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SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES VIA k-FRACTIONAL INTEGRALS CONCERNING DIFFERENTIABLE GENERALIZED RELATIVE SEMI-$(r; m, p, q, h_1, h_2)$-PREINVEX MAPPINGS

Abstract. In this article, we first presented a new identity concerning differentiable mappings defined on m-invex set via k-fractional integrals. By using the notion of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via k-fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article.

Key words: Hermite-Hadamard type inequality, MT-convex function, Hölder's inequality, Minkowski inequality, power mean inequality, k-fractional integrals, m-invex.

AMS Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [35]-[20], [25], [28], [30], [41]). For some applications to special means of this foundamental inequalities (see [29], [36]-[38]). Fractional calculus [21], was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Let us recall some special functions and evoke some basic definitions as follows.

Definition 2. *The Euler beta function is defined for $a, b > 0$ as*

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 3. *For $k \in \mathbb{R}^+$ and $x, y \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as*

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{2}$$

For $k = 1$, (2) gives integral representation of beta function.

Definition 4. *For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by*

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha).$$

For $k = 1$, (4) gives integral representation of gamma function.

Theorem 2. *Let $x, y > 0$. Then, for k-gamma and k-beta function the following equality holds:*

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)}. \tag{5}$$

Definition 5 ([23]). *Let $f \in L_1[a, b]$. Then, k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$, are defined as*

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x - t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t - x)^{\frac{\alpha}{k}-1} f(t)dt, \quad b > x.$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Definition 6 ([40]). *A set $M_\varphi \subseteq \mathbb{R}^n$ is said to be a relative convex (φ-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,*

$$t\varphi(x) + (1 - t)\varphi(y) \in M_\varphi, \ \forall \ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, \ t \in [0, 1]. \tag{6}$$

Definition 7 ([40]). *A function f is said to be a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,*

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)), \tag{7}$$

$\forall \ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]$.

Definition 8 ([1]). *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true (see [1], [39]).

Definition 9 ([27]). *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0, 1]$, we have that*

$$f\left(x + t\eta(y, x)\right) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Definition 10 ([22]). *Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if*

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{8}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Definition 11 ([32]). *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function, $h \not\equiv 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex, if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$, one has*

$$f\big(tx + (1-t)y\big) \leq h(t)f(x) + h(1-t)f(y). \tag{9}$$

Definition 12 ([31]). *Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, we say that $f : K \longrightarrow \mathbb{R}$ is a tgs-convex function on K if the inequality*

$$f\Big((1-t)x + ty\Big) \leq t(1-t)[f(x) + f(y)] \tag{10}$$

holds for all $x, y \in K$ and $t \in (0,1)$. We say that f is tgs-concave if $(-f)$ is tgs-convex.

Definition 13 ([24]). *A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, with $m \in [0,1]$, satisfies the following inequality*

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{11}$$

Definition 14 ([26]). *Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is a generalized (m, h_1, h_2)-preinvex with respect to η, if*

$$f\big(mx + t\eta(y,x,m)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{12}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, with $m \in (0,1]$. If the inequality (12) reverses, then f is said to be (m, h_1, h_2)-preincave on K.

Definition 15 ([11]). *A set $K \subseteq \mathbb{R}$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ grips for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1. In Definition 15, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We are in position to introduce the notion of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex functions.

Definition 16. *Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. A function $f : K \longrightarrow (0, +\infty)$ is said to be generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex, if*

$$f\big(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\big) \leq M_r(h_1(t), h_2(t); mf(x), f(y), p, q) \tag{13}$$

holds for all $x, y \in I$ and $t \in [0, 1]$, for $p, q > -1$ and some fixed $m \in (0, 1]$, where

$$\begin{aligned} &M_r(h_1(t), h_2(t); mf(x), f(y), p, q) \\ &\quad := \begin{cases} \Big[mh_1^p(t)f^r(x) + h_2^q(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } \ r \neq 0 \\ \big[mf(x)\big]^{h_1^p(t)}\big[f(y)\big]^{h_2^q(t)}, & \text{if } \ r = 0, \end{cases} \end{aligned}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2. In Definition 16, if we choose $r = p = q = 1$ and $\varphi(x) = x$, then we get Definition 14. If we choose $r = p = q = 1$, $\varphi(x) = x$, $\forall x \in I$ and $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{m\sqrt{t}}{2\sqrt{1-t}}$, then we get MT_m-preinvex function (see [12], [13]).

Remark 3. For $r = p = q = 1$, let us discuss some special cases in Definition 16 as follows.

(I) If taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0, 1]$, then we get generalized relative semi-$(m, s)-$ Breckner-preinvex functions.

(II) If taking $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m, P)-preinvex functions.

(III) If taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized relative semi-(m, s)-Godunova-Levin-Dragomir-preinvex functions.

(IV) If taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m, h)-preinvex functions.

(V) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m, tgs)-preinvex functions.

(VI) If taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex functions.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Motivated by the above literatures, the main objective of this article is to establish some new estimates on generalizations to Hermite-Hadamard type inequalities via k-fractional integrals associated with differentiable generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings on m-invex set. It is pointed out that some new special cases will be deduced from main results of the article.

2. Main results

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for differentiable generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex functions via k-fractional integrals, we need the following new integral identity:

Lemma 1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ be a differentiable function on K° and $f' \in L_1(K)$. Then for any $\alpha,\ k > 0$, $\lambda, \mu \in \mathbb{R}$ and $r \in [0,1]$, the following integral identity holds:*

$$
\begin{aligned}
(14)\qquad & \frac{\eta(\varphi(b),\varphi(a),m)}{4}\left\{\frac{(r+1)\left(\lambda\left[f\left(m\varphi\left(\frac{a+b}{2}\right)\right)\right.\right.}{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}\right. \\
& -\frac{\left.\left. f\left(m\varphi\left(\frac{a+b}{2}\right)+\frac{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}{r+1}\right)\right]-f\left(m\varphi\left(\frac{a+b}{2}\right)\right)\right)}{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)} \\
& +\frac{(r+1)^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)} I^{\alpha,k}_{\left(m\varphi\left(\frac{a+b}{2}\right)+\frac{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}{r+1}\right)^-} \\
& -f\left(m\varphi\left(\frac{a+b}{2}\right)\right)\Bigg) \\
& +\frac{(r+1)\left(\mu\left[f\left(m\varphi(b)+\frac{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}{r+1}\right)-f(m\varphi(b))\right]\right.}{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)} \\
& -\frac{\left. f\left(m\varphi(b)+\frac{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}{r+1}\right)\right)}{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}
\end{aligned}
$$

$$+\frac{(r+1)^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)} I^{\alpha,k}_{\left(m\varphi(b)+\frac{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}{r+1}\right)} - f\left(m\varphi(b)\right)\Bigg\}$$

$$=\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\int_0^1\left(1-\lambda-t^{\frac{\alpha}{k}}\right)$$

$$\times f'\left(m\varphi\left(\frac{a+b}{2}\right)+\left(\frac{t}{r+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)dt$$

$$+\int_0^1\left(\mu-t^{\frac{\alpha}{k}}\right)f'\left(m\varphi(b)+\left(\frac{t}{r+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)dt\Bigg\}.$$

Proof. Integrating by parts and changing variables of definite integrals yield

$$\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\int_0^1\left(1-\lambda-t^{\frac{\alpha}{k}}\right)f'\left(m\varphi\left(\frac{a+b}{2}\right)\right.$$

$$\left.+\left(\frac{t}{r+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)dt$$

$$+\int_0^1\left(\mu-t^{\frac{\alpha}{k}}\right)f'\left(m\varphi(b)+\left(\frac{t}{r+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)dt\Bigg\}$$

$$=\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\frac{(r+1)\left(1-\lambda-t^{\frac{\alpha}{k}}\right)}{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}f\left(m\varphi\left(\frac{a+b}{2}\right)\right.$$

$$\left.+\left(\frac{t}{r+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)\Bigg|_0^1+\frac{\alpha(r+1)}{k\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}$$

$$\times\int_0^1 t^{\frac{\alpha}{k}-1}f\left(m\varphi\left(\frac{a+b}{2}\right)+\left(\frac{t}{r+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)dt$$

$$+\frac{(r+1)\left(\mu-t^{\frac{\alpha}{k}}\right)}{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}f\left(m\varphi(b)+\left(\frac{t}{r+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)\Bigg|_0^1$$

$$+\frac{\alpha(r+1)}{k\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}$$

$$\times\int_0^1 t^{\frac{\alpha}{k}-1}f\left(m\varphi(b)+\left(\frac{t}{r+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)dt\Bigg\}$$

$$=\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\frac{(r+1)\Bigg(\lambda\Bigg[f\left(m\varphi\left(\frac{a+b}{2}\right)\right)}{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}$$

$$-\frac{f\left(m\varphi\left(\frac{a+b}{2}\right)+\frac{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}{r+1}\right)\Bigg]-f\left(m\varphi\left(\frac{a+b}{2}\right)\right)\Bigg)}{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}$$

$$+\frac{(r+1)^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}I^{\alpha,k}_{\left(m\varphi\left(\frac{a+b}{2}\right)+\frac{\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)}{r+1}\right)}$$

$$-f\left(m\varphi\left(\frac{a+b}{2}\right)\right)+\frac{(r+1)\left(\mu\left[f\left(m\varphi(b)+\frac{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}{r+1}\right)\right.\right.}{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}$$

$$-\frac{\left.f\left(m\varphi(b)\right)\right]-f\left(m\varphi(b)+\frac{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}{r+1}\right)\Big)}{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}$$

$$+\frac{(r+1)^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}I^{\alpha,k}_{\left(m\varphi(b)+\frac{\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)}{r+1}\right)}-f\left(m\varphi(b)\right)\Bigg\}.$$

So, the proof of this lemma is completed. ■

Remark 4. In Lemma 1, if we choose $\alpha=k=m=1$, $r=0$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$ and $\varphi(x)=x$, $\forall x\in I$, we get ([35], Lemma 2.1).

Throughout this paper we denote

$$(15)\quad I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,r,m,a,b):=\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\int_0^1\left(1-\lambda-t^{\frac{\alpha}{k}}\right)$$

$$\times f'\left(m\varphi\left(\frac{a+b}{2}\right)+\left(\frac{t}{r+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)dt$$

$$+\int_0^1\left(\mu-t^{\frac{\alpha}{k}}\right)f'\left(m\varphi(b)+\left(\frac{t}{r+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)dt\Bigg\}.$$

Using relation (15), the following results can be obtained for the corresponding version for power of the first derivative.

Theorem 3. *Let α, $k>0$, λ, μ, $r_1\in[0,1]$, $0<r\le 1$ and $p_1,p_2>-1$. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Assume that $f:K\longrightarrow(0,+\infty)$ be a differentiable function on K°, where $\eta(\varphi(b),\varphi(a),m)>0$. If $f'(x)^q$ is generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex functions, $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$(16)\qquad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,r_1,m,a,b)\right|\le\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{A_1^{\frac{1}{p}}(\lambda,\alpha,k,p)$$

$$\times\left[mf'\left(\frac{a+b}{2}\right)^{rq} I^{r}(h_1(t);r,r_1,p_1)+f'(a)^{rq} I^{r}(h_2(t);r,r_1,p_2)\right]^{\frac{1}{rq}}$$

$$+A_2^{\frac{1}{p}}(\mu,\alpha,k,p)\Bigg[mf'(b)^{rq} I^{r}(h_1(t);r,r_1,p_1)$$

$$+\ f'\left(\frac{a+b}{2}\right)^{rq} I^{r}(h_2(t);r,r_1,p_2)\Bigg]^{\frac{1}{rq}}\Bigg\},$$

where

$$A_1(\lambda,\alpha,k,p):=\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|^p dt,\quad A_2(\mu,\alpha,k,p):=\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|^p dt$$

and

$$I(h_i(t);r,r_1,p_i):=\int_0^1 h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right)dt,\ \ \forall\, i=1,2.$$

Proof. Suppose that $q>1$, λ, μ, $r_1\in[0,1]$ and $0<r\leq 1$. From Lemma 1, generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvexity of $f'(x)^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,r_1,m,a,b)\right|\leq\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|\Bigg|$$

$$\times f'\left(m\varphi\left(\frac{a+b}{2}\right)+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)\Bigg|dt$$

$$+\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|\left|f'\left(m\varphi(b)+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)\right|dt\Bigg\}$$

$$\leq\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(f'\left(m\varphi\left(\frac{a+b}{2}\right)+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(f'\left(m\varphi(b)+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)\right)^q dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq\frac{\eta(\varphi(b),\varphi(a),m)}{4}\times\Bigg\{\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)f'\left(\frac{a+b}{2}\right)^{rq}+h_2^{p_2}\left(\frac{t}{r_1+1}\right)f'(a)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\mu - t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)f'(b)^{rq} + h_2^{p_2}\left(\frac{t}{r_1+1}\right)f'\left(\frac{a+b}{2}\right)^{rq}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \frac{\eta(\varphi(b),\varphi(a),m)}{4} \times \Bigg\{\left(\int_0^1 \left|1-\lambda - t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\int_0^1 m^{\frac{1}{r}} f'\left(\frac{a+b}{2}\right)^q h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1 f'(a)^q h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\Bigg]^{\frac{1}{rq}} + \left(\int_0^1 \left|\mu - t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\int_0^1 m^{\frac{1}{r}} f'(b)^q h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1 f'\left(\frac{a+b}{2}\right)^q h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}$$

$$= \frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{A_1^{\frac{1}{p}}(\lambda,\alpha,k,p)$$

$$\times\left[mf'\left(\frac{a+b}{2}\right)^{rq} I^r(h_1(t);r,r_1,p_1) + f'(a)^{rq} I^r(h_2(t);r,r_1,p_2)\right]^{\frac{1}{rq}}$$

$$+ A_2^{\frac{1}{p}}(\mu,\alpha,k,p)\Bigg[mf'(b)^{rq} I^r(h_1(t);r,r_1,p_1)$$

$$+ f'\left(\frac{a+b}{2}\right)^{rq} I^r(h_2(t);r,r_1,p_2)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is completed. ■

We point out some special cases of Theorem 3.

Corollary 1. *In Theorem 3 for* $r_1 = 0$, $h_1(t) = h(1-t)$ *and* $h_2(t) = h(t)$, *we have the following inequality for generalized relative semi-*$(r;m,p_1,p_2,h)$*-preinvex functions:*

$$(17)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,0,m,a,b)\right| \leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{A_1^{\frac{1}{p}}(\lambda,\alpha,k,p)$$

$$\times\left[mf'\left(\frac{a+b}{2}\right)^{rq} I^r(h(1-t);r,0,p_1) + f'(a)^{rq} I^r(h(t);r,0,p_2)\right]^{\frac{1}{rq}}$$

$$+ A_2^{\frac{1}{p}}(\mu, \alpha, k, p)\left[mf'(b)^{rq} I^r(h(1-t); r, 0, p_1)\right.$$

$$\left.\left. + f'\left(\frac{a+b}{2}\right)^{rq} I^r(h(t); r, 0, p_2)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2. *In Theorem 3 for* $r_1 = 0$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, s)$*-Breckner-preinvex functions:*

$$\left|I_{f,\eta,\varphi}(\lambda, \mu, \alpha, k, 0, m, a, b)\right| \leq \frac{\eta(\varphi(b), \varphi(a), m)}{4} \tag{18}$$

$$\times\left\{A_1^{\frac{1}{p}}(\lambda, \alpha, k, p)\left[mf'\left(\frac{a+b}{2}\right)^{rq}\left(\frac{r}{r+sp_1}\right)^r\right.\right.$$

$$\left. + f'(a)^{rq}\left(\frac{r}{r+sp_2}\right)^r\right]^{\frac{1}{rq}}$$

$$+ A_2^{\frac{1}{p}}(\mu, \alpha, k, p)\left[mf'(b)^{rq}\left(\frac{r}{r+sp_1}\right)^r\right.$$

$$\left.\left. + f'\left(\frac{a+b}{2}\right)^{rq}\left(\frac{r}{r+sp_2}\right)^r\right]^{\frac{1}{rq}}\right\}.$$

Corollary 3. *In Theorem 3 for* $r_1 = 0$, $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *and* $r > \max\{sp_1, sp_2\}$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, s)$*-Godunova-Levin-Dragomir-preinvex functions:*

$$\left|I_{f,\eta,\varphi}(\lambda, \mu, \alpha, k, 0, m, a, b)\right| \leq \frac{\eta(\varphi(b), \varphi(a), m)}{4} \tag{19}$$

$$\times\left\{A_1^{\frac{1}{p}}(\lambda, \alpha, k, p)\left[mf'\left(\frac{a+b}{2}\right)^{rq}\left(\frac{r}{r-sp_1}\right)^r\right.\right.$$

$$\left. + f'(a)^{rq}\left(\frac{r}{r-sp_2}\right)^r\right]^{\frac{1}{rq}}$$

$$+ A_2^{\frac{1}{p}}(\mu, \alpha, k, p)\left[mf'(b)^{rq}\left(\frac{r}{r-sp_1}\right)^r\right.$$

$$\left.\left. + f'\left(\frac{a+b}{2}\right)^{rq}\left(\frac{r}{r-sp_2}\right)^r\right]^{\frac{1}{rq}}\right\}.$$

Corollary 4. *In Theorem 3 for $r_1 = 0$, $h_1(t) = h_2(t) = t(1-t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, tgs)$-preinvex functions:*

$$(20)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,0,m,a,b)\right| \leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}$$
$$\times\Bigg\{A_1^{\frac{1}{p}}(\lambda,\alpha,k,p)\Bigg[mf'\left(\frac{a+b}{2}\right)^{rq}\beta^r\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)$$
$$+f'(a)^{rq}\beta^r\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}$$
$$+A_2^{\frac{1}{p}}(\mu,\alpha,k,p)\Bigg[mf'(b)^{rq}\beta^r\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)$$
$$+f'\left(\frac{a+b}{2}\right)^{rq}\beta^r\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 5. *In Theorem 3 for $r_1 = 0$, $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $r > \frac{\max\{p_1,p_2\}}{2}$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2)$-MT-preinvex functions:*

$$(21)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,0,m,a,b)\right| \leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}$$
$$\times\Bigg\{A_1^{\frac{1}{p}}(\lambda,\alpha,k,p)\Bigg[mf'\left(\frac{a+b}{2}\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$
$$+f'(a)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}$$
$$+A_2^{\frac{1}{p}}(\mu,\alpha,k,p)\Bigg[mf'(b)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$
$$+f'\left(\frac{a+b}{2}\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Theorem 4. *Let α, $k > 0$, λ, μ, $r_1 \in [0,1]$, $0 < r \leq 1$ and $p_1, p_2 > -1$. Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K \longrightarrow (0,+\infty)$ be a differentiable function on K°, where $\eta(\varphi(b),\varphi(a),m) > 0$. If $f'(x)^q$ is* *generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex functions, $q\geq 1$, then the following inequality holds:*

$$
\begin{aligned}
(22)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,r_1,m,a,b)\right| &\leq \frac{\eta(\varphi(b),\varphi(a),m)}{4} \\
&\times\Bigg\{A_1^{1-\frac{1}{q}}(\lambda,\alpha,k,1)\Bigg[mf'\left(\frac{a+b}{2}\right)^{rq}B^r(h_1(t);r,r_1,\lambda,\alpha,k,p_1) \\
&+f'(a)^{rq}B^r(h_2(t);r,r_1,\lambda,\alpha,k,p_2)\Bigg]^{\frac{1}{rq}} \\
&+A_2^{1-\frac{1}{q}}(\mu,\alpha,k,1)\Bigg[mf'(b)^{rq}C^r(h_1(t);r,r_1,\mu,\alpha,k,p_1) \\
&+f'\left(\frac{a+b}{2}\right)^{rq}C^r(h_2(t);r,r_1,\mu,\alpha,k,p_2)\Bigg]^{\frac{1}{rq}}\Bigg\},
\end{aligned}
$$

where

$$B(h_i(t);r,r_1,\lambda,\alpha,k,p_i):=\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right)dt,$$

$$C(h_i(t);r,r_1,\mu,\alpha,k,p_i):=\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right)dt,\quad \forall\, i=1,2$$

and $A_1(\lambda,\alpha,k,1)$ and $A_2(\mu,\alpha,k,1)$ are defined as in Theorem 3, for $p=1$.

Proof. Suppose that $q\geq 1$, λ, μ, $r_1\in[0,1]$ and $0<r\leq 1$. From Lemma 1, generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvexity of $f'(x)^q$, power mean inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
\left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,r_1,m,a,b)\right| &\leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right| \\
&\times\left|f'\left(m\varphi\left(\frac{a+b}{2}\right)+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)\right|dt \\
&+\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|\left|f'\left(m\varphi(b)+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)\right|dt\Bigg\} \\
&\leq\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|\left(f'\left(m\varphi\left(\frac{a+b}{2}\right)\right.\right.\right. \\
&+\left.\left.\left.\left(\frac{t}{r_1+1}\right)\eta\left(\varphi(a),\varphi\left(\frac{a+b}{2}\right),m\right)\right)\right)^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|\left(f'\left(m\varphi(b)\right.\right.\right.$$

$$\left.\left.\left.+\left(\frac{t}{r_1+1}\right)\eta\left(\varphi\left(\frac{a+b}{2}\right),\varphi(b),m\right)\right)\right)^q dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq\frac{\eta(\varphi(b),\varphi(a),m)}{4}\Bigg\{\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|\right.$$

$$\left.\times\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)f'\left(\frac{a+b}{2}\right)^{rq}+h_2^{p_2}\left(\frac{t}{r_1+1}\right)f'(a)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|\left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)f'(b)^{rq}\right.\right.$$

$$\left.\left.+h_2^{p_2}\left(\frac{t}{r_1+1}\right)f'\left(\frac{a+b}{2}\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq\frac{\eta(\varphi(b),\varphi(a),m)}{4}\times\Bigg\{\left(\int_0^1\left|1-\lambda-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}}f'\left(\frac{a+b}{2}\right)^q\left|1-\lambda-t^{\frac{\alpha}{k}}\right|h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1 f'(a)^q\left|1-\lambda-t^{\frac{\alpha}{k}}\right|h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right]^{\frac{1}{rq}}$$

$$+\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 m^{\frac{1}{r}}f'(b)^q\left|\mu-t^{\frac{\alpha}{k}}\right|h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1 f'\left(\frac{a+b}{2}\right)^q\left|\mu-t^{\frac{\alpha}{k}}\right|h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right]^{\frac{1}{rq}}\Bigg\}$$

$$=\frac{\eta(\varphi(b),\varphi(a),m)}{4}$$

$$\times\Bigg\{A_1^{1-\frac{1}{q}}(\lambda,\alpha,k,1)\left[mf'\left(\frac{a+b}{2}\right)^{rq}B^r(h_1(t);r,r_1,\lambda,\alpha,k,p_1)\right.$$

$$\left.+f'(a)^{rq}B^r(h_2(t);r,r_1,\lambda,\alpha,k,p_2)\right]^{\frac{1}{rq}}$$

$$+A_2^{1-\frac{1}{q}}(\mu,\alpha,k,1)\left[mf'(b)^{rq}C^r(h_1(t);r,r_1,\mu,\alpha,k,p_1)\right.$$

$$\left.+f'\left(\frac{a+b}{2}\right)^{rq}C^r(h_2(t);r,r_1,\mu,\alpha,k,p_2)\right]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is completed. ■

We point out some special cases of Theorem 4.

Corollary 6. *In Theorem 4 for $r_1 = 0$, $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, h)$-preinvex functions:*

$$(23)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,0,m,a,b)\right| \leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}$$
$$\times\Bigg\{A_1^{1-\frac{1}{q}}(\lambda,\alpha,k,1)\Bigg[mf'\left(\frac{a+b}{2}\right)^{rq}B^r(h(1-t);r,0,\lambda,\alpha,k,p_1)$$
$$+f'(a)^{rq}B^r(h(t);r,0,\lambda,\alpha,k,p_2)\Bigg]^{\frac{1}{rq}}$$
$$+A_2^{1-\frac{1}{q}}(\mu,\alpha,k,1)\Bigg[mf'(b)^{rq}C^r(h(1-t);r,0,\mu,\alpha,k,p_1)$$
$$+f'\left(\frac{a+b}{2}\right)^{rq}C^r(h(t);r,0,\mu,\alpha,k,p_2)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 7. *In Theorem 4 for $r_1 = 0$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, s)$-Breckner-preinvex functions:*

$$(24)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,0,m,a,b)\right| \leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}$$
$$\times\Bigg\{A_1^{1-\frac{1}{q}}(\lambda,\alpha,k,1)\Bigg[mf'\left(\frac{a+b}{2}\right)^{rq}B^r((1-t)^s;r,0,\lambda,\alpha,k,p_1)$$
$$+f'(a)^{rq}B^r(t^s;r,0,\lambda,\alpha,k,p_2)\Bigg]^{\frac{1}{rq}}$$
$$+A_2^{1-\frac{1}{q}}(\mu,\alpha,k,1)\Bigg[mf'(b)^{rq}C^r((1-t)^s;r,0,\mu,\alpha,k,p_1)$$
$$+f'\left(\frac{a+b}{2}\right)^{rq}C^r(t^s;r,0,\mu,\alpha,k,p_2)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 8. *In Theorem 4 for $r_1 = 0$, $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, s)$-Godunova-Levin-Dragomir-preinvex functions:*

$$\left|I_{f,\eta,\varphi}(\lambda, \mu, \alpha, k, 0, m, a, b)\right| \leq \frac{\eta(\varphi(b), \varphi(a), m)}{4} \tag{25}$$

$$\times\left\{A_1^{1-\frac{1}{q}}(\lambda, \alpha, k, 1)\left[mf'\left(\frac{a+b}{2}\right)^{rq} B^r((1-t)^{-s}; r, 0, \lambda, \alpha, k, p_1)\right.\right.$$

$$\left.+ f'(a)^{rq} B^r(t^{-s}; r, 0, \lambda, \alpha, k, p_2)\right]^{\frac{1}{rq}}$$

$$+ A_2^{1-\frac{1}{q}}(\mu, \alpha, k, 1)\left[mf'(b)^{rq} C^r((1-t)^{-s}; r, 0, \mu, \alpha, k, p_1)\right.$$

$$\left.\left.+ f'\left(\frac{a+b}{2}\right)^{rq} C^r(t^{-s}; r, 0, \mu, \alpha, k, p_2)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 9. *In Theorem 4 for $r_1 = 0$, $h_1(t) = h_2(t) = t(1-t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, tgs)$-preinvex functions:*

$$\left|I_{f,\eta,\varphi}(\lambda, \mu, \alpha, k, 0, m, a, b)\right| \leq \frac{\eta(\varphi(b), \varphi(a), m)}{4} \tag{26}$$

$$\times\left\{A_1^{1-\frac{1}{q}}(\lambda, \alpha, k, 1)\left[mf'\left(\frac{a+b}{2}\right)^{rq} B^r(t(1-t); r, 0, \lambda, \alpha, k, p_1)\right.\right.$$

$$\left.+ f'(a)^{rq} B^r(t(1-t); r, 0, \lambda, \alpha, k, p_2)\right]^{\frac{1}{rq}}$$

$$+ A_2^{1-\frac{1}{q}}(\mu, \alpha, k, 1)\left[mf'(b)^{rq} C^r(t(1-t); r, 0, \mu, \alpha, k, p_1)\right.$$

$$\left.\left.+ f'\left(\frac{a+b}{2}\right)^{rq} C^r(t(1-t); r, 0, \mu, \alpha, k, p_2)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 10. *In Theorem 4 for $r_1 = 0$, $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we have the following inequality for generalized relative semi-$(r; m,$*

$p_1, p_2)$*-MT-preinvex functions:*

$$(27)\quad \left|I_{f,\eta,\varphi}(\lambda,\mu,\alpha,k,0,m,a,b)\right| \leq \frac{\eta(\varphi(b),\varphi(a),m)}{4}$$
$$\times\Bigg\{A_1^{1-\frac{1}{q}}(\lambda,\alpha,k,1)\Bigg[mf'\left(\frac{a+b}{2}\right)^{rq}B^r\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,0,\lambda,\alpha,k,p_1\right)$$
$$+f'(a)^{rq}B^r\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,0,\lambda,\alpha,k,p_2\right)\Bigg]^{\frac{1}{rq}}$$
$$+A_2^{1-\frac{1}{q}}(\mu,\alpha,k,1)\Bigg[mf'(b)^{rq}C^r\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,0,\mu,\alpha,k,p_1\right)$$
$$+f'\left(\frac{a+b}{2}\right)^{rq}C^r\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,0,\mu,\alpha,k,p_2\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Remark 5. For $k = 1$, we get some new Hermite-Hadamard type inequalities via fractional integrals associated with generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex functions.

Remark 6. *Applying our Theorems 3 and 4, we can deduce some new inequalities using special means associated with generalized relative semi-*$(r; m, p_1, p_2, h_1, h_2)$*-preinvex functions.*

3. Conclusions

In this article, we first presented a new identity concerning differentiable mappings defined on m-invex set via k-fractional integrals. By using the notion of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via k-fractional integrals are established. It is pointed out that some new special cases are deduced from main results of the article. Motivated by this new interesting class of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p, q)-calculus, time scale calculus and conformable fractional integrals.

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