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# SOME COMMON COUPLED FIXED POINT THEOREMS IN $S_b$-METRIC SPACES

Abstract. In this paper, we prove some common coupled fixed point theorems for mapping satisfying a nonlinear contraction in $S_b$-metric space and some results are also given in the form of corollary. Also, some examples are given to verify the main results.

Key words: coupled fixed point, common coupled fixed point, $S_b$-metric space.

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## 2. Introduction and preliminaries

As metric spaces play a very important role in mathematics and applied sciences, many authors have been generalizing the concept of metric space in several ways. Some of them are 2-metric space [4], $D$-metric space [3], $G$-metric space [8], $D^*$-metric space [9], $G_b$-metric space [2] etc..As one of the generalization of the metric space, Sedghi et al. [10] introduced the concept of $S$-metric space and proved a fixed point theorem. After that, many papers about fixed point theory in $S$-metric spaces appeared (see [11], [7] etc.).

Very recently, Souayah and Mlaiki [12] introduced the concept of $S_b$-metric space as a generalization of the $b$-metric space and proved some fixed point results. $\mathbb{R}$

Sedghi et al. [13] also introduced the concept of $S_b$-metric space and their definition of $S_b$-metric space is different from the definition of $S_b$-metric space given by Souayah and Mlaiki [12]. Sedghi et al. [13] defined the definition of $S_b$-metric space without condition $(ii)$ of definition (1) whereas Souayah and Mlaiki [12] considered condition $(ii)$ of definition (1) to be a part of the definition. Y. Rohen et al. [14] also proved some coupled fixed point theorem in $S_b$-metric space.

Now, we consider the following definitions.

**Definition 1** ([12]). *Let $X$ be a nonempty set and let $s \geq 1$ be a given real number. A function $S_b : X^3 \to [0, \infty)$ is said to be $S_b$-metric if and only if for all $x, y, z, t \in X$: the following conditions hold:*

(*i*) $S_b(x, y, z) = 0$ *if and only if* $x = y = z$,

(*ii*) $S_b(x, x, y) = S_b(y, y, x)$ *for all* $x, y \in X$,

(*iii*) $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$

*The pair $(X, S_b)$ is called a $S_b$-metric space.*

**Remark 1** ([12]). Note that the class of $S_b$-metric spaces is larger than the class of $S$-metric spaces. Indeed, every $S$-metric spaces is an $S_b$-metric space with $s = 1$. However, the converse is not always true.

**Definition 2** ([12]). *Let $(X, S_b)$ be an $S_b$-metric space and $\{x_n\}$ be a sequence in $X$. Then*

(*i*) *A sequence $\{x_n\}$ is called convergent if and only if there exist $z \in X$ such that $S_b(x_n, x_n, z) \to 0$ as $n \to \infty$. In this case we write $\lim_{n\to\infty} x_n = z$.*

(*ii*) *A sequence $\{x_n\}$ is called Cauchy sequence if and only if $S_b(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.*

(*iii*) *$(X, S_b)$ is said to be a complete $S_b$-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that*

$$\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = \lim_{n,m\to\infty} S_b(x_n, x_n, x) = S_b(x, x, x).$$

(*iv*) *Define the diameter of a subset $Y$ of $X$ by*

$$diam(Y) := \sup\{S_b(x, y, z) \mid x, y, z \in Y\}.$$

**Definition 3** ([5]). *An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

**Definition 4** ([6]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and $(gx, gy)$ is called a coupled point of coincidence.*

**Definition 5** ([6]). *An element $(x, y) \in X \times X$ is called a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.*

**Definition 6** ([6]). *Let $X$ be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say $F$ and $g$ are commutative if $g(F(x, y)) = F(gx, gy)$, for all $x, y \in X$.*

**Definition 7** ([1]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are called $w$-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.*

Next we prove the following Lemma.

**Lemma 1.** *Let $(X, S_b)$ be an $S_b$-metric space, and suppose that the sequence $\{x_n\}$ is convergent to $x$. Then*

$$\frac{1}{s}S_b(y,y,x) \leq \liminf_{n\to\infty} S_b(y,y,x_n) \leq \limsup_{n\to\infty} S_b(y,y,x_n) \leq sS_b(y,y,x)$$

*In particular, if $x = y$, then we get $\lim_{n\to\infty} S_b(y,y,x_n) = 0$*

**Proof.** Using condition $(ii)$ and $(iii)$ of definition of $S_b$-metric space, we get

$$\begin{aligned} S_b(y,y,x_n) &= S_b(x_n,x_n,y) \\ &\leq s[2S_b(x_n,x_n,x) + S_b(y,y,x)]. \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{1}{s}S_b(y,y,x) &= \frac{1}{s}S_b(x,x,y) \\ &\leq 2S_b(x,x,x_n) + S_b(y,y,x_n). \end{aligned} \tag{2}$$

Taking the upper limit as $n \to \infty$ in (1) and the lower limit as $n \to \infty$ in (2), we get the required result. ∎

The aim of this paper is to establish some common coupled fixed point results for mapping satisfying a nonlinear contraction in $S_b$-metric spaces. Also, we establish some examples to verify the results.

## 2. Main results

Let $\Psi$ denote the class of all function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi$ is increasing, continuous, $\psi(t) < \frac{t}{2}$ for all $t > 0$ and $\psi(0) = 0$. It is easy to see that for every $\psi \in \Psi$, we can choose a $k$ in $(0, \frac{1}{2})$ such that $\psi(t) \leq kt$.

Now we prove the following theorem.

**Theorem 1.** *Let $(X, S_b)$ be an $S_b$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$S_b(F(x,y), F(u,v), F(a,b)) \leq \frac{1}{s}\psi(S_b(gx,gu,ga) + S_b(gy,gv,gb))$$

*for some $\psi \in \Psi$ and for all $x,y,u,v,a,b \in X$. Assume that $F$ and $g$ satisfy the following conditions:*

*(a) $F(X \times X) \subseteq g(X)$,*

*(b) $g(X)$ is complete, and*

*(c) $F$ and $g$ are $w$-compatible.*

*Then $F$ and $g$ have a unique common coupled fixed point, and which is of the form $(x,x)$, that is, there is a unique $x \in X$ such that $F(x,x) = gx = x$.*

**Proof.** Let $x_0, y_0 \in X$ be any two elements. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing this process, we can construct two sequences $\{x_n\}$, $\{y_n\}$ in $X$ such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$, for all $n \geq 0$.

For $n \in \mathbf{N}$, we have

$$\begin{aligned} S_b(gx_n, gx_n, gx_{n+1}) &= S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \frac{1}{s}\psi(S_b(gx_{n-1}, gx_{n-1}, gx_n) \\ &\quad + S_b(gy_{n-1}, gy_{n-1}, gy_n)). \end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned} S_b(gy_n, gy_n, gy_{n+1}) \leq \frac{1}{s}\psi(S_b(gx_{n-1}, gx_{n-1}, gx_n) \\ + S_b(gy_{n-1}, gy_{n-1}, gy_n)). \end{aligned} \tag{4}$$

From (3) and (4), we have

$$\begin{aligned} P_n &:= S_b(gx_n, gx_n, gx_{n+1}) + S_b(gy_n, gy_n, gy_{n+1}) \\ &\leq \frac{2}{s}\psi(S_b(gx_{n-1}, gx_{n-1}, gx_n) + S_b(gy_{n-1}, gy_{n-1}, gy_n)) \\ &= \frac{2}{s}\psi(P_{n-1}) \end{aligned}$$

for all $n \in \mathbf{N}$. Thus, we get a $k$ in $(0, \frac{1}{2})$ such that

$$P_n \leq \frac{2}{s}\psi(P_{n-1}) \leq \frac{2k}{s}P_{n-1} = hP_{n-1},$$

for $h = \frac{2k}{s}$. Hence we have

$$P_n \leq hP_{n-1} \leq h^2 P_{n-2} \leq ... \leq h^n P_0.$$

For $m, n \in \mathbf{N}$ with $m > n$, we have

$$\begin{aligned} &S_b(gx_n, gx_n, gx_m) + S_b(gy_n, gy_n, gy_m) \\ &\quad \leq s[2S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_m, gx_m, gx_{n+1})] \\ &\qquad + s[2S_b(gy_n, gy_n, gy_{n+1}) + S_b(gy_m, gy_m, gy_{n+1})] \\ &\quad = 2sP_n + s[S_b(gx_{n+1}, gx_{n+1}, gx_m) + S_b(gy_{n+1}, gy_{n+1}, gy_m)] \\ &\quad \leq 2sP_n + 2s^2 P_{n+1} + ... + 2s^{m-n} P_{m-1} \\ &\quad \leq 2sh^n P_0 + 2s^2 h^{n+1} P_0 + 2s^3 h^{n+2} P_0 + ... \\ &\quad = 2sh^n P_0 (1 + sh + s^2 h^2 + ...) \\ &\quad = \frac{2sh^n P_0}{1 - sh}. \end{aligned}$$

On taking limit as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty}\{S_b(gx_n, gx_n, gx_m) + S_b(gy_n, gy_n, gy_m)\} = 0.$$

Thus, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, we get $\{gx_n\}$ and $\{gy_n\}$ are convergent to some $gx$ and $gy$ in $g(X)$ respectively.

Consider

$$\begin{aligned} S_b(gx_{n+1}, gx_{n+1}, F(x,y)) &= S_b(F(x_n,y_n), F(x_n,y_n), F(x,y)) \\ &\leq \frac{1}{s}\psi(S_b(gx_n, gx_n, gx) + S_b(gy_n, gy_n, gy)). \end{aligned}$$

On taking limit as $n \to \infty$, we obtain $S_b(gx, gx, F(x,y)) = 0$. Hence, we get $gx = F(x,y)$. Similarly, we get $gy = F(y,x)$. Thus, $(gx, gy)$ is coupled point of coincidence of $F$ and $g$.

Next, we show that $F$ and $g$ have unique coupled point of coincidence. Assume that $(gx^*, gy^*)$ is also a coupled point of coincidence of $F$ and $g$, that is, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$.

Consider

$$\begin{aligned} S_b(gx, gx, gx^*) &= S_b(F(x,y), F(x,y), F(x^*,y^*)) \\ &\leq \frac{1}{s}\psi(S_b(gx, gx, gx^*) + S_b(gy, gy, gy^*)). \end{aligned} \tag{5}$$

Similarly,

$$S_b(gy, gy, gy^*) \leq \frac{1}{s}\psi(S_b(gx, gx, gx^*) + S_b(gy, gy, gy^*)). \tag{6}$$

From (5) and (6), we have

$$\begin{aligned} S_b(gx, gx, gx^*) + S_b(gy, gy, gy^*) &\leq \frac{2}{s}\psi(S_b(gx, gx, gx^*) + S_b(gy, gy, gy^*)) \\ &\leq \frac{2k}{s}(S_b(gx, gx, gx^*) + S_b(gy, gy, gy^*)) \end{aligned}$$

As $0 < k < \frac{1}{2}$ , we get $S_b(gx, gx, gx^*) + S_b(gy, gy, gy^*) = 0$. Hence, $gx = gx^*$ and $gy = gy^*$. Thus, $F$ and $g$ have a unique coupled point of coincidence. Next, we show that $gx = gy$. For this consider

$$\begin{aligned} S_b(gx, gx, gy) &= S_b(F(x,y), F(x,y), F(y,x)) \\ &\leq \frac{1}{s}\psi(S_b(gx, gx, gy) + S_b(gy, gy, gx)) \\ &= \frac{1}{s}\psi(2S_b(gx, gx, gy)) \\ &\leq \frac{2k}{s}S_b(gx, gx, gy). \end{aligned}$$

As $0 < k < \frac{1}{2}$, we get $S_b(gx, gx, gy) = 0$. Hence, $gx = gy$. Thus, we obtain

$$F(x, y) = gx = gy = F(y, x).$$

As $F$ and $g$ are $w$-compatible, by taking $u = gx$, we get

$$gu = ggx = gF(x, y) = F(gx, gy) = F(u, u).$$

This shows that $(gu, gu)$ is a coupled point of coincidence of $F$ and $g$. As coupled point of coincidence of $F$ and $g$ is unique, we have $gu = gx = u$. Hence, $u = gu = F(u, u)$, that is, $F$ and $g$ have a unique common coupled fixed point. ■

**Corollary 1.** *Let $(X, S_b)$ be an $S_b$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$S_b(F(x, y), F(u, v), F(a, b)) \leq \frac{k}{s}(S_b(gx, gu, ga) + S_b(gy, gv, gb))$$

*for all $x, y, u, v, a, b \in X$. Assume that $F$ and $g$ satisfy the following conditions:*
*(a) $F(X \times X) \subseteq g(X)$,*
*(b) $g(X)$ is complete and*
*(c) $F$ and $g$ are $w$-compatible.*
*If $k \in (0, \frac{1}{2})$, then $F$ and $g$ have a unique common coupled fixed point, and which is of the form $(x, x)$, that is, there is a unique $x \in X$ such that $F(x, x) = gx = x$.*

**Proof.** The result follows from Theorem 1 by taking $\psi(t) = kt$. ■

**Corollary 2.** *Let $(X, S_b)$ be a complete $S_b$-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$S_b(F(x, y), F(u, v), F(a, b)) \leq \frac{1}{s}\psi(S_b(x, u, a) + S_b(y, v, b))$$

*for all $x, y, u, v, a, b \in X$. If $k \in (0, \frac{1}{2})$, then $F$ has a unique coupled fixed point, and which is of the form $(x, x)$, that is, there is a unique $x \in X$ such that $F(x, x) = x$.*

**Proof.** The result follows from Theorem 1 by taking $g = I$ (the identity mapping on $X$). ■

**Corollary 3.** *Let $(X, S_b)$ be a complete $S_b$-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$S_b(F(x,y), F(u,v), F(a,b)) \leq \frac{k}{s}(S_b(x,u,a) + S_b(y,v,b))$$

*for all $x, y, u, v, a, b \in X$. If $k \in (0, \frac{1}{2})$, then $F$ has a unique coupled fixed point, and which is of the form $(x, x)$, that is, there is a unique $x \in X$ such that $F(x, x) = x$.*

**Proof.** The result follows from Theorem 1 by taking $g = I$ (the identity mapping on $X$) and $\psi(t) = kt$. ■

In the next theorem, we consider $g$ is continuous and commutes with $F$ instead of the condition that $F$ and $g$ are $w$-compatible.

**Theorem 2.** *Let $(X, S_b)$ be a complete $S_b$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$S_b(F(x,y), F(u,v), F(a,b)) \leq \frac{1}{s^2}\psi(S_b(gx,gu,ga) + S_b(gy,gv,gb))$$

*for some $\psi \in \Psi$ and for all $x, y, u, v, a, b \in X$. Assume that $F$ and $g$ satisfy the following conditions:*
*(a) $F(X \times X) \subseteq g(X)$,*
*(b) $g(X)$ is complete, and*
*(c) $g$ is continuous and commutes with $F$.*
*Then $F$ and $g$ have a unique common coupled fixed point, and which is of the form $(x, x)$, that is, there is a unique $x \in X$ such that $F(x, x) = gx = x$.*

**Proof.** Let $x_0, y_0 \in X$ be any two element. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing this process, we can construct two sequences $\{x_n\}$, $\{y_n\}$ in $X$ such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ , for all $n \geq 0$.

For $n \in \mathbf{N}$, we have

$$\begin{aligned} (7)\quad S_b(gx_n, gx_n, gx_{n+1}) &= S_b(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \frac{1}{s^2}\psi(S_b(gx_{n-1}, gx_{n-1}, gx_n) \\ &\quad + S_b(gy_{n-1}, gy_{n-1}, gy_n)). \end{aligned}$$

Similarly,

$$\begin{aligned} S_b(gy_n, gy_n, gy_{n+1}) &\leq \frac{1}{s^2}\psi(S_b(gx_{n-1}, gx_{n-1}, gx_n) \\ (8)\qquad &\quad + S_b(gy_{n-1}, gy_{n-1}, gy_n)). \end{aligned}$$

From (7) and (8), we have

$$\begin{aligned} Q_n &:= S_b(gx_n, gx_n, gx_{n+1}) + S_b(gy_n, gy_n, gy_{n+1}) \\ &\leq \frac{2}{s^2}\psi(S_b(gx_{n-1}, gx_{n-1}, gx_n) + S_b(gy_{n-1}, gy_{n-1}, gy_n)) \\ &= \frac{2}{s^2}\psi(Q_{n-1}) \end{aligned}$$

for all $n \in \mathbf{N}$. Thus, we get a $k$ in $(0, \frac{1}{2})$ such that

$$Q_n \leq \frac{2}{s^2}\phi(Q_{n-1}) \leq \frac{2}{s}\phi(Q_{n-1}) \leq \frac{2k}{s}Q_{n-1} = pQ_{n-1},$$

for $p = \frac{2k}{s}$. Hence we have

$$Q_n \leq pQ_{n-1} \leq p^2 Q_{n-2} \leq ... \leq p^n Q_0.$$

For $m, n \in \mathbf{N}$ with $m > n$, we have

$$\begin{aligned} &S_b(gx_n, gx_n, gx_m) + S_b(gy_n, gy_n, gy_m) \\ &\quad \leq s[2S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_m, gx_m, gx_{n+1})] \\ &\qquad + s[2S_b(gy_n, gy_n, gy_{n+1}) + S_b(gy_m, gy_m, gy_{n+1})] \\ &\quad = 2sQ_n + s[S_b(gx_{n+1}, gx_{n+1}, gx_m) + S_b(gy_{n+1}, gy_{n+1}, gy_m)] \\ &\quad \leq 2sQ_n + 2s^2 Q_{n+1} + ... + 2s^{m-n} Q_{m-1} \\ &\quad \leq 2sp^n Q_0 + 2s^2 p^{n+1} Q_0 + 2s^3 p^{n+2} Q_0 + ... \\ &\quad = 2sp^n Q_0 (1 + sp + s^2 p^2 + ...) \\ &\quad = \frac{2sp^n Q_0}{1 - sp}. \end{aligned}$$

On taking limit as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \{S_b(gx_n, gx_n, gx_m) + S_b(gy_n, gy_n, gy_m)\} = 0.$$

Thus, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, $\{gx_n\}$ and $\{gy_n\}$ are convergent to some $x$ and $y$ in $X$ respectively. Since $g$ is continuous, $\{ggx_n\}$ and $\{ggy_n\}$ are convergent to $gx$ and $gy$ respectively. Also, since $g$ and $F$ commute, we have

$$ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n).$$

Thus

$$\begin{aligned} S_b(F(x,y),F(x,y),ggx_{n+1}) &= S_b(F(x,y),F(x,y),F(gx_n,gy_n)) \\ &\leq \frac{1}{s^2}\psi(S_b(gx,gx,ggx_n)+S_b(gy,gy,ggy_n). \end{aligned}$$

Applying the Lemma 1, we have

$$\begin{aligned} \frac{1}{s}S_b(F(x,y),F(x,y),gx) &\leq \limsup_{n\to\infty} S_b(F(x,y),F(x,y),F(gx_n,gy_n)) \\ &\leq \limsup_{n\to\infty} \frac{1}{s^2}\psi(S_b(gx,gx,ggx_n)+S_b(gy,gy,ggy_n)) \\ &\leq \frac{1}{s^2}\psi(s(S_b(gx,gx,gx)+S_b(gy,gy,gy)))=0. \end{aligned}$$

Thus, $S_b(F(x,y),F(x,y),gx)=0$. Hence, $gx=F(x,y)$. Similarly, we get $gy=F(y,x)$. Thus $(x,y)$ is a coupled coincidence point of $F$ and $g$.

Now

$$\begin{aligned} S_b(gx,gx,gy) &= S_b(F(x,y),F(x,y),F(y,x)) \\ &\leq \frac{1}{s^2}\psi(S_b(gx,gx,gy)+S_b(gy,gy,gx)) \\ &= \frac{1}{s^2}\psi(2S_b(gx,gx,gy)) \\ &\leq \frac{2k}{s^2}S_b(gx,gx,gy). \end{aligned}$$

As $0<k<\frac{1}{2}$, we get $S_b(gx,gx,gy)=0$, which implies that $gx=gy$. Thus we obtain

$$F(x,y)=gx=gy=F(y,x).$$

Again, using the Lemma 1, we have

$$\begin{aligned} \frac{1}{s}S_b(gx,gx,x) &\leq \limsup_{n\to\infty} S_b(gx,gx,gx_{n+1}) \\ &= \limsup_{n\to\infty} S_b(F(x,y),F(x,y),F(x_n,y_n)) \\ &\leq \limsup_{n\to\infty} \frac{1}{s^2}\psi(S_b(gx,gx,gx_n)+S_b(gy,gy,gy_n)) \\ &\leq \frac{1}{s^2}\psi(s(S_b(gx,gx,x)+S_b(gy,gy,y))). \end{aligned}$$

which implies that

$$S_b(gx,gx,x)\leq \frac{1}{s}\psi(s(S_b(gx,gx,x)+S_b(gy,gy,y))). \tag{9}$$

Similarly,

$$S_b(gy,gy,y) \le \frac{1}{s}\psi(s(S_b(gx,gx,x)+S_b(gy,gy,y))). \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} S_b(gx,gx,x)+S_b(gy,gy,y) &\le \frac{2}{s}\psi(s(S_b(gx,gx,x)+S_b(gy,gy,y))) \\ &\le 2k(S_b(gx,gx,x)+S_b(gy,gy,y)). \end{aligned}$$

As $0<k<\frac{1}{2}$, we get $S_b(gx,gx,x)+S_b(gy,gy,y)=0$. Thus we obtain $x=gx$ and $y=gy$.

Hence we have

$$x = gx = F(x,x).$$

To prove uniqueness, let $x^* \in X$ with $x^* \neq x$ such that

$$x^* = gx^* = F(x^*,x^*).$$

Consider

$$\begin{aligned} S_b(x,x,x^*) &= S_b(F(x,x),F(x,x),F(x^*,x^*)) \\ &\le \frac{1}{s^2}\psi(S_b(gx,gx,gx^*)+S_b(gx,gx,gx^*)) \\ &= \frac{1}{s^2}\psi(2S_b(x,x,x^*)) \\ &\le \frac{2k}{s^2}S_b(x,x,x^*). \end{aligned}$$

As $0<k<\frac{1}{2}$, we get $S_b(x,x,x^*)=0$. Thus we obtain $x=x^*$. Hence, $F$ and $g$ have a unique common coupled fixed point. ■

**Corollary 4.** *Let $(X,S_b)$ be a complete $S_b$-metric space. Let $F: X\times X \to X$ and $g: X\to X$ be two mappings such that*

$$S_b(F(x,y),F(u,v),F(a,b)) \le \frac{k}{s^2}(S_b(gx,gu,ga)+S_b(gy,gv,gb))$$

*for all $x,y,u,v,a,b\in X$. Assume that $F$ and $g$ satisfy the following conditions:*

*(a) $F(X\times X)\subseteq g(X)$,*

*(b) $g(X)$ is complete and*

*(c) $g$ is continuous and commutes with $F$.*

*If $k\in(0,\frac{1}{2})$, then $F$ and $g$ have a unique common coupled fixed point, and which is of the form $(x,x)$, that is, there is a unique $x\in X$ such that $F(x,x)=gx=x$.*

**Proof.** The result follows from Theorem 2 by taking $\psi(t) = kt$. ■

**Corollary 5.** *Let $(X, S_b)$ be a complete $S_b$-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$S_b(F(x,y), F(u,v), F(a,b)) \leq \frac{1}{s^2}\psi(S_b(x,u,a) + S_b(y,v,b))$$

*for all $x, y, u, v, a, b \in X$. If $k \in (0, \frac{1}{2})$, then $F$ has a unique coupled fixed point, and which is of the form $(x, x)$, that is, there is a unique $x \in X$ such that $F(x, x) = x$.*

**Proof.** The result follows from Theorem 2 by taking $g = I$ (the identity mapping on $X$). ■

**Corollary 6.** *Let $(X, S_b)$ be a complete $S_b$-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$-S_b(F(x,y), F(u,v), F(a,b)) \leq \frac{k}{s^2}(S_b(x,u,a) + S_b(y,v,b))$$

*for all $x, y, u, v, a, b \in X$. If $k \in (0, \frac{1}{2})$, then $F$ has a unique coupled fixed point, and which is of the form $(x, x)$, that is, there is a unique $x \in X$ such that $F(x, x) = x$.*

**Proof.** The result follows from Theorem 2 by taking $g = I$ (the identity mapping on $X$) and $\psi(t) = kt$. ■

**Example 1.** Let $X = [0, \infty)$. Define $S_b : X^3 \to \mathbf{R}^+$ by $S_b(x,y,z) = (|x-z| + |y-z|)^2$. Clearly, $S_b(x,x,y) = S_b(y,y,x)$, for all $x, y \in X$. Thus, $(X, S_b)$ is an $S_b$-metric space with $s = 4$. Define $F(x,y) = \frac{x}{128} + \frac{y}{128}$, $g(x) = \frac{x}{4}$ and $\psi(t) = \frac{t}{8}$. Next we show that $F$ and $g$ are $w$-compatible. In fact

$$\begin{cases} F(x,y) = gx; \\ F(y,x) = gy, \end{cases} \Leftrightarrow x = y = 0.$$

Thus, $(g0, g0)$ is the unique coupled point of coincidence of the mappings $F$ and $g$. Obviously, $F(g0, g0) = g(F(0,0)) = 0$, therefore $F$ and $g$ are $w$-compatible.

Now

$$\begin{aligned} S_b(F(x,y), F(x,y), F(y,x)) &= S_b\left(\frac{x}{128} + \frac{y}{128}, \frac{u}{128} + \frac{v}{128}, \frac{a}{128} + \frac{b}{128}\right) \\ &= \left(\left|\frac{x}{128} + \frac{y}{128} - \frac{a}{128} - \frac{b}{128}\right| + \left|\frac{u}{128} + \frac{v}{128} - \frac{a}{128} - \frac{b}{128}\right|\right)^2 \\ &\leq \left(\left|\frac{x}{128} - \frac{a}{128}\right| + \left|\frac{y}{128} - \frac{b}{128}\right| + \left|\frac{u}{128} - \frac{a}{128}\right| + \left|\frac{v}{128} - \frac{b}{128}\right|\right)^2 \end{aligned}$$

$$\leq \frac{1}{32^2}\left[\left(\left|\frac{x}{4}-\frac{a}{4}\right|+\left|\frac{u}{4}-\frac{a}{4}\right|\right)+\left(\left|\frac{y}{4}-\frac{b}{4}\right|+\left|\frac{v}{4}-\frac{b}{4}\right|\right)\right]^2$$
$$\leq \frac{2}{32^2}\left[\left(\left|\frac{x}{4}-\frac{a}{4}\right|+\left|\frac{u}{4}-\frac{a}{4}\right|\right)^2+\left(\left|\frac{y}{4}-\frac{b}{4}\right|+\left|\frac{v}{4}-\frac{b}{4}\right|\right)^2\right]$$
$$= \frac{2}{32^2}[(|gx-ga|+|gu-ga|)^2+(|gy-gb|+|gv-gb|)^2]$$
$$\leq \frac{1}{4}\frac{(S_b(gx,gu,ga)+S_b(gy,gv,gb))}{8}$$
$$= \frac{1}{4}\psi(S_b(gx,gu,ga)+S_b(gy,gv,gb))$$

Thus $F$ and $g$ satisfies all the conditions of Theorem 1. Hence, $F$ and $g$ have a unique common coupled fixed point. In fact, $F(0,0)=g(0)=0$.

**Example 2.** Let $X=[0,\infty)$. Define $S_b : X^3 \to \mathbf{R}^+$ by $S_b(x,y,z) = (|x-z|+|y-z|)^2$. Clearly, $S_b(x,x,y)=S_b(y,y,x)$, for all $x,y\in X$. Thus, $(X,S_b)$ is a complete $S_b$-metric space with $s=4$. Define $F(x,y)=\frac{x}{64}+\frac{y}{64}$, $g(x)=\frac{x}{4}$ and $\psi(t)=\frac{t}{3}$.

Now

$$S_b(F(x,y),F(x,y),F(y,x)) = S_b\left(\frac{x}{64}+\frac{y}{64},\frac{u}{64}+\frac{v}{64},\frac{a}{64}+\frac{b}{64}\right)$$
$$= \left(\left|\frac{x}{64}+\frac{y}{64}-\frac{a}{64}-\frac{b}{64}\right|+\left|\frac{u}{64}+\frac{v}{64}-\frac{a}{64}-\frac{b}{64}\right|\right)^2$$
$$\leq \left(\left|\frac{x}{64}-\frac{a}{64}\right|+\left|\frac{y}{64}-\frac{b}{64}\right|+\left|\frac{u}{64}-\frac{a}{64}\right|+\left|\frac{v}{64}-\frac{b}{64}\right|\right)^2$$
$$\leq \frac{1}{16^2}\left[\left(\left|\frac{x}{4}-\frac{a}{4}\right|+\left|\frac{u}{4}-\frac{a}{4}\right|\right)+\left(\left|\frac{y}{4}-\frac{b}{4}\right|+\left|\frac{v}{4}-\frac{b}{4}\right|\right)\right]^2$$
$$\leq \frac{2}{16^2}\left[\left(\left|\frac{x}{4}-\frac{a}{4}\right|+\left|\frac{u}{4}-\frac{a}{4}\right|\right)^2+\left(\left|\frac{y}{4}-\frac{b}{4}\right|+\left|\frac{v}{4}-\frac{b}{4}\right|\right)^2\right]$$
$$= \frac{2}{16^2}[(|gx-ga|+|gu-ga|)^2+(|gy-gb|+|gv-gb|)^2]$$
$$\leq \frac{1}{4^2}\frac{(S_b(gx,gu,ga)+S_b(gy,gv,gb))}{3}$$
$$= \frac{1}{4^2}\psi(S_b(gx,gu,ga)+S_b(gy,gv,gb))$$

Thus $F$ and $g$ satisfies all the conditions of Theorem 2. Hence, $F$ and $g$ have a unique common coupled fixed point. In fact, $F(0,0)=g(0)=0$.

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## References


[1] Abbas M., Khan M.A., Radenović S., Common coupled fixed point theorems in cone metric spaces for $w$-compatible mappings, *Appl. Math. Comput.,* 217(2010), 195-202.

[2] Aghajani A., Abbas M., Roshan J.R., Common fixed point of generalized weak contractive mappings in partially ordered $G_b$-metric spaces, *Filomat,* 28(6)(2014), 1087-1101.

[3] Dhage B.C., Generalized metric spaces mappings with fixed point, *Bull. Calcutta Math. Soc.,* 84(1992), 329-336.

[4] Gähler S., 2-metrische Räume und iher topoloische Struktur, *Math. Nachr.,* 26(1963), 115-148.

[5] Guo D., Lakshmikantham V., Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal.,* 11(1987), 623-632.

[6] Lakshmikantham V., Ciric L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.,* 70(2009), 4341-4349.

[7] Liu L., Gu F., Some common coupled fixed point results in two $S$-metric spaces and applications to integral equations, *J. Nonlinear Sci. Appl.,* 9(2016), 3527-3544.

[8] Mustafa Z., Sims B., A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.,* 7(2006), 289-297.

[9] Sedghi S., Shobe N., Zhou H., A common fixed point theorem in $D^*$-metric spaces, *Fixed Point Theory Appl.,* Vol. 2007, Article ID 27906, 13 pages.

[10] Sedghi Sh., Shobe N, Aliouche A., A generalization of fixed point theorems in $S$-metric spaces, *Mat. Vesnik,* 64(3)(2012), 258-266.

[11] Sedghi Sh., Dung N.V., Fixed point theorems on S-metric spaces, *Mat. Vesnik,* 66(1)(2014), 113-124.

[12] Souayah N., Mlaiki N., A fixed point theorem in $S_b$-metric spaces, *J. Math. Computer Sci.,* 16(2016), 131-139.

[13] Sedghi Sh., Gholidahneh A., Dosenović T., Esfahani J., Radenović S., Common fixed point of four maps in $S_b$-metric spaces, *JLTA,* 5(2)(2016), 93-104.

[14] Rohen Y., Dosenović T., Radenović S., A note on the paper "A fixed point theorems in $S_b$-metric spaces by Nizar Souayah and Nabil Malaiki, *Filomat,* 31(11)(2017), 3335-3346.



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