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Kinga Cichoń, Mieczysław Cichoń, and Bianca Satco

ON REGULATED FUNCTIONS

Abstract. In this paper we investigate the space of regulated functions on a compact interval $[0, 1]$. When equipped with the topology of uniform convergence this space is isometrically isomorphic to some space of continuous functions. We study some of its properties, including the characterization of the dual space, weak and strong compactness properties of sets. Finally, we investigate some compact and weakly compact operators on the space of regulated functions. The paper is complemented by an existence result for the Hammerstein-Stieltjes integral equation with regulated solutions.

Key words: regulated functions, discontinuous functions, compactness.

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1. Introduction

In some differential and integral problems only discontinuous functions are expected to be their solutions (such as impulsive problems or measure perturbed problems, cf. [9, 38]). It is evident, that impulsive problems cannot be solved in the space of continuous functions. Moreover, it is known that solutions of problems described in terms of Stieltjes-type integrals need not be continuous. The class of regulated functions is often used to study problems of this kind. Otherwise, some extra assumptions are imposed to solve the problem in narrower space, i.e. solutions have some extra properties.

The space of regulated functions consists of functions having finite one-side limits at every point (i.e. with discontinuities of the first kind) and therefore it contains the space of continuous functions. In fact, a function is regulated if and only if it is a uniform limit of step functions ([12, (7.6.1)]. It is worthwhile to note that a function of bounded variation is regulated, while regulated functions are not necessarily of bounded variation. Our research allows to study, by unified manner, both cases mentioned above (i.e. continuous and BV solutions) and to extend earlier results to a wider space. In this paper we will investigate the space of regulated functions on a finite interval and some of its subspaces.

Non-separable Banach spaces cannot be embedded isometrically in the separable space $C([0,1])$, but for every Banach space X one can find a compact Hausdorff space S and an isometric linear embedding j of X into the space $C(S)$ of scalar continuous functions on S.

In the present paper, we will investigate the space $G([0,1],X)$ of vector-valued regulated functions as an interesting example of non-separable Banach spaces. We indicate an isometric isomorphism between the space of vector-valued regulated functions and a non-metrizable space of continuous functions on a compact Hausdorff space (the Alexandroff arrow space, see [22], for instance). Note that if X is a Banach algebra, then the space of vector-valued regulated functions is a Banach algebra too and related properties were investigated in [16]. Here we concentrate on the compactness properties of subsets of $G([0,1],X)$ and on operators acting on this space.

Afterwards, we will apply the result about the existence of an isomorphism between spaces of regulated and continuous functions. By proving some results so far known only for spaces of continuous functions, we obtain new properties of the space of regulated functions including the characterization of its dual and some new results about weak and strong compactness of its subsets. The entire discussion will explain why the space of regulated functions has so many properties similar to that of the space of continuous functions. It means that several proofs for differential or integral problems having continuous solutions can be easily extended to the case of regulated ones.

Finally, we study some properties of operators acting on the space of regulated functions, as well as, with values in this space. The above mentioned results allows us to treat the considered operators in a similar manner as in a classical theory for $C(I,X)$ and are then applied in a proof of an existence theorem for the Hammerstein integral equation of Stieltjes-type (with regulated solutions).

2. Regulated functions seen as continuous ones

Let $(X,\|\cdot\|)$ be a Banach space and denote by $B_r(x)$ the open ball in X centered at x with a radius r (and B_r will denote the ball with its center at the origin θ). A function $f:[0,1]\to X$ is said to be regulated if there exist the limits $f(t+)$ and $f(s-)$ for any point $t\in[0,1)$ and $s\in(0,1]$. Their name was introduced by Dieudonné (see [12, Section VII.6]).

Lemma 1 ([21], Chapter 1, Corollary 3.2, Theorem 3.5)**.** *The set of discontinuities of a regulated function is at most countable. Regulated functions are bounded and the space $G([0,1],X)$ of regulated functions defined on $[0,1]$ with values in a Banach space X is a Banach space too, endowed with the topology of uniform convergence, i.e. with the norm $\|f\|_\infty = \sup_{t\in[0,1]}\|f(t)\|$.*

The space $G([0,1],X)$ is not separable, contains as a proper subset the space of continuous functions and, as claimed in the Introduction, it can be represented (as an isomorphic copy) as a space of continuous functions on some Hausdorff compact non-metrizable space $\mathbb{K}$ (different than $[0,1]$ as $G([0,1],X)$ is not separable). Put

$$\mathbb{K} = \{(t,0) : 0 < t \le 1\} \cup \{(t,1) : 0 \le t \le 1\} \cup \{(t,2) : 0 \le t < 1\}$$

equipped with the order topology given by the lexicographical order (i.e. $(s,i) \prec (t,j)$ if either $s < t$, or $s = t$ and $i < j$), which is known as the Alexandroff (or the Alexandroff-Urysohn) arrow space (cf. also [21] for some isometries of $G([0,1],X)$ with different spaces). It will be useful to note that the neighborhoods of the point (t,τ) are generated by

$$\begin{aligned} V_b'(t,0) &= \{(s,r) : b < s < t, r = 0,1,2\} \cup \{(t,0)\} \\ V_c'(t,2) &= \{(s,r) : t < s < c, r = 0,1,2\} \cup \{(t,2)\} \\ V_d'(t,1) &= \{(t,1)\} \end{aligned}$$

for $\tau = 0,2$ and 1, respectively.

We are able to prove the following result given in [25, Proposition 3.5] for real-valued functions (without proof), cf. also [10] for basic ideas:

Theorem 1. *The Banach spaces $G([0,1],X)$ and $C(\mathbb{K},X)$ are isometrically isomorphic in the following sense: given functions $f \in G([0,1],X)$, and $\kappa(f) = g \in C(\mathbb{K},X)$ they correspond to each other, if $g(t,r) = \lim_{s\to t-} f(s)$ if $r = 0$ and $t \in (0,1]$, $g(t,r) = f(t)$ if $r = 1$ and $t \in [0,1]$ and $g(t,r) = \lim_{s\to t+} f(s)$ if $r = 2$ and $t \in [0,1)$.*

Proof. Recall that both spaces $G([0,1],X)$ and $C(\mathbb{K},X)$ are considered with the uniform topology, i.e. with sup-norm $\|\cdot\|_\infty$. It is clear that the map κ is injective and $g(t,r)$ is well-defined for all $(t,r) \in \mathbb{K}$.

Let $f \in G([0,1],X)$. We need to show, that $g = \kappa(f) \in C(\mathbb{K},X)$. Since a singleton $\{(t,1)\}$ is a neighborhood of $(t,1)$ for $0 \le t \le 1$, κ is continuous at those points.

Consider now the points of the form $(t,0)$ and $(t,2)$ for $t \in [0,1]$. Then neighborhoods of a point $(t,0)$ are generated by the sets of the form $V_\varepsilon' = \{(s,r) : t < s < t+\varepsilon, r = 0,1,2\} \cup \{(t,2)\}$, and neighborhoods of a point $(t,2)$ are generated by the sets of the form $W_\varepsilon' = \{(s,r) : t-\varepsilon < s < t\ ,\ r =$

$0,1,2\} \cup \{(t,0)\}$. Note that $\{V_b'(t,0), V_c'(t,2), V_d'(t,1)\}$ is a subbase of the neighborhood of the point $(t,r) \in \mathbb{K}$.

Fix arbitrary point $(t,0)$ and a neighborhood V_ε' of this point. Put $\delta = \varepsilon$. We need to show, that $g^{-1}(V_\varepsilon') \subset (t-\delta, t+\delta)$. By the definition of κ it is clear for the level $r=1$. Again by the definition of κ for any point $(t',0)$, $t' \in (t-\delta,t)$ there exists an open interval $(\tau,t') \subset (t-\delta,t)$ such that for any s from this interval $g(s,1) \in V_\varepsilon'$, so $g(s,0) \in V_\varepsilon'$. Similarly, for any point $(t',2) \in V_\varepsilon'$ there exists an open interval $(t',t'') \subset (t-\delta,t)$ such that for any s from this interval $g(s,1) \in V_\varepsilon'$ and then $g(s,2) \in V_\varepsilon'$. Finally, $g^{-1}(V_\varepsilon') \subset (t-\delta,t+\delta)$. Thus, the continuity of g at $(t,0)$ is just a consequence of the definition of κ and the existence of the finite left limit of f at t. For a neighborhood W_ε' we have analogous property and hence the continuity of g at $(t,2)$ is related to the existence of right limits of f at t. Thus, κ maps $G([0,1],X)$ into $C(\mathbb{K},X)$.

Take a continuous function g on $\mathbb{K}$. Put $f(t) = g(t,1)$. We need to show that f is regulated. But it is a simple consequence of the choice of the topology on $\mathbb{K}$ and the form of neighborhoods of points in this space.

Finally, we need to show that $\|\kappa(f)\|_\infty = \|f\|_\infty$ for any $f \in G([0,1],X)$. It is quite standard, but in view of the definition of the arrow space we will do a sketch here. Since $f(t) = \kappa(f)(t,1) = g(t,1)$, $\|f\|_\infty \leq \|\kappa(f)\|_\infty$.

Fix arbitrary $\varepsilon > 0$. By the continuity of g, for any $(t,0)$ there exists a point $s < t$ such that $(s,1) \in V_\varepsilon'$ and $\|g(t,0) - g(s,1)\| < \varepsilon$. It means that $\|g(t,0)\| \leq \|g(s,1)\| + \varepsilon$ and finally $\|g(t,0)\| \leq \|f(s)\| + \varepsilon \leq \|f\|_\infty + \varepsilon$. Since ε is arbitrary, $\|g(t,0)\| \leq \|f\|_\infty$. Similarly we can show that $\|g(t,2)\| \leq \|f\|_\infty$. Thus

$$\|\kappa(f)\|_\infty = \sup_{(t,r)\in\mathbb{K}} \|g(t,r)\| = \|f\|_\infty.$$

The operator κ is linear and bounded, so we are done. ■

As we proved that the set $\{(t,1) : t \in [0,1]\}$ is dense in $\mathbb{K}$, the proof of the isometry property can be also carried out using density argument.

If $X = \mathbb{R}$ the space $G([0,1],\mathbb{R}) = G([0,1])$ is also a Banach algebra [4] (with pointwise multiplication, cf. also the case when X is a Banach algebra [15]). As a consequence of the above result we get (see also [15, Theorem 1], [16, Theorem 2.3] or [4]):

Corollary 1. *If X is a commutative Banach algebra, then $G([0,1],X)$ is also a commutative Banach algebra and is isometrically isomorphic to $C(\mathbb{K},X)$ as Banach algebras.*

Note that the space of vector-valued regulated functions seen as a Banach algebra, as well as, some linear integral functionals on $G([0,1],X)$ are studied in details in [16]. We will extend the earlier results for spaces X being not necessarily that Banach algebras.

3. Compactness in $G([0,1],X)$

The classical Ascoli theorem characterizes the compactness of subsets of the space of continuous functions via the notion of equi-continuity. Similarly, as proved by [17], compactness in the space of regulated functions is characterized (Proposition 1 below) by the concept of equi-regularity. Let us recall the notion and a very useful characterization of equi-regularity proved in [17] for $X = \mathbb{R}^d$.

Definition 1. *A set $\mathcal{A} \subset G([0,1],X)$ is said to be equi-regulated at the point $t_0 \in [0,1]$ if for every $\varepsilon > 0$ we have*

i) if $t_0 \in (0,1]$ there exists $\delta > 0$ such that for all $f \in \mathcal{A}$ and any $t_0 - \delta < s < t_0$, $\|f(s) - f(t_0-)\| < \varepsilon$;

ii) if $t_0 \in [0,1)$ there exists $\delta > 0$ such that for all $f \in \mathcal{A}$ and any $t_0 < \tau < t_0 + \delta$, $\|f(\tau) - f(t_0+)\| < \varepsilon$.

Also, $\mathcal{A}$ is called equi-regulated if it has this property at all points in $[0,1]$.

Proposition 1. *For any $\mathcal{A} \subset G([0,1],X)$ the following statements are equivalent:*
i) $\mathcal{A} \subset G([0,1],X)$ is relatively compact;
ii) $\mathcal{A}$ is equi-regulated and, for every $t \in [0,1]$, $\mathcal{A}(t) = \{f(t) : f \in \mathcal{A}\}$ is relatively compact in X.

At this point, having in mind Theorem 1, it is clear that the above proposition can be interpreted as a particular case of the Ascoli theorem and this clarifies the use of equi-regularity condition in earlier papers (e.g. [31]).

Theorem 2. *A subset $\mathcal{A} \subset G([0,1],X)$ is equi-regulated if and only if $\kappa(\mathcal{A}) \subset C(\mathbb{K},X)$ is equicontinuous in $C(\mathbb{K},X)$.*

Proof. $(\Rightarrow)$ The set $\mathcal{A}$ is equi-regulated, so for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any point $t_0 > 0$ such that $t_0 - \delta < s < t_0$: $\|f(s) - f(t_0-)\| < \varepsilon/2$ for any $f \in \mathcal{A}$. By using this condition we are able to prove equicontinuity of $\kappa(\mathcal{A})$ at any point on the "bottom" level, i.e. in $(t_0, 0)$. Take an arbitrary $f \in \mathcal{A}$.

Recall that any neighborhood of the point $(t_0, 0)$ $(t_0 > 0)$ is generated by sets of the form $V'_\delta(t_0,0) = \{(s,r) : t_0 - \delta < s < t_0, r = 0,1,2\} \cup \{(t_0,0)\}$. Thus, for any neighborhood of the point $\kappa(f)(t_0,0)$ in X (a ball $B_\varepsilon(\kappa(f)(t_0,0))$) we need to find a neighborhood $V'(t_0,0)$ such that $\kappa(f)(V'(t_0,0)) \subset B_\varepsilon(\kappa(f)(t_0,0))$. Since $f(t_0-) = \kappa(f)(t_0,0)$ and $f(s) = \kappa(f)(s,1)$, directly from the definition of equi-regularity we get $\delta > 0$ such that $\kappa(f)(s,1) \subset B_{\varepsilon/2}(f(t_0))$ for any $s \in (t_0 - \delta, t_0)$.

Let us check now the points $(s,0) \in V'_\delta(t_0,0)$. By the definition of κ for a point $(s,0) \in \mathbb{K}$ there exists an interval (s', s) such that $\|f(\tau) - f(s-)\| < \varepsilon/2$

for $\tau \in (s', s)$ (left limits of f). Then $\|\kappa(f)(s,0) - \kappa(f)(t_0,1)\| = \|f(s-) - f(t_0)\| \leq \|f(\tau) - f(s-)\| + \|f(\tau) - f(t_0)\| \leq \varepsilon$ for $\tau \in (s', s) \subset (t_0 - \delta, t_0)$. Thus, $\kappa(f)(s,0) \subset B_\varepsilon(f(t_0))$ for any $f \in \mathcal{A}$.

Similar calculation holds true for all points $(t,2)$. We need only to use the fact that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any point t such that $t_0 < s < t_0 + \delta$: $\|f(s) - f(t_0+)\| < \varepsilon/2$ and the remaining part of the proof will be the same.

Finally, we need to investigate the points in $\mathbb{K}$ of the form $(t_0, 1)$. Recall that the singletons are also the neighborhoods for that points. It means that $\kappa(f)\{(t_0,1)\} \subset B_r(f(t_0))$ and the equicontinuity of $\kappa(f)$ holds true for any $f \in \mathcal{A}$.

We obtained that $\kappa(f)(V'(t_0,0)) \subset B_\varepsilon(f)(t_0)$, so $\kappa(\mathcal{A})$ is equicontinuous in $C(\mathbb{K}, X)$.

($\Leftarrow$) Assume now, that $\mathcal{B}$ is an equicontinuous set. We need to show, that the set $\mathcal{A}$ such that $\kappa(\mathcal{A}) = \mathcal{B}$ is equi-regulated.

As in a previous part we will investigate equicontinuity at every levels $r = 0, 1, 2$ separately. For $r = 1$ we have only the information, that a function is everywhere defined. Since the set is equicontinuous at the points $(t, 0)$ it means that for any $\varepsilon > 0$ there exists a neighborhood V'_δ of the point $(t,0) \in \mathbb{K}$ such that $\kappa(f)(V'_\delta) \subset B_\varepsilon(\kappa(f)(t,0))$ for any $f \in \mathcal{A}$. But V'_δ contains the set $(t-\delta, t) \times \{1\}$. It means that for any $s \in (t - \delta, t)$ $\kappa(f)(s,1) \in B_\varepsilon(\kappa(f)(t,0)$. Thus, $\|f(s) - f(t-)\| < \varepsilon$ for any $t - \delta < s < t$. It is just the condition (i) from Definition 1.

The case of $r = 2$ with analogical proof allows us to check the condition (ii) from the definition of equi-regularity, so we omit the details. ■

As an immediate consequence of the above Theorem and Theorem 1 we get a new equivalent condition for Proposition 1. Although the space $G([0,1], X)$ consists not only of continuous functions we are ready to use the Ascoli Theorem in $C(\mathbb{K}, X)$ due to the following theorem (cf. [17, Proposition 2.3])

Theorem 3. *A subset $\mathcal{A} \subset G([0,1], X)$ is relatively compact if and only if $\kappa(\mathcal{A}) \subset C(\mathbb{K}, X)$ is relatively compact in $C(\mathbb{K}, X)$.*

As the compact Hausdorff space $\mathbb{K}$ is not metrizable, we get a special case of the above theorem (cf. also Theorem 2), which implies also Proposition 1:

Corollary 2. *A subset $\kappa(\mathcal{A}) \subset C(\mathbb{K}, X)$ is relatively compact in $C(\mathbb{K}, X)$ in the topology induced by the supremum norm $\|\cdot\|_\infty$ if and only if it is equicontinuous and has relatively compact sections $\kappa(\mathcal{A})(t,r)$ for all $(t,r) \in \mathbb{K}$.*

4. Weak compactness in $G([0,1],X)$

We are also interested in providing a characterization of weak compactness in the space of regulated functions. To this aim, we need some auxiliary results. The problem how to describe the topological dual for the space of regulated functions is important and has a few solutions based on different approaches. To the best of our knowledge, the first paper was published in 1934 by Kaltenborn [23]. We present here a useful theorem about weak convergence in $G([0,1],X)$ and we collect some important results.

Denote by $\mathcal{B}_o(\mathbb{K})$ the σ-algebra of all Borel measurable subsets of $\mathbb{K}$ and by $cabv(\Omega,X))$ $[rcabv(\Omega,X)]$ the space of all [regular] countably additive vector-valued Borel measures with scalar bounded variation on a compact Hausdorff space Ω with values in a Banach space X. To indicate a choice of a Borel σ-algebra considered on Ω we will sometimes write $rcabv(\mathcal{B}_o(\Omega),X))$.

It is well-known, that isometric isomorphisms preserve weak compactness. Either we can use our isomorphism κ and then the Riesz-Singer representation theorem ([13, Theorem 2] or we can apply [24, Theorem 6.1.5]) by considering spaces of vector-valued measures or some special function spaces. As a consequence of our Theorem 1, as well as the Riesz-Singer theorem we have:

Proposition 2. *The dual space of $G([0,1],X)$ is isometrically isomorphic to the space $rcabv(\mathcal{B}_o(\mathbb{K}),X^*)$ of regular countably additive X^*-valued Borel vector measures on $\mathbb{K}$ with bounded variation.*

Unfortunately, it means that we need to investigate Borel σ-algebra and Borel measures on $\mathbb{K}$. As the space $C([0,1],X)$ is isometrically embedded into $C(\mathbb{K},X)$ $(r=1)$ and for regulated functions we have at most countable number of discontinuity points, we conclude, that Borel σ-algebra on $\mathbb{K}$ contains a family of sets consisting of the sums of countable sets and images of Borel sets in $[0,1]$ through this embedding.

However, it will be easier and more natural to keep the original space, i.e. $[0,1]$. Let us recall that for a given countably additive measure $m \in cabv([0,1],X^*)$ of bounded variation it is easy to define an integral for simple functions $\sum_i x_i\chi_{A_i}$ as the sum $\sum_i m(A_i)x_i$. Since m is a measure with bounded variation the integral can be extended to all X-valued function being uniform limits of Borel measurable simple functions, so for regulated functions too. The integral is well-defined for regulated functions from $[0,1]$ to X. If we restrict our attention to regular measures we have only one such a measure defining continuous linear functional over $C([0,1],X)$.

As claimed above, regulated functions are uniform limits of step functions, then they are Borel measurable and the integrals with respect to measures described below are well-defined. It is possible to repeat the proof of the Riesz-Singer theorem for $C([0,1],X)$ (cf. [24, Theorem 6.1.5]) for regulated functions and consequently we have the following ([4, Theorem 13], [8, Theorem 6.1.5] or [6]):

Proposition 3 (cf. [4], Theorem 13, cf. [6])**.** *The dual space of $G([0,1],X)$ is isometrically isomorphic to the space $rcabv(\mathcal{B}_o([0,1]),X^*)$ of regular countably additive X^* valued Borel vector measures on $[0,1]$ with bounded variation.*

Our results in the previous section allows one to check the (strong) compactness property in the space $G([0,1],X)$. Following the idea from [13] (see also [24, Theorem 6.1.6] and [8, Proposition 1.7.1]) we can now also investigate the weak compactness. We characterize a weak convergence of bounded sequences in this space as a weak pointwise convergence (as in $C([0,1],X)$ in [13]):

Theorem 4. *A sequence (f_n) of regulated functions $f_n \in G([0,1],X)$ is weakly convergent to f in $G([0,1],X)$ if and only if it is (norm) bounded and for any $t \in [0,1]$ a sequence $(f_n(t))$ is weakly convergent to $f(t)$ in X for each $t \in [0,1]$.*

Proof. ($\Rightarrow$) Let (f_n) be a sequence weakly convergent in $G([0,1],X)$ to a regulated function f. Take a point-functional, i.e. measure of the form $\mu_{t,x^*} = \delta_t x^*$. Here $t \in [0,1]$, $x^* \in X^*$ and δ_t is a Dirac measure concentrated at t. Clearly $\mu_{t,x^*} \in (G([0,1],X))^*$.

Thus

$$\lim_{n\to\infty} x^*(f_n(t)) = \lim_{n\to\infty} \int_{[0,1]} f_n \, d\mu_{t,x^*} = \int_{[0,1]} f \, d\mu_{t,x^*} = x^*(f(t)).$$

It means that $(f_n(t))$ is weakly convergent in X for every $t \in [0,1]$.

($\Leftarrow$) Assume now, that (f_n) is bounded in $G([0,1],X)$ with $(f_n(t))$ being weakly convergent in X for any $t \in [0,1]$.

Take an arbitrary measure $m \in rcabv(\mathcal{B}_o,X^*)$. Note that since m is of bounded variation then it is a regular measure if and only if all scalar measures $m_x(A) = m(A)x$ are regular measures for all $x \in X$ ([24, Proposition 6.1.3]). All of them are also absolutely continuous with respect to the variation $\|m\|$ of the measure m. Without loss of generality we may assume, that the measure $\|m\|$ is complete and by the Radon-Nikodým theorem for liftings ([8, Theorem 1.5.2]) there exists a weak* measurable and bounded function $m_h : [0,1] \to X^*$ such that

$$m(A)x = \int_A < x, m_h > d\|m\|$$

for each Borel measurable set A and all $x \in X$. Thus, for any X-valued simple function s (Borel measurable) we get $\int_A s(t)\, dm(t) = \int_A < s(t), m_h(t) > d\|m\|(t)$. Since m is a regular measure, $m(A)$ is uniquely determined.

We recall that an arbitrary regulated function g is a uniform limit of simple functions (h_n) and the integral can be extended to regulated functions and we are able to use the Lebesgue dominated convergence theorem for the measure $\|m\|$, so we get

$$\begin{aligned}\int_{[0,1]} h_n(t)dm(t) &= \int_{[0,1]} < h_n(t), m_h(t) > d\|m\|(t)\\ &\to \int_{[0,1]} < g(t), m_h(t) > d\|m\|(t) = \int_{[0,1]} g(t)\, dm(t).\end{aligned}$$

By the above consideration (with g replaced by f_n) we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_{[0,1]} f_n(t)\, dm(t) &= \lim_{n\to\infty} \int_{[0,1]} < f_n(t), m_h(t) > \ d\|m\|(t)\\ &= \int_{[0,1]} < f(t), m_h(t) > \ d\|m\|(t) = \int_{[0,1]} f(t)\, dm(t).\end{aligned}$$

The weak convergence is then proved. ■

It should be noted, that we do not assume, that X^* has the Radon-Nikodym property (RNP) because we do not prove the existence of the integrable density m_h (and we are unable to use the classical Radon-Nikodym theorem for vector-valued functions). In particular, it means that the proof of the above theorem in the form suggested by Dobrakov ([13, Theorem 9]) still requires the RNP for X^*. However, his result is correct being a special case of Theorem 4. Thus we are able to characterize the *wbo*-convergence studied by Brokate and Krejčí ([6, Definition 2.4]) of bounded sequences of functions from $G([0,1], X)$. They considered an additional assumption of weak pointwise convergence of sequences in $G([0,1], X)$, so the *wbo*-convergence is, in fact, weak convergence in $G([0,1], X)$ together with uniform bounded oscillation of the sequence [6, Theorem 2.5]. Moreover, as a consequence of the Eberlein-Šmulian Theorem we have

Theorem 5. *A subset A of $G([0,1], X)$ is weakly relatively compact if and only if it is sequentially compact in the weak topology*

$$\sigma(G([0,1], X), (G([0,1], X)^*)) = \sigma(G([0,1], X), rcabv(\mathcal{B}_o(\mathbb{K}), X^*)).$$

Example 1. Let us present a simple example of a weakly compact set in $G([0,1])$. Put $x_t = \chi_{\{t\}}$ for any $t \in [0,1]$ and $A = \{x_t : t \in [0,1]\} \cup \{\theta\}$. Clearly, this set is uniformly equi-regulated (the number δ in Definition 1 can be arbitrarily chosen). Moreover, $\kappa(x_s)$ is a function g_s from $\mathbb{K}$ to $\mathbb{R}$ such that $g_s(t,r) = 0$ for $r = 0$, $r = 2$ and $r = 1, s \neq t$. Finally $g_s(t,1) = 1$ for $s = t$.

Thus, as the set $\kappa(A)$ is weakly compact in $C(\mathbb{K},\mathbb{R})$, as a consequence of the above theorem we get that A is weakly compact in $G([0,1])$. Any sequence from A is convergent to the zero function $\theta(t) \equiv 0$ (see Theorem 4).

Note that it implies, that weakly compact subsets of the space of regulated functions need not be separable.

5. Subspaces of the space of regulated functions

There are several subspaces of the space of regulated functions that have been studied in the literature. Here, we will focus only on some spaces which can be treated in the same manner as $G([0,1],X)$, namely on the space $D([0,1],X)$ consisting in the right-continuous regulated functions and on the space $PC([0,1],X)$ of piecewise continuous functions.

Some other interesting subspaces of $G([0,1],X)$ with their inclusions and properties can be found in [6, Lemma 1.2].

5.1. The space $D([0,1],X)$

The space of X-valued functions right-continuous admit finite left-limits at every point will be denoted by $D([0,1],X)$. Such functions are sometimes called càdlàg functions (”continue à droit, limite à gauche”, in French) or RCLL (right continuous with left limits). Similarly we can define a subspace of $G([0,1],X)$ consisting of all left-continuous regulated functions (see [21], for instance).

In such a case we are able to simplify the space $\mathbb{K}$ in the construction of an isometrically isomorphic copy of $D([0,1],X)$. Since the values of a function in this space is strictly related to right limits, the “middle” level in a model of the arrow space is superfluous. Instead of the Alexandroff arrow space, we consider the set

$$\mathbb{L} = \{(t,0) : 0 < t \leq 1\} \cup \{(t,1) : 0 \leq t < 1\}$$

(it is a modification of the two arrow space) again equipped with the order topology given by the lexicographical order. We have (see [25, Proposition 2.1] for real-valued functions):

Theorem 6. *The Banach spaces $D([0,1],X)$ and $C(\mathbb{L},X)$ are isometrically isomorphic in the following way: given functions $f \in D([0,1],X)$, and $\kappa(f) = g \in C(\mathbb{L},X)$, as corresponding to each other if $g(t,r) = \lim_{s\to t-} f(s)$ if $r = 0$ and $t \in (0,1]$, $g(t,r) = f(t)$ if $r = 1$ and $t \in [0,1)$.*

As in Theorem 1 the map κ allows to identify the values of a function $\kappa(f)$ at level $r = 0$ with left-limits of f and at the level $r = 1$ with values of f (and the right-side limits simultaneously). It is based on the observation of the fact that the neighborhoods of the point (t, r) are of the form

$$\begin{aligned} V_b(t,0) &= \{(s,r) : b < s < t, r = 0,1,2\} \cup \{(t,0)\} \\ V_c(t,1) &= \{(s,r) : t < s < c, r = 0,1,2\} \cup \{(t,1)\}. \end{aligned}$$

Thus all our main results can be expressed in terms of $C(\mathbb{L}, X)$, allowing to put $\mathbb{K}$ instead of $\mathbb{L}$ and $G([0,1], X)$ instead of $D([0,1], X)$ without essential changes in the proofs, so we will omit the details.

5.2. The space of piecewise continuous functions

Another subspace of $G([0,1], X)$ occurs when studying impulsive differential equations. Fix a finite number of points $t_k \in [0,1]$ $(k = 1, 2, ..., n)$. A function $f : [0,1] \to X$ is called piecewise continuous if it is continuous on the intervals $(0, t_1), ..., (t_{i-1}, t_i)$ (for $i = 2, ..., n$) and on $(t_n, 1)$ and moreover has finite one-side limits at each t_i (for $i = 1, ..., n$) (and right limit at 0 and left limit at 1).

The space of all such functions $PC([0,1], X)$ is a Banach space when endowed with the norm $\|f\|_\infty$. Sometimes it is assumed additionally, that the functions are right- or left-continuous. Note that the set of discontinuity points is fixed. If we assume that the piecewise continuous functions have a finite, but not fixed, number of discontinuity points we obtain the space $PC_{fin}([0,1], X)$ which is dense in $G([0,1], X)$. All compactness results stated in earlier papers are consequences of our theorems for the space $G([0,1], X)$.

6. Operators on $G([0,1], X)$

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, the Nemytskii superposition operator $S_f : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ is defined by the formula

$$S_f(x)(t) = f(t, x(t)).$$

It is one of the most important nonlinear operators, which is investigated in different function spaces (with its acting conditions, boundedness or continuity, for instance). Let us recall some properties of the Nemytskii operator acting on $G([0,1])$. The following useful theorem is proved by Michalak [25].

Theorem 7 ([26], Proposition 2.2). *The operator S_f maps $G([0,1])$ into itself if and only if the function f has the following properties:*

(1) *the limit $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ exists for every $(s,x) \in (0,1]\times\mathbb{R}$,*

(2) *the limit* $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ *exists for every* $(t,x)\in[0,1)\times\mathbb{R}$.

In particular, it implies that if $f(t,x(t)) = f(x(t))$ then the composition operator (i.e. autonomous superposition operator) S_f maps $G([0,1])$ into itself if and only if f is continuous (see [25, Corollary 3.7]).

Corollary 3 ([2], Theorem 2.3)**.** *Suppose that the function* $h(\cdot,u)$ *is regulated on* $[0,1]$ *for all* $u\in\mathbb{R}$, *and the function* $h(t,\cdot)$ *is continuous on* $\mathbb{R}$, *uniformly with respect to* $t\in I$. *Then the operator* S_h *maps* $G([0,1])$ *into itself and is (norm) bounded.*

Theorem 8 ([25], Corollary 3.6)**.** *The operator* $S_f : G([0,1]) \to G([0,1])$ *is continuous if and only if a function* $\tilde{f} : \mathbb{R} \to G([0,1])$ *given by the formula* $\tilde{f}(x)(t) = f(t,x)$ *is continuous.*

The Nemytskii operator is, in general, neither compact nor weakly compact when acting on infinite dimensional spaces. Since compactness is very important in the theory of linear operators, now we will investigate this class of operators. Let us begin by presenting an acting condition for linear operators on $G([0,1])$, i.e. for real-valued functions.

Theorem 9 ([33], Theorem 1)**.** *Assume, that* $K : [0,1]\times[0,1] \to \mathbb{R}$ *satisfies*

1. $K(t,\cdot)$ *is a function of bounded variation for every* $t\in[0,1]$ *with* $\|K(t,\cdot)\|_{BV} \le M$, *for some* $M > 0$,
2. $K(\cdot,s) \in G([0,1])$.

Then the linear operator

$$H(x)(t) = \int_0^1 x(s)\; d_sK(t,s)$$

maps $G([0,1])$ *into itself and is bounded with* $\|H\| \le 2\sup_{t\in[0,1]} \|K(t,\cdot)\|_{BV}$. *This integral is taken in the Kurzweil-Stieltjes sense.*

We will show, that our results are useful when studying compact and weakly compact operators on $G([0,1],X)$ being counterparts of classical results for $C([0,1],X)$ (see [3, 14]).

6.1. General form of linear opeartors on $G([0,1],X)$

First of all, let us indicate an immediate consequence of Theorem 1. Let Y be a Banach space and let an operator $T : G([0,1],X) \to Y$ be linear. By applying [3, p.217] (cf. [11, p.182], [30, Lemma 2] or [14, Theorem VI.7.2], for real-valued functions), our earlier results (Theorem 1, in particular) can be used for the superposition $T\circ\kappa : C(\mathbb{K},X) \to Y$. Now, by applying the characterization of the dual space $G([0,1],X)$ (Proposition 2) we are able to prove a more general result about representing measures for linear continuous operators acting on a space of regulated functions and then to obtain the general form of linear integral operators on $G(0,1],X)$. By $SemiVar(\mu)$ we will denote the semivariation of a measure μ, which is given by $SemiVar(\mu)(E) = \sup\{\mu(E)y^* : \|y^*\| \leq 1\}$. Recall that κ denotes the isomorphical isometry between $G([0,1],X)$ and $C(\mathbb{K},X)$. Consequently, we get

Theorem 10. *Let X, Y be arbitrary Banach spaces and let T be a bounded linear operator acting on $G([0,1],X)$ with values in Y. Then there exists a unique vector measure $\mu : \mathcal{B}_o(\mathbb{K}) \to L(X, Y^{**})$ such that*

(*a*) *μ is finitely additive, $SemiVar(\mu)(\mathbb{K}) < \infty$ and μ is weakly regular, i.e. for each $y^* \in Y^*$ and μ is X^*-regular (so countably additive),*

(*b*) *the mapping $y^* \to \mu(\cdot)y^*$ of Y^* into $rcabv(\mathcal{B}_o(\mathbb{K}), X^*)$ [$= (G([0,1],X))^*$ by Proposition 2] is continuous with respect to $\sigma(Y^*,Y)$ topology of Y^* and the weak-* $C(\mathbb{K},X)$ topology of $G([0,1],X)^*$,*

(*c*) *$Tf = \int_{\mathbb{K}} \kappa(f)(s)\, \mu(ds), \quad f \in G([0,1],X)$,*

(*d*) *$\|T\| = SemiVar(\mu)(\mathbb{K})$,*

(*e*) *$T^*y^* = \mu(\cdot)y^*$ in the isometric sense of $rcabv(\mathcal{B}_o(\mathbb{K}), X^*)$ and $(G([0,1],X))^*$.*

Conversely, any vector measure μ on $\mathcal{B}_o(\mathbb{K}) \to L(X, Y^{**})$ which satisfies (a) and (b), then the equation (c) defines a linear mapping from $G([0,1],X)$ into Y with its norm given by (d), and such that (e) follow.

Note that the above theorem allows us to investigate regulated functions seen as continuous ones. The formula (d) can be also rewritten as the Kurzweil-Stieltjes integral of a regulated function $\int_{[0,1]} f(s)\, \tilde{\mu}(ds)$ with respect to the measure dg generated by a function of bounded variation g or with a measure $\tilde{\mu} : \mathcal{B}_o([0,1]) \to L(X, Y^{**})$ (cf. Proposition 2). This direct approach was also considered in earlier papers cf. [6, 28, 36].

Let us now concentrate on a special case of linear bounded operators $T : G([0,1],X) \to G([0,1],X)$. If we consider some particular measures, then we can obtain classical integral operators. Let us stress the role of the Kurzweil-Stieltjes (Henstock-Stieltjes) (see [21, 35, 36]) integral in the formulae below. It is necessary to note that in the vector-valued context the above theorem is strictly related to the Kurzweil-Stieltjes integrals investigated in [28] (cf. also [32, 33, 35, 36], for instance). This kind of integral is very useful in applications (as in linear integral equations) and the above theorem gives us a full characterization of operators acting on the space of regulated functions, possibly with this integral (if the representing measure for the operator is in the general form) - cf. [34, Proposition 2.1], for the direct proof of linearity and continuity.

We should present some classical Stieltjes-type operators, which are useful in the study of discontinuous solutions for some integral equations:

A): Let $H_1 : G([0,1]) \to G([0,1])$ be given by

$$H_1(x)(t) = \int_{[0,t)} x(s)\, dg(s),$$

where g is a function of bounded variation and the integral is the Henstock-Stieltjes one. Clearly, by Theorem 9 it is bounded linear operator from $G([0,1])$ into itself.

B): Now, let $H_2 : G([0,1]) \to G([0,1])$

$$H_2(x)(t) = \int_{[0,1]} K(t,s)x(s)\, dg(s),$$

where $g \in BV([0,1])$ and the kernel satisfies the following assumptions
(1) $K(t,\cdot)$ is a function of bounded variation for every $t \in [0,1]$ with $\|K(t,\cdot)\|_{BV} \leq M$, for some $M > 0$,
(2) $K(\cdot,s) \in G([0,1])$.
Clearly, the above Theorem implies that H_2 is a continuous linear operator.

C): Let us also recall the vector-valued case (cf. [28, 34], for instance). Let $F : [0,1] \to L(X)$ and $g : [0,1] \to X$. Assume, that F has bounded semi-variation (see [28]). Then the operator $T : G([0,1],X) \to G([0,1],X)$ given by

$$T(x)(t) = \int_{[0,t)} x(s)\, dF(s)$$

is linear and continuous.

6.2. Weakly compact operators

Since $G([0,1],X)$ is isometrically isomorphic to a space $C(\mathbb{K},X)$ and $\mathbb{K}$ is compact Hausdorff space, by using the result from [3] and our Theorem 1 we get one more direct consequence for weakly compact linear operators acting on $G([0,1],X)$:

Corollary 4 (cf. [3], Theorem 9)**.** *Let X be a reflexive Banach space and assume, that the space Y does not contain isomorphic copy of c_0. Then any bounded linear operator from the space $G([0,1],X)$ into a weakly complete Banach space Y is automatically weakly compact.*

If we are interested in operators with values in the space of regulated functions, then the above Corollary cannot be applied. In many applications (such as solving of integral equations) we consider operators dominated by weakly compact ones. The following proposition will be useful (cf. [7, Theorem 5.3]):

Proposition 4. *Let X, Y be arbitrary Banach spaces and let L, T be bounded linear operators acting on $G([0,1], X)$ with values in Y. If X^* and X^{**} have the Radon-Nikodym property, L is weakly compact, and the representing measure for T is strongly absolutely continuous with respect to representing measure for L, then T is also a weakly compact operator.*

In particular, if the space X is reflexive, the space Y does not contain isomorphic copy of c_0 and the representing measure for the operator T is strongly bounded (s-bounded), then T is weakly compact (cf. [3]).

Many interesting results of this type can be found in [5]. The current status of the research for weakly compact (and compact) linear operators acting on the space of continuous functions (so potentially generalizable to the space of regulated functions) can be found in a recent paper [29] (see also references therein).

6.3. Compact operators

In the case of (strongly) compact operators we have a lot of interesting characterizations (parallel to those for weakly compact ones), so let us restrict ourselves to some special cases. First, we study the case when the operator is dominated (cf. again [7, Theorem 5.3]):

Proposition 5. *Let X, Y be an arbitrary Banach spaces and let L, T be bounded linear operators acting on $G([0,1], X)$ with values in Y. If L is compact and the representing measure for T is strongly absolutely continuous with respect to the representing measure for L, then T is also compact.*

Since we have a compactness criterion for $G([0,1], X)$ (Corollary 1), the following corollary seems to be useful:

Corollary 5. *Let X, Y be arbitrary Banach spaces. A bounded linear operator $T : G([0,1], X) \to G([0,1], X)$ is compact if and only if the representing vector measure μ takes its values in a compact subset of $G([0,1], X)$.*

Let us present a simple example:

Corollary 6. *Let $H : G([0,1]) \to G([0,1])$ be a linear continuous operator*

$$H(x)(t) = \int_{[0,1]} x(s)\, dK(t,s),$$

where $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies:

(1) *$K(t, \cdot)$ is a function of bounded variation for every $t \in [0,1]$ with $\|K(t,\cdot)\|_{BV} \le M$, for some $M > 0$,*

(2) *$K(\cdot, s) \in G([0,1])$.*

(3) *the kernel K satisfies the following conditions*

$$\lim_{\varepsilon\to 0^+}\left(\sup\left\{Var_0^1[K(t^-,\cdot)-K(\tau,\cdot)]:\ t\in(0,1],\tau\in(t-\varepsilon,t)\right\}\right)=0$$

$$\lim_{\varepsilon\to 0^+}\left(\sup\left\{Var_0^1[K(t^+,\cdot)-K(\tau,\cdot)]:\ t\in[0,1),\tau\in(t,t+\varepsilon)\right\}\right)=0.$$

Then the operator H is compact.

Proof. Indeed, as a consequence of Proposition 5 we need only to prove, that the range of H is relatively compact as a subset of $G([0,1])$. By Theorem 3 it is sufficient to prove, that it is uniformly equi-regulated subset of $G([0,1])$.

In what follows, let us take a nonempty bounded set $A\subset B_r$ in $G([0,1])$. Further, fix arbitrarily a number $\varepsilon>0$ and choose an arbitrary $x\in A$, $t\in(0,1]$ and $\tau\in(t-\varepsilon,t)$. Since x is regulated and $H: G([0,1])\to G([0,1])$, one-side limits $H(x)(t^-)$ and $H(x)(t^+)$ exist at every point t. Let us estimate:

$$\begin{aligned}|H(x)(t^-)-H(x)(\tau)| &\le \left|\int_0^1 x(s)\,d_sK(t^-,s)-\int_0^1 x(s)\,d_sK(\tau,s)\right|\\ &\le \left|\int_0^1 x(s)\,d_s[K(t^-,s)-K(\tau,s)]\right|\\ &\le \|x\|_\infty\cdot Var_0^1[K(t^-,\cdot)-K(\tau,\cdot)]\\ &\le \|x\|_\infty\cdot \sup_{\tau\in(t-\varepsilon,t)}\{Var_0^1[K(t^-,\cdot)-K(\tau,\cdot)]\}\\ &\le \|x\|_\infty\cdot\gamma_r^-(\varepsilon),\end{aligned}$$

where

$$\gamma_r^-(\varepsilon)=\sup_{t\in(0,1],\rho\in(t-\varepsilon,t)}\left\{Var_0^1[K(t^-,\cdot)-K(\rho,\cdot)]\right\}.$$

Similar estimation holds true for the right limit:

$$|H(x)(t^-)-H(x)(\tau)|\le\|x\|_\infty\cdot\gamma_r^+(\varepsilon),$$

$$\gamma_r^+(\varepsilon)=\sup_{t\in(0,1],\rho\in(t,t+\varepsilon)}\left\{Var_0^1[K(t^+,\cdot)-K(\rho,\cdot)]\right\}.$$

Since A is bounded, the assumption (3) implies that $H(A)$ is equi-regulated uniformly with respect to $x\in A$ and then $H(A)$ is uniformly equi-regulated subset of $G([0,1])$, so by Proposition 1 it is relatively compact subset of this space. The operator H is compact. ■

7. An application

Let us complete the paper by presenting a simple example of application of the obtained results for the Hammerstein integral equation in the space of regulated functions. It is important to note that if discontinuous solutions were investigated, then the solutions should have an additional property, i.e. should be of bounded variation. It was the case when solutions were expected in $BV([0,1])$. But in such a case it is necessary to assume, an extra condition, that the considered operators transform this space into itself. In our approach such a condition is superfluous. Consider the equation:

$$x(t) = g(t) + \lambda \int_0^1 f(s, x(s))\, d_s K(t,s), \tag{1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping. Put the following set of assumptions:

$(H1)$ Assume, that $g \in G([0,1])$.

$(H2)$ Suppose that the function $f(\cdot, x)$ is regulated on $[0,1]$ for all $x \in \mathbb{R}$, and the function $f(t, \cdot)$ satisfies with the constant L the Lipschitz condition with respect to the second argument,

$(H3)$ Assume, that $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies $K(t, \cdot)$ is a function of bounded variation for every $t \in [0,1]$ with $\|K(t,\cdot)\|_{BV([0,1])} \leq M$, for some $M > 0$, $K(\cdot, s) \in G([0,1])$ and

$$\lim_{\varepsilon \to 0^+} \left(\sup \left\{ Var_0^1 [K(t^-, \cdot) - K(\tau, \cdot)] :\ t \in (0,1], \tau \in (t - \varepsilon, t) \right\} \right) = 0$$

$$\lim_{\varepsilon \to 0^+} \left(\sup \left\{ Var_0^1 [K(t^+, \cdot) - K(\tau, \cdot)] :\ t \in [0,1), \tau \in (t, t + \varepsilon) \right\} \right) = 0,$$

$(H4)$ Let $a(t) = f(t, 0)$. Assume, that λ is sufficiently small, i.e. $1 - \lambda ML > 0$ and for some positive constant r we have

$$\|g\|_\infty + \lambda \cdot M \cdot (\|a\|_\infty + L \cdot r) \leq r$$

Theorem 11. *Let assumptions $(H1)$–$(H4)$ be satisfied. Then there exists a solution $x \in G([0,1])$ for the equation (1), i.e its regulated solution.*

Proof. First observe, that the equation (1) can be rewritten in an operator form:

$$x = g + \lambda \cdot (H \circ S_f)(x),$$

where H is a linear operator as in Corollary 6 and S_f is a Nemytskii superposition operator described in Theorem 7. It we denote the right-hand side of this equation by T, then it suffices to show, that $T : G([0,1]) \to G([0,1])$ has a fixed point.

Note that the assumption $(H2)$ implies, that S_f maps $G([0,1])$ into itself (Theorem 7) and since the Lipschitz condition with respect to the second argument implies, that $\tilde{f}(x)(t) = f(t, x(t))$ is continuous (Theorem 8), we get the continuity of S_f too. At the same time, our assumptions on K are sufficient to apply Corollary 6 and we get, that the operator H is compact. Recall that the composition of two operators is compact whenever at least one of them is compact, so T is compact too.

We will apply the Schauder fixed point theorem. In order to do it one must find an invariant closed convex bounded set.

Fix an arbitrary $x \in G([0,1])$ and $t \in [0,1]$. Since by $(H2)$ $f(t,x)$ satisfies the Lipschitz condition with constant L, then $\|f(t,x)\| \leq \|f(t,x) - f(t,0)\| + \|f(t,0)\| \leq L \cdot \|x\| + \|f(t,0)\|$. Thus

$$\|f(t,x)\| \leq a(t) + L \cdot \|x\|$$

for the regulated real-valued and non-negative function $a(t) = \|f(t,0)\|$. In view of the assumptions $(H1)$-$(H3)$ and by using the properties of the Henstock-Stieltjes integral, we have

$$\begin{aligned} |(T(x))(t)| &\leq |g(t)| + \lambda \int_0^1 |f(s, x(s))| \, d_s K(t,s) \\ &\leq |g(t)| + \lambda \|K(t,\cdot)\|_{BV([0,1])} \|F(x)\|_\infty \\ &\leq \|g\|_\infty + \lambda \cdot M \cdot (\|a\|_\infty + L \cdot \|x\|_\infty). \end{aligned}$$

Then $\|(T(x))\|_\infty \leq \|g\|_\infty + \lambda \cdot M \cdot (\|a\|_\infty + L \cdot \|x\|_\infty)$. If we take a ball $B_r \subset G([0,1])$ as a domain for T with $r > 0$ given in the assumption $(H4)$, then we get that $T : B_r \to B_r$. Obviously, the set B_r is nonempty bounded closed and convex.

Now, the proof runs as in the case of continuous functions (see [1] for a survey about the methods how to solve Hammerstein integral equations in function spaces) and by the Schauder fixed point theorem we find a fixed point for T, so we have a regulated solution for the considered equation. ■

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KINGA CICHOŃ
INSTITUTE OF MATHEMATICS
FACULTY OF ELECTRICAL ENGINEERING
POZNAN UNIVERSITY OF TECHNOLOGY
PIOTROWO 3A, 60–965 POZNAŃ, POLAND
e-mail: kinga.cichon@put.poznan.pl

MIECZYSŁAW CICHOŃ
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
A. MICKIEWICZ UNIVERSITY
UMULTOWSKA 87, 61-614 POZNAŃ, POLAND
e-mail: mcichon@amu.edu.pl

BIANCA SATCO
STEFAN CEL MARE UNIVERSITY OF SUCEAVA
FACULTY OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE
INTEGRATED CENTER FOR RESEARCH

Development and Innovation in Advanced Materials
Nanotechnologies and Distributed Systems for
Fabrication and Control (MANSiD)
Universitatii 13 - Suceava, Romania
e-mail: bianca.satco@eed.usv.ro

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