this radius cannot be replaced by larger one.*

Proof. Consider $\operatorname{Re}\left(1+\frac{zs_2''(z)}{s_2'(z)}\right) = 1+\frac{2a_2z}{1+2a_2z}$. Then by Theorem 1, we have $|a_2| < \frac{2+\alpha}{2}$ and hence $\operatorname{Re}\left(1+\frac{zs_2''(z)}{s_2'(z)}\right) > 1-\frac{2|a_2||z|}{1-2|a_2||z|} \geq 1-\frac{(2+\alpha)|z|}{1-(2+\alpha)|z|}$; $\operatorname{Re}\left(1+\frac{zs_2''(z)}{s_2'(z)}\right) > \frac{1-2(2+\alpha)|z|}{1-(2+\alpha)|z|} = 0$ at $|z| = \frac{1}{2(2+\alpha)}$. By Maximum Principle for harmonic functions, we have $\operatorname{Re}\left(1+\frac{zs_2''(z)}{s_2'(z)}\right)>0$ for $|z| \leq \frac{1}{2(2+\alpha)}$. Now for $n > 2$, we obtain

$$\operatorname{Re}\left(1+\frac{zs_n''(z)}{s_n'(z)}\right) > 1-\left|\frac{zf''(z)}{f'(z)}\right| - \frac{\left|\frac{zf''(z)}{f'(z)}\right||\sigma_n'(z)|+|z\sigma_n''(z)|}{|f'(z)|-|\sigma_n'(z)|}.$$

Then by Theorem 2, we have

$$> 1-\frac{(2+\alpha)r}{1-r} - \frac{\frac{r^{n+2}\Gamma(n+3+\alpha)}{(1-r)^{n+4+\alpha}\Gamma(2+\alpha)(n+1)!}[2+\alpha+n+3+\alpha]}{\left(\frac{\Gamma(2+\alpha)(n+1)!(1-r)^{n+3+\alpha}-(1+r)^{2+\alpha}\Gamma(n+3+\alpha)r^{n+1}}{(1+r)^{2+\alpha}\Gamma(2+\alpha)(n+1)!(1-r)^{n+3+\alpha}}\right)}.$$

$$\operatorname{Re}\left(1+\frac{zs_n''(z)}{s_n'(z)}\right) > \frac{1-(3+\alpha)r}{1-r} - \frac{r^{n+2}A_0(1+r)^{2+\alpha}[n+5+2\alpha]}{(1-r)((1-r)^{n+3+\alpha}-A_0(1+r)^{2+\alpha}r^{n+1})},$$

where $A_0 = \frac{\Gamma(n+3+\alpha)}{\Gamma(2+\alpha)(n+1)!}$. Then $\operatorname{Re}\left(1+\frac{zs_n''(z)}{s_n'(z)}\right) > 0$ provided

$$\frac{1-(3+\alpha)r}{1-r} > \frac{r^{n+2}A_0(1+r)^{2+\alpha}[n+5+2\alpha]}{(1-r)((1-r)^{n+3+\alpha}-(1+r)^{2+\alpha}A_0r^{n+1})}.$$

Therefore, we have to prove

$$\begin{aligned} r^{n+2}A_0(1+r)^{2+\alpha}[n+5+2\alpha] &< [1-(3+\alpha)r]((1-r)^{n+3+\alpha} \\ &\quad -(1+r)^{2+\alpha}A_0r^{n+1}) \\ r^{n+1}A_0(1+r)^{2+\alpha}[1+(n+2+\alpha)r] &< [1-(3+\alpha)r](1-r)^{n+3+\alpha}. \end{aligned}$$

On $|z| = \frac{1}{2(2+\alpha)}$, above inequality becomes

$$A_0(5+2\alpha)^{2+\alpha}[n+6+3\alpha] < (1+\alpha)(3+2\alpha)^{n+3+\alpha}$$

which is true for all $n \geq 3$ and $0 \leq \alpha \leq 1$. By Maximum Principle for harmonic functions, we have $\mathrm{Re}\left(1+\frac{zs_n''(z)}{s_n'(z)}\right) > 0$ in the disk $|z| \leq \frac{1}{2(2+\alpha)}$.
Hence the radius of convexity of sections of $f \in \mathcal{F}_\alpha$ is $|z| = \frac{1}{2(2+\alpha)}$. ■

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