sciendo FASCICULI MATHEMATICI

Nr 60 2018

DOI:10.1515/fascmath-2018-0007

D. A. Mojdeh and A. S. Emadi

CONNECTED DOMINATION POLYNOMIAL OF GRAPHS

Abstract. Let G be a simple graph of order n. The connected domination polynomial of G is the polynomial $D_c(G,x) = \sum_{i=\gamma_c(G)}^{|V(G)|} d_c(G,i)x^i$, where $d_c(G,i)$ is the number of connected dominating sets of G of size i and $\gamma_c(G)$ is the connected domination number of G. In this paper we study $D_c(G,x)$ of any graph. We classify many families of graphs by studying their connected domination polynomial.

Key words: connected domination polynomial, graphs, connected dominating set, Petersen graphs.

AMS Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V,E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ-set. The family of all γ-sets of a graph G is denoted by $\Gamma(G)$. A connected dominating set S is a dominating set whose induced subgraph $\langle S \rangle$ is connected. Since a dominating set must contain at least one vertex from each component of G, it follows that only connected graphs have a connected dominating set. The minimum cardinality of a connected dominating set is the connected domination number $\gamma_c(G)$. The family of all γ_c-sets of a graph G is denoted by $\Gamma_c(G)$. Obviously, $\gamma(G) \leq \gamma_c(G)$, for more information of these parameters, refer to [7].

The i-subset of $V(G)$ is a subset of $V(G)$ of size i. Let $D(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i) = |D(G,i)|$. The domination polynomial $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G.

We denote the family of all dominating sets of G with cardinality i containing a vertex v by $D_v(G,i)$, and $d_v(G,i) = |D_v(G,i)|$ (see [2], [3], [4]).

Definition 1. *Let $D_c(G,i)$ be the family of connected dominating sets of cardinality i in G and let $d_c(G,i) = |D_c(G,i)|$. The connected domination polynomial $D_c(G,x)$ of G is defined as $D_c(G,x) = \sum_{i=\gamma_c(G)}^{|V(G)|} d_c(G,x)x^i$, where $\gamma_c(G)$ is the connected domination number of G. We denote the family of all connected dominating sets of G with cardinality i containing a vertex v by $D_{cv}(G,i)$, and $d_{cv}(G,i) = |D_{cv}(G,i)|$ and the disconnected dominating sets of G containing a vertex v by $\overline{D}_{cv}(G,i)$.*

The minimum degree of G is denoted by $\delta(G)$ and the maximum degree of G is denoted by $\Delta(G)$. A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Like wise, a cycle on three or more vertices is a simple graph whose vertices can be arranged in a cycle sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. The length of a path or a cycle is the number of its edges. A path or cycle of length k is called k-path or k-cycle.

A vertex v of G is a cut vertex if $G - v$, the subgraph of G induced by $V(G) - v$ has components more than of G. A subset S of $V(G)$ is a vertex cut of G if $G - S$, the subgraph of G induced by $V(G) - S$ has components more than of G. If S has k vertices we denote the set S by k-vertex cut. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. The complement G^c of a simple graph G is the simple graph with vertex set V, such that two vertices are adjacent in G^c if and only if they are not adjacent in G.

The corona of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex $v^{'}$ and a pendant edge $vv^{'}$ are added.

A vertex-transitive graph is a graph G such that for every pair of vertices v and w of G, there exists an automorphism θ such that $\theta(v) = w$.

The notion of generalized Petersen graph is introduced as follows: given integers $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$, the graph $GP(n,k)$ is defined on the set

$\{x_i, y_i \,|i \in \mathbb{Z}_n\}$ of $2n$ vertices, with the adjacencies given by x_ix_{i+1}, x_iy_i, y_iy_{i+k} for all i. In this notation, the Petersen graph is $GP(5, 2)$, we denote the Petersen graph by $P = P(5, 2)$.

$GP(n, k)$ is vertex-transitive if and only if $k^2 \equiv +1, -1 \pmod{n}$ or $(n, k) = (10, 2)$. For more information refer to [5], [6]. We use the notation $\binom{n}{r} = \frac{p(n,r)}{r!} = \frac{n!}{r! \times (n-r)!}$.

Recently, it has been studied of the domination polynomials of cubic graphs of order 10 and it has been obtained the domination polynomial of Petersen graph [3].

In this paper, we study the connected dominating sets and connected domination polynomial of any graphs in particular we classify the connected domination polynomial of paths, cycles, corona of a graph and Petersen graph $P(5, 2)$ and generalized Petersen graph $GP(6, 1)$.

2. The connected domination polynomial of graph

In this section, we study the connected dominating sets and connected domination polynomial of any graph. In particular paths, cycles, corona of a graph.

Observation 1. *For any connected graph G of order n we have.*
(i) $d_c(G, n) = 1$.
(ii) $d_c(G, n-1) = n - k$, *where k is the number of cut vertex of G.*
(iii) *If G has some vertices of degree $n-1$, then* $d_c(G, 1) = |\{v \in V(G) \mid d(v) = n-1\}|$.

Proof. (i) and (iii) are clear.

(ii) If a vertex v is a cut vertex of G, then the induced subgraph $\langle V(G) - \{v\}\rangle$ is not connected. Therefore $\langle V(G) - \{v\}\rangle$ is not a connected dominating set of G. This implies that, any vertex set of size $(n-1)$ is a connected dominating set if and only if the nth vertex is not a cut vertex. Thus $d_c(G, n-1) = n - k$ where k is the number of cut vertices of G. ■

Lemma 1. *Let G be a connected graph of order n. If G has k cut vertices and r 2-vertex cut sets, then* $d_c(G, n-2) = \binom{n-k}{2} - r$.

Proof. Let $B = \{\{w_1, u_1\}, \{w_2, u_2\}, \ldots, \{w_r, u_r\}\}$ be all 2-vertex cut of G and $C = \{v_1, v_2, \ldots, v_k\}$ be the cut vertex-set of G. Then any $n-2$-connected dominating set of G should contains C and B. Therefore $D_c = V(G) \setminus \{v, w\}$ is a connected dominating set if and only if $\{v, w\} \cap C = \emptyset$ and $\{v, w\} \notin B$. $d_c(G, n-2) = \binom{n-k}{2} - r$. ■

Corollary 1. *If G a graph of order $2n$ and $D_c(G, x) = x^n(x+1)^n$, then G has n cut vertices.*

Proof. By Observation 1, part (ii) and the coefficient of x^{2n-1} in the polynomial $x^n(1+x)^n$ which is $\binom{n}{1}$, the number of cut vertices,

$$k = 2n - d_c(G, 2n-1) = 2n - \binom{n}{1} = n$$

■

It is well known that the minimum size of connected dominating set of a path P_n and a cycle C_n is $n-2$. Hence we have;

Corollary 2. (i) *For every* $n \geq 2,\ d_c(P_n, n-1) = 2.$
(ii) *For every* $n \geq 3,\ d_c(P_n, n-2) = 1.$
(iii) *For every* $n \geq 3,\ D_c(P_n, x) = x^n + 2x^{n-1} + x^{n-2}.$

Corollary 3. *For every* $n \geq 3$
(i) $d_c(C_n, n-1) = n.$
(ii) $d_c(C_n, n-2) = n.$
(iii) $D_c(C_n, x) = x^n + nx^{n-1} + nx^{n-2}.$

Let G be a connected graph with n vertex set and $G \circ K_1$ be the corona of G. It is easy to see that $\gamma_c(G \circ K_1) = n$. Therefore $d_c(G \circ K_1, m) = 0$, for every $m < n$. So we shall compute $d_c(G \circ K_1, m)$ for each $m,\ n \leq m \leq 2n$.

Theorem 1. *For any connected graph G of order n and every $m,\ n \leq m \leq 2n$, we have $d_c(G \circ K_1, m) = \binom{n}{m-n}$. Hence*

$$D_c(G \circ K_1, x) = x^n(1+x)^n.$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G and $\{u_1, u_2, \cdots, u_n\}$ be the n pendant vertices of $G \circ K_1$. Suppose that D is a connected dominating set of $G \circ K_1$ of size m, then $|D \cap V(G)| = n$ and $D \cap V(G) = \{v_1, \cdots, v_n\}$. Therefore D contains $m-n$ vertices from the set $\{u_1, \ldots, u_n\}$. Without loss of generality $D \cap \{u_1, u_2, \cdots, u_n\} = \{u_{n+1}, \ldots, u_m\}$. Thus $D = \{v_1, \cdots, v_n, u_{n+1}, \cdots, u_m\}$. We have $\binom{n}{m-n}$ possibilities for finding D. Therefore $d_c(G \circ K_1, m) = \binom{n}{m-n}$ and $D_c(G \circ K_1, x) = x^n(1+x)^n$. ■

Since for finding the connected dominating set of $G \circ H$ for a connected graph G, we should choose all vertices of G, then using the method of the proof of Theorem 1 we have.

Theorem 2. *For any connected graph G of order n and any graph H of order m, k, $n \leq k \leq n(m+1)$, we have $d_c(G \circ H, k) = \binom{nm}{k-n}$ and $D_c(G \circ H, x) = x^n(1+x)^{mn}$.*

In particular we have.

Theorem 3. *For any connected graph G of order n, and every k, $n \leq k \leq n(m+1)$, we have $d_c(G \circ K_m, k) = \binom{nm}{k-n}$ and $D_c(G \circ K_m, x) = x^n(1+x)^{mn}$.*

Now Theorem 2 implies that.

Corollary 4. *Let G be a connected graph of order n. Then* $D_c(G \circ \overline{K}_m, x) = D_c(G \circ K_m, x)$.

Corollary 5. *For any connected graph G of order n and any graph H with m vertices, we have* $D_c(G \circ \overline{H}, x) = D_c(G \circ H, x)$.

Theorem 4. *For a complete bipartite graph $K_{m,n}$ $(2 \leq n < m)$, we have;*

$$D_c(K_{m,n}, x) = ((1+x)^n - 1)\,((1+x)^m - 1)\,.$$

Proof. Suppose D is a connected dominating set with cardinality i, since D is a connected set, we have $1 < i \leq n+m$ and

$$d_c(K_{n,m}, i) = \begin{cases} \binom{m+n}{i} & i > m \\ \binom{m+n}{i} - \binom{m}{i} - \binom{n}{i} & 2 \leq i \leq n \\ \binom{m+n}{i} - \binom{m}{i} & n < i \leq m. \end{cases}$$

Since coefficient x^i in the connected domination polynomial is the same as $d_c(K_{m,n}, i)$, therefore $D_c(K_{m,n}, x) = ((1+x)^n - 1)\,((1+x)^m - 1)$. ■

3. Connected domination polynomial of the Petersen graph $P(5,2)$

In this section we shall investigate the connected domination polynomial of the Petersen graph. First, we need the following results.

Lemma 2 ([3], Lemma 1)**.** *Let G be a vertex transitive graph of order n and $v \in V(G)$. For any $1 \leq i \leq n$, $d(G,i) = \frac{n}{i} d_v(G,i)$.*

Lemma 3 ([1], Lemma 4)**.** *Let G be a graph of order n with domination polynomial $D(G,x) = \sum_{i=1}^{n} d(G,i)x^i$. If $d(G,x) = \binom{n}{j}$ for some j, then $\delta(G) \geq n-j$, more precisely, $\delta(G) = n-l$ where $l = \min\{j | d(G,j) = \binom{n}{j}\}$, and there are at least $\binom{n}{l-1} - d(G,l-1)$ vertices of degree $\delta(G)$ in G.*

Furthermore, if for every two vertices of degree $\delta(G)$, say u and v we have $N[u] \neq N[v]$, then there are exactly $\binom{n}{l-1} - d(G,l-1)$ vertices of degree $\delta(G)$.

It can be easily investigated that the connected domination number of the Petersen graph is 4 and in the following we show that $d_c(P,4) = 10$.

The four connected dominating sets are listed below by Figure 1.

$$\begin{aligned} D_c(P,4) \;=\; & \{\{1,2,5,7\},\{1,2,3,8\},\{1,4,5,6\},\{1,7,9,10\},\{2,8,6,10\}, \\ & \{2,3,4,9\},\{3,9,7,6\},\{3,4,5,10\},\{4,10,7,8\},\{5,6,8,9\}\}. \end{aligned}$$

Note that we denote vertices a_1, a_2, a_3, a_4, a_5 for outer 5-cycle and vertices b_1, b_2, b_3, b_4, b_5 for inner 5-cycle.

Lemma 4. *For the Petersen graph P, $d_c(P, 5) = 72$.*

Proof. First, we list all connected dominating sets of P of cardinality 5 containing one vertex, say the vertex labeled 1, which are the γ_c sets of the labeled Petersen graph given in Figure 1.

To determine $d_{c_1}(P, 5)$, we consider 5 cases.

Case 1. We select 5 vertices on the outer 5-cycle. This case is possible in 1 way.

Case 2. We select 4 vertices on the outer 5-cycle and 1 vertex on the inner 5-cycle. If 4 vertices are selected on the outer 5-cycle, then 1 vertex on the inner 5-cycle is not dominated. Since 5 selected vertices are connected, selection 1 vertex of the inner 5-cycle is possible in 2 ways. If the vertex a_i is not selected, then the vertex b_i is not dominated, so there exist two choose domination b_i, we add the vertex b_{i-2} or b_{i-3}.

The number of selection 4 vertices on the outer 5-cycle containing $a_1 = 1$ is 4, therefore this case is possible in 8 ways.

Case 3. We select 3 vertices on the outer 5-cycle and 2 vertices on the inner 5-cycle.

Since each of 5 selected vertices is connected, 3 selected vertices on the outer 5-cycle should be adjacent. So 2 vertices on the inner 5-cycle are not dominated and 2 vertices are selected of the inner 5-cycle such that all vertices are dominated. It is possible in 5 ways, for example, if a_1, a_2, a_3 are be selected, then b_4, b_5 are not dominated. We can select b_1, b_2; b_5, b_2; b_4, b_2; b_3, b_2 or b_1, b_3.

The number of selection 3 vertices on the outer 5-cycle containing $a_1 = 1$ is 3, therefore this case is possible in 15 ways.

Case 4. We select 2 vertices on the outer 5-cycle and 3 vertices on the inner 5-cycle.

If 2 vertices are adjacent, for example a_1, a_2, then a_4 is not dominated. So b_4 should be selected. In this case b_3, b_5 are not dominated, we can dominate them in 3 ways, b_2, b_5; b_1, b_2 or b_1, b_3. If 2 vertices are not adjacent, for example a_1, a_3, then b_2, b_4, b_5 are not dominated. Since 5 selected vertices should be connected, two vertices b_1, b_3 should be selected such that b_4, b_5 are dominated. By selecting b_4 or b_5, vertex b_2 is dominated.

Therefore in this case the number of connected dominating set with 5 vertices containing $a_1 = 1$, is 10.

Case 5. We select 1 vertex on the outer 5-cycle and 4 vertices on the inner 5-cycle.

By selecting vertex $a_1 = 1$, vertices a_3, a_4 are not dominated. So we should select b_1, b_3, b_4 and another vertex is b_2 or b_5. In this case the number of 5 selected vertices is 2.

Therefore $d_{c_1}(P,5) = 36$. These connected dominating sets are listed below.

$$\begin{aligned} D_{c_1}(P,5) = \; & \{\{1,2,3,4,5\},\{1,2,3,4,8\},\{1,2,3,4,9\}, \\ & \{1,2,4,5,6\},\{1,2,4,5,7\},\{1,2,3,5,7\},\{1,2,3,5,8\}, \\ & \{1,3,4,5,6\},\{1,3,4,5,10\},\{1,2,3,8,7\},\{1,2,3,8,6\}, \\ & \{1,2,3,8,10\},\{1,2,3,8,9\},\{1,2,3,7,9\},\{1,2,5,7,6\}, \\ & \{1,2,5,7,8\},\{1,2,5,7,9\},\{1,2,5,7,10\},\{1,2,5,6,8\}, \\ & \{1,4,5,6,7\},\{1,4,5,6,8\},\{1,4,5,6,9\},\{1,4,5,6,10\}, \\ & \{1,4,5,7,10\},\{1,2,10,8,6\},\{1,2,10,7,8\},\{1,2,10,7,9\}, \\ & \{1,3,7,9,6\},\{1,3,7,9,10\},\{1,4,7,10,8\},\{1,4,7,10,9\}, \\ & \{1,5,9,7,10\},\{1,5,9,7,6\},\{1,5,9,6,8\},\{1,7,10,9,8\}, \\ & \{1,7,10,9,6\}\}. \end{aligned}$$

Since the Petersen graph is vertex-transitive graph, therefore the number of these connected dominating sets is $36 \times 10 = 360$. By Lemma 2 and note that the above five dominating sets are connected, $d_c(P,5) = \frac{360}{5} = 72$. ■

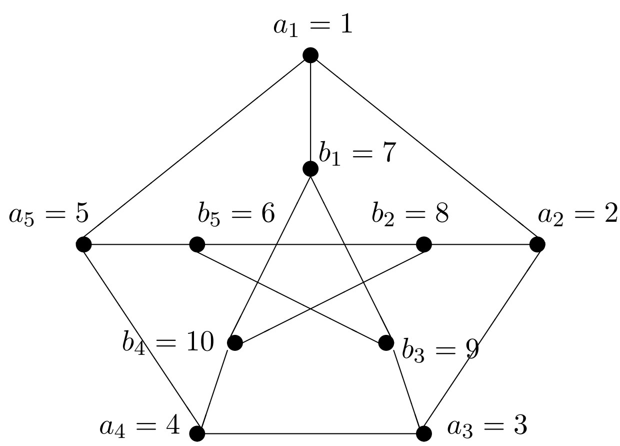

Figure 1. *Petersen graph with label*

Lemma 5. *For the Petersen graph P, $d_c(P,6) = 135$.*

Proof. To determine $d_{c_1}(P,6)$, we consider 5 cases.

Case 1. We select 5 vertices on the outer 5-cycle and 1 vertex on the inner 5-cycle. Assume that 5 vertices on the outer 5-cycle were be selected, then we can select one vertex on the inner 5-cycle in 5 ways. So selection 5 vertices on the outer 5-cycle containing $a_1 = 1$ is possible in 5 ways.

Case 2. We select 4 vertices on the outer 5-cycle and 2 vertices on the inner 5-cycle. Assume that 4 vertices were be selected on the outer 5-cycle for example, a_1, a_2, a_3, a_4 then b_5 is not dominated. Since 6 selected vertices should be connected, b_2 or b_3 should be selected and we can select another vertex among remained vertices. It is possible in 7 ways.

So, if 4 vertices are selected on the outer 5-cycle containing $a_1 = 1$, then this case is possible in 28 ways.

Case 3. We select 3 vertices on the outer 5-cycle and 3 vertices on the inner 5-cycle. If 3 vertices on the outer 5-cycle are adjacent, for example a_1, a_2, a_3, then b_4, b_5 are not dominated. So we consider 2 cases for selecting 3 vertices on the inner 5-cycle.

(i) By selecting vertex b_2, then b_4, b_5 are dominated and two another vertex are selected so that induced the subgraph with 6 vertices are connected. It is possible in $\binom{4}{2} = 6$ ways.

(ii) By selecting the vertices b_1 and b_3, then b_4 and b_5 are dominated and another vertex can be b_4 or b_5, therefore this is possible with 2 ways.

If 3 vertices of the outer 5-cycle are not formed path. Like vertices a_1, a_2, a_4, then b_3, b_5 are not dominated, b_4 should be selected and two other vertices are b_1, b_2; b_3, b_1 or b_2, b_5.

Therefore the number of selection of 3 vertices containing $a_1 = 1$ is 33.

Case 4. We select 2 vertices on the outer 5-cycle and 4 vertices on the inner 5-cycle. If 2 vertices on the outer 5-cycle are adjacent, for example, a_1 and a_2; the vertex a_4 is not dominated. So the vertex b_4 should be selected and 3 other vertices can select of among 4 remained vertices in $\binom{4}{3} = 4$ ways.

If 2 vertices on the outer 5-cycle are not adjacent, for example by selecting a_1, a_3; then vertices b_1 and b_3 should be selected and 3 other vertices can be selected among 3 remained vertices in $\binom{3}{2} = 3$ ways.

So the number of selection 2 vertices containing $a_1 = 1$ is possible by 14 ways.

Case 5. We select 1 vertex on the outer 5-cycle and 5 vertices on the inner 5-cycle. For every vertex on the outer 5-cycle, we can select 5 vertices on the inner 5-cycle in 1 way. So for vertex $a_1 = 1$ is possible in 1 way.

Therefore $d_{c_1}(P, 6) = 81$, these dominating sets are listed below.

$$\begin{aligned} D_{c_1}(P,6) \;=\; & \{\{1,2,3,4,5,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,8\},\{1,2,3,4,5,9\},\\ & \{1,2,3,4,5,10\},\{1,2,3,4,8,7\},\{1,2,3,4,8,6\},\{1,2,3,4,8,10\},\\ & \{1,2,3,4,8,9\},\{1,2,3,4,9,7\},\{1,2,3,4,9,6\},\{1,2,3,4,9,10\}, \end{aligned}$$

$\{1,2,4,5,7,6\}, \{1,2,4,5,7,8\}, \{1,2,4,5,7,9\}, \{1,2,4,5,7,10\},$
$\{1,2,4,5,6,8\}, \{1,2,4,5,6,9\}, \{1,2,4,5,6,10\}, \{1,3,4,5,6,8\},$
$\{1,3,4,5,6,7\}, \{1,3,4,5,6,9\}, \{1,3,4,5,6,10\}, \{1,3,4,5,10,8\},$
$\{1,3,4,5,10,7\}, \{1,3,4,5,10,9\}, \{1,2,3,5,7,6\}, \{1,2,3,5,7,10\},$
$\{1,2,3,5,7,8\}, \{1,2,3,5,7,9\}, \{1,2,3,5,8,6\}, \{1,2,3,5,8,10\},$
$\{1,2,3,5,8,9\}, \{1,2,3,8,6,9\}, \{1,2,3,8,6,7\}, \{1,2,3,8,7,9\},$
$\{1,2,3,7,9,6\}, \{1,2,3,8,10,7\}, \{1,2,3,8,10,9\}, \{1,2,3,8,6,10\},$
$\{1,2,3,7,9,10\}, \{1,2,4,7,10,8\}, \{1,2,4,7,10,9\}, \{1,2,4,8,10,6\},$
$\{1,2,5,7,6,8\}, \{1,2,5,7,8,9\}, \{1,2,5,7,8,10\}, \{1,2,5,7,9,10\},$
$\{1,2,5,7,9,6\}, \{1,2,5,7,10,6\}, \{1,2,5,6,8,10\}, \{1,2,5,6,8,9\},$
$\{1,3,4,9,7,10\}, \{1,3,4,7,10,8\}, \{1,3,4,7,9,6\}, \{1,3,5,9,7,10\},$
$\{1,3,5,9,7,6\}, \{1,3,5,9,6,8\}, \{1,4,5,6,7,8\}, \{1,4,5,6,7,9\},$
$\{1,4,5,6,7,10\}, \{1,4,5,6,8,9\}, \{1,4,5,6,8,10\}, \{1,4,5,6,9,10\},$
$\{1,4,5,7,10,8\}, \{1,4,5,7,10,9\}, \{1,2,10,7,8,6\}, \{1,2,10,7,9,6\},$
$\{1,2,7,10,8,9\}, \{1,2,8,10,6,9\}, \{1,3,7,9,10,8\}, \{1,3,7,9,10,6\},$
$\{1,3,7,9,8,6\}, \{1,4,7,10,8,6\}, \{1,4,7,10,8,9\}, \{1,4,7,10,9,6\},$
$\{1,5,9,7,6,8\}, \{1,5,9,7,6,10\}, \{1,5,9,6,8,10\}, \{1,5,9,8,10,7\},$
$\{1,7,10,8,6,9\}\}.$

By Lemma 2 and note that the above six dominating sets are connected, $d_c(P,6) = \frac{810}{6} = 135$. ■

Lemma 6. *For the Petersen graph P, $d_c(P,7) = \binom{10}{7} - 10$.*

Proof. It suffices to determine the number of 7-subsets of vertices which are not connected dominating set. Suppose that $S \subseteq V(P)$, $|S| = 7$, and S is not a connected dominating set for P. Thus there exists $v \in V(P)$ such that $N[v] \cap S = \emptyset$ or S is not connected. By Lemma 3, S is a dominating set. Now, note that for every $x \in V(P)$, $V(P) \setminus N(x)$ is not connected. The number of 7-subsets of $V(P)$ which are not connected dominating sets is 10. So we have $d_c(P,7) = \binom{10}{7} - 10$. ■

Lemma 7. *For the Petersen graph P, $d_c(P,i) = \binom{10}{i}$ for $i = 8,9,10$.*

Proof. Note that every i selected vertices for $i = 8,9,10$ is connected. So the result follows by Lemma 3. ■

Now, we have the main theorem that have straightforward proof by above lemmas.

Theorem 5. *The connected domination polynomial of the Petersen graph is:*

$$D_c(P,X) = X^{10} + 10X^9 + 45X^8 + 110X^7 + 135X^6 + 72X^5 + 10X^4.$$

Proof. The result follows from Lemmas 4 - 7. ■

4. Connected domination polynomial of the generalized Petersen graph $GP(6,1)$

In this section we shall investigate the connected domination polynomial of the $GP(6,1)$. It can be easily investigated that the connected domination number of the $GP(6,1)$ is 6. First we obtain all connected dominating sets of size 6 by the following lemma. Note that we denote vertices $a_1, a_2, a_3, a_4, a_5, a_6$ for outer 6-cycle and vertices $b_1, b_2, b_3, b_4, b_5, b_6$ for inner 6-cycle.

Lemma 8. *For the $G = GP(6,1)$, $d_c(G,6) = 74$.*

Proof. First, we list all connected dominating sets of G of cardinality 6 containing one vertex, say the vertex labeled 1, which are the γ_c sets of the labeled $GP(6,1)$ given in Figure 2. To determine $d_{c_1}(G,6)$, we consider 6 cases.

Case 1. We select 6 vertices on the outer 6-cycle. This case is possible in 1 way.

Case 2. We select 5 vertices on the outer 6-cycle and 1 vertex on the inner 6-cycle. If 5 vertices are selected, then one of vertices on the inner 6-cycle is not dominated, since 6 selected vertices are connected, for selecting 1 vertex on the inner 6-cycle is possible in 2 ways.

For example if a_1, a_2, a_3, a_4, a_5 are selected, then vertex b_6 is not dominated and vertices b_1 or b_5 should be selected. Therefore the number of selection 5 vertices on the outer 6-cycle and containing $a_1 = 1$ is 10.

Case 3. We select 4 vertices on the outer 6-cycle and 2 vertices on the inner 6-cycle. If 4 vertices are adjacent, then 2 vertices on the inner 6-cycle are not dominated. For example if a_1, a_2, a_3, a_4 are selected, then the vertices b_5, b_6 are not dominated. Since 6 selected vertices are connected, it is possible in 3 ways, b_5, b_4; b_6, b_1 or b_1, b_4. If 4 vertices are not adjacent, we can not select 6 selected vertices connected dominating sets. Therefore the number of selection 4 vertices on the outer 6-cycle and containing $a_1 = 1$ is 12.

Case 4. We select 3 vertices on the outer 6-cycle and 3 vertices on the inner 6-cycle. If 3 vertices are adjacent, then one of vertices on the outer

6-cycle is not dominated, so its adjacent vertex on the inner 6-cycle should be selected and two other vertices be selected such that induced 6 selected vertices are connected. For example by selecting a_1, a_2, a_3; the vertex a_5 is not dominated, so vertex b_5 should be selected and two other vertices are b_1, b_6 or b_3, b_4. If 3 vertices are not adjacent, we can not select 6 selected vertices connected dominating sets. Since the number of selection 3 adjacent vertices on the outer 6-cycle and containing $a_1 = 1$ is 3, so this case is possible in 6 ways.

Case 5. We select 2 vertices on the outer 6-cycle and 4 vertices on the inner 6-cycle. If 2 vertices are adjacent, 2 vertices on the outer 6-cycle are not dominated, then 2 vertices adjacent to 2 above vertices on the inner 6-cycle should be selected and two other vertices should be selected such that 6 selected vertices be connected. For example by selecting a_1, a_2; vertices b_4 and b_5 and vertices b_1, b_6 or b_2, b_3 should be selected. Since the number of selection 2 adjacent vertices and including $a_1 = 1$ is 2, this case is possible in 4 ways. If 2 vertices are not adjacent and there is one of vertices on the outer 6-cycle that is not dominated, then we can not select 4 vertices on the inner 6-cycle. If two vertices are not adjacent and all vertices are dominated, then 4 vertices on the inner 6-cycle are not dominated and selection 4 vertices on the inner 6-cycle is possible in 2 ways. In this case the number of 6 selected vertices dominating sets including $a_1 = 1$ is 6.

Case 6. We select 1 vertex on the outer 6-cycle and 5 vertices on the inner 6-cycle. If 1 vertex on the outer 6-cycle is selected, then 3 vertices on outer 6-cycle are not dominated. If we select $a_1 = 1$, then b_3, b_4, b_5 should be selected and two other vertices b_1, b_6 or b_1, b_2. In this case the number of 6 selected vertices dominating sets and containing $a_1 = 1$ is 2. Therefore the number of 6 selected vertices connected dominating sets containing $a_1 = 1$ is 37.

These connected dominating sets are listed below.

$$
\begin{aligned}
D_{c_1}(G,6) = \; & \{\{1,2,3,4,5,6\},\{1,2,3,4,5,8\},\{1,2,3,4,5,12\},\\
& \{1,2,3,4,6,7\},\{1,2,3,4,6,11\},\{1,2,3,5,6,10\},\\
& \{1,2,3,5,6,12\},\{1,2,4,5,6,9\},\{1,2,4,5,6,11\},\\
& \{1,3,4,5,6,8\},\{1,3,4,5,6,10\},\{1,2,3,4,7,8\},\\
& \{1,2,3,4,11,12\},\{1,2,3,4,8,11\},\{1,2,3,6,7,12\},\\
& \{1,2,3,6,10,11\},\{1,2,3,6,7,10\},\{1,2,5,6,9,10\},\\
& \{1,2,5,6,12,11\},\{1,4,5,6,8,9\},\{1,4,5,6,11,10\},\\
& \{1,2,5,6,9,12\},\{1,4,5,6,8,11\},\{1,2,3,8,7,12\},\\
& \{1,2,3,10,11,12\},\{1,2,6,11,12,7\},\{1,2,6,11,10,9\},\\
& \{1,5,6,10,9,8\},\{1,5,6,10,11,12\},\{1,2,11,12,7,8\},
\end{aligned}
$$

$\{1,2,11,12,10,9\},\{1,4,11,12,7,8\},\{1,4,11,10,9,8\},$
$\{1,6,10,11,12,7\},\{1,6,10,11,9,8\},\{1,8,7,12,11,10\},$
$\{1,8,9,10,11,12\}\}.$

Since the $G = GP(6,1)$ is vertex-transitive graph, therefore the number of connected dominating sets is $12 \times 37 = 444$. By Lemma 2 and note that the above six dominating sets are connected, $d_c(G,6) = \frac{444}{6} = 74$. ■

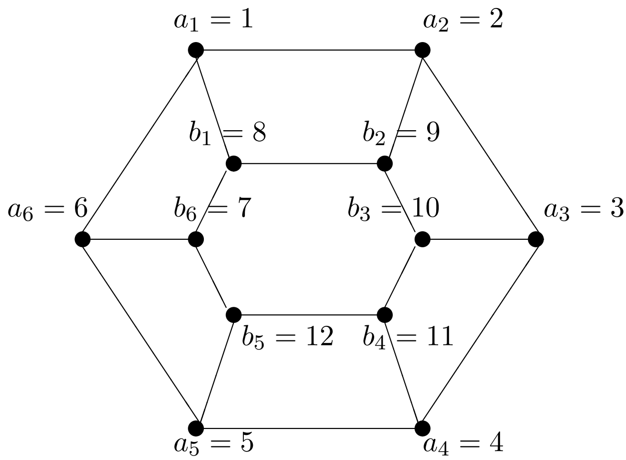

Figure 2. $GP(6,1)$ *with label*

Lemma 9. *9 For the $G = GP(6,1)$, $d_c(G,7) = 264$.*

Proof. First, we list all connected dominating sets of G of cardinality 7 containing one vertex, say the vertex labeled 1, which are the γ_c-sets of the labeled $GP(6,1)$ given in Figure 2.

To determine $d_{c_1}(G,7)$, we consider 6 cases.

Case 1. We select 6 vertices on the outer 6-cycle and 1 vertex on the inner 6-cycle. If 6 vertices are selected, then all vertices are dominated and we can select one of vertices on the inner 6-cycle. In this case the number of 7 selected vertices is 6.

Case 2. We select 5 vertices on the outer 6-cycle and 2 vertices on the inner 6-cycle. If 5 vertices are selected, then one of vertices on the inner 6-cycle is not dominated. For example if a_1, a_2, a_3, a_4, a_5 are selected, then vertex b_6 is not dominated and we consider 2 cases for selecting two vertices on the inner 6-cycle.

(i) By selecting b_6, then another vertex is b_1 or b_5. So this case is possible in 2 ways.

(*ii*) By selecting b_1 or b_5, the vertex b_6 is dominated and another vertex is selected among remaind vertices except b_6. This case is possible in 7 ways. Therefore the number of selection 5 vertices on the outer 6-cycle and including $a_1 = 1$ is 45.

Case 3. We select 4 vertices on the outer 6-cycle and 3 vertices on the inner 6-cycle. If 4 vertices are adjacent, then 2 vertices on the inner 6-cycle are not dominated. For example if a_1, a_2, a_3, a_4 are selected, then vertices b_5 and b_6 are not dominated. 3 vertices on the inner 6-cycle should be selected such that 7 selected vertices are connected and b_5, b_6 are dominated. It is possible in 10 ways, b_6, b_1, b_4; b_6, b_1, b_2; b_6, b_1, b_3; b_6, b_1, b_5; b_1, b_2, b_4; b_6, b_4, b_5; b_1, b_3, b_4; b_1, b_4, b_5; b_2, b_4, b_5 or b_3, b_4, b_5.
If 3 vertices are adjacent and 1 vertex is not adjacent to them, then two vertices on the inner 6-cycle are not dominated.
For example if a_1, a_2, a_3, a_5 are selected, then vertices b_4 and b_6 are not dominated. 3 vertices on the inner 6-cycle should be selected such that 7 selected vertices are connected and b_4, b_6 are dominated. It is possible in 2 ways, b_1, b_6, b_5 or b_3, b_4, b_5.
If 2 vertices are adjacent with each other and 2 other vertices are adjacent with each other, too, then two nonadjacent vertices on the inner 6-cycle are not dominated. We cannot select 3 vertices on the inner 6-cycle.
Therefore in this case, the number of selection 4 vertices on the outer 6-cycle containing $a_1 = 1$ is 48.

Case 4. We select 3 vertices on the outer 6-cycle and 4 vertices on the inner 6-cycle. If 3 vertices are adjacent, then one of vertices on the outer 6-cycle is not dominated, so its adjacent vertex on the inner 6-cycle should be selected and three other vertices be selected such that induced 7 selected vertices are connected. For example to select a_1, a_2, a_3; the vertex a_5 is not dominated. So vertex b_5 should be selected and three other vertices be selected in 6 ways, b_6, b_1, b_2; b_6, b_1, b_3; b_6, b_1, b_4; b_4, b_3, b_2; b_4, b_3, b_1 or b_4, b_3, b_6. If 2 vertices are adjacent and 1 vertex is not adjacent to them, then three vertices on the inner 6-cycle are not dominated and 4 vertices on the inner 6-cycle be selected in 3 ways. For example to select a_1, a_2, a_4; vertices b_3, b_5, b_6 are not dominated. The vertex b_4 should be selected and three other vertices be selected in 3 ways, b_5, b_6, b_1; b_3, b_2, b_1 or b_3, b_2, b_5. If 3 vertices are not adjacent, we cannot select 7 selected vertices connected dominating sets. In this case the number of selection 3 vertices on the outer 6-cycle and including $a_1 = 1$ is 36.

Case 5. We select 2 vertices on the outer 6-cycle and 5 vertices on the inner 6-cycle. If 2 vertices are adjacent, then 2 vertices on the outer 6-cycle are not dominated, so 2 its adjacent vertices on the inner 6-cycle should be selected and 3 other vertices be selected among remind vertices in $\binom{4}{3} = 4$

ways. For example by selecting a_1, a_2; vertices b_4, b_5 should be selected and three vertices be selected in 4 ways, b_1, b_2, b_3; b_1, b_6, b_3; b_2, b_3, b_6 or b_2, b_6, b_1. If 2 vertices are not adjacent and distance between of them is 2, then one vertex on the outer 6-cycle is not dominated. So its adjacent vertex on the inner 6-cycle should be selected and four other vertices be selected in 3 ways. For example by selecting a_1, a_3; vertex b_5 should be selected and four vertices be selected in 3 ways, b_1, b_6, b_4, b_3; b_1, b_2, b_3, b_4 or b_3, b_2, b_1, b_6. If the distance between to selected vertices from outer 6-cycle is 3 and a_1, a_4 are selected, then b_2, b_3, b_6, b_5 are not dominated. Since 7 selected vertices are connected, the vertices b_1, b_4 should be selected and three other vertices be selected in $\binom{4}{3} = 4$ ways, b_6, b_5, b_3; b_2, b_3, b_5; b_2, b_3, b_6 or b_2, b_5, b_6. In this case the number of 7 selected vertices connected dominating sets including $a_1 = 1$ is 18.

Case 6. We select 1 vertex on the outer 6-cycle and 6 vertices on the inner 6-cycle. If 1 vertex on the outer 6-cycle is selected, then we can select 6 vertices on the inner 6-cycle is one way. In this case the number of 7 selected vertices connected dominating sets including $a_1 = 1$ is 1. Therefore the number of 7 selected vertices connected dominating sets including $a_1 = 1$ is 154.

These connected dominating sets are listed below.

$$
\begin{aligned}
D_{c_1}(G,7) = \; & \{\{1,2,3,4,5,6,8\},\{1,2,3,4,5,6,9\},\{1,2,3,4,5,6,10\},\\
& \{1,2,3,4,5,6,11\},\{1,2,3,4,5,6,12\},\{1,2,3,4,5,6,7\},\\
& \{1,2,3,4,5,7,8\},\{1,2,3,4,5,7,12\},\{1,2,3,4,5,9,8\},\\
& \{1,2,3,4,5,11,12\},\{1,2,3,4,5,8,10\},\{1,2,3,4,5,8,11\},\\
& \{1,2,3,4,5,8,12\},\{1,2,3,4,5,12,10\},\{1,2,3,4,5,12,9\},\\
& \{1,2,3,4,6,12,7\},\{1,2,3,4,6,12,11\},\{1,2,3,4,6,7,8\},\\
& \{1,2,3,4,6,7,9\},\{1,2,3,4,6,7,10\},\{1,2,3,4,6,7,11\},\\
& \{1,2,3,4,6,11,10\},\{1,2,3,4,6,11,9\},\{1,2,3,4,6,11,8\},\\
& \{1,2,3,5,6,11,10\},\{1,2,3,5,6,11,12\},\{1,2,3,5,6,10,9\},\\
& \{1,2,3,5,6,10,8\},\{1,2,3,5,6,10,7\},\{1,2,3,5,6,10,12\},\\
& \{1,2,3,5,6,12,7\},\{1,2,3,5,6,12,8\},\{1,2,3,5,6,12,9\},\\
& \{1,2,4,5,6,10,9\},\{1,2,4,5,6,10,11\},\{1,2,4,5,6,9,8\},\\
& \{1,2,4,5,6,9,7\},\{1,2,4,5,6,9,12\},\{1,2,4,5,6,9,11\},\\
& \{1,2,4,5,6,11,12\},\{1,2,4,5,6,11,7\},\{1,2,4,5,6,11,8\},\\
& \{1,3,4,5,6,9,8\},\{1,3,4,5,6,9,10\},\{1,3,4,5,6,8,7\},\\
& \{1,3,4,5,6,8,12\},\{1,3,4,5,6,8,11\},\{1,3,4,5,6,8,10\},\\
& \{1,3,4,5,6,10,11\},\{1,3,4,5,6,10,12\},\{1,3,4,5,6,10,7\},
\end{aligned}
$$

$\{1,2,3,4,7,8,11\}, \{1,2,3,4,7,8,12\}, \{1,2,3,4,7,8,9\},$
$\{1,2,3,4,8,9,11\}, \{1,2,3,4,7,11,12\}, \{1,2,3,4,7,8,10\},$
$\{1,2,3,4,8,10,11\}, \{1,2,3,4,8,11,12\}, \{1,2,3,4,9,11,12\},$
$\{1,2,3,4,10,11,12\}, \{1,2,3,5,10,11,12\}, \{1,2,3,5,7,8,12\},$
$\{1,2,3,6,7,8,12\}, \{1,2,3,6,7,9,12\}, \{1,2,3,6,7,8,10\},$
$\{1,2,3,6,7,10,12\}, \{1,2,3,6,7,11,12\}, \{1,2,3,6,8,10,11\},$
$\{1,2,3,6,9,10,11\}, \{1,2,3,6,10,11,12\}, \{1,2,3,6,7,9,10\},$
$\{1,2,3,6,7,10,11\}, \{1,3,4,5,7,8,12\}, \{1,3,4,5,8,9,10\},$
$\{1,4,5,6,7,8,9\}, \{1,4,5,6,8,7,11\}, \{1,4,5,6,7,10,11\},$
$\{1,4,5,6,8,9,10\}, \{1,4,5,6,8,9,11\}, \{1,4,5,6,8,9,12\},$
$\{1,4,5,6,8,10,11\}, \{1,4,5,6,9,10,11\}, \{1,4,5,6,10,11,12\},$
$\{1,4,5,6,8,11,12\}, \{1,2,4,6,11,12,7\}, \{1,2,4,6,11,10,9\},$
$\{1,2,5,6,7,9,10\}, \{1,2,5,6,7,11,12\}, \{1,2,5,6,8,9,10\},$
$\{1,2,5,6,8,11,12\}, \{1,2,5,6,9,10,11\}, \{1,2,5,6,9,10,12\},$
$\{1,2,5,6,9,11,12\}, \{1,2,5,6,10,11,12\}, \{1,2,5,6,7,9,12\},$
$\{1,2,5,6,8,9,12\}, \{1,3,5,6,10,9,8\}, \{1,3,5,6,10,11,12\},$
$\{1,2,3,12,7,8,9\}, \{1,2,3,12,7,8,10\}, \{1,2,3,12,7,8,11\},$
$\{1,2,3,12,11,10,9\}, \{1,2,3,12,11,10,8\}, \{1,2,3,12,11,10,7\},$
$\{1,2,4,11,12,7,8\}, \{1,2,4,11,10,9,8\}, \{1,2,4,11,10,9,12\},$
$\{1,2,5,12,7,8,9\}, \{1,2,5,12,7,8,11\}, \{1,2,5,12,11,10,9\},$
$\{1,2,6,10,9,12\}, \{1,2,6,11,12,7,8\}, \{1,2,6,11,12,7,9\},$
$\{1,2,6,11,12,7,10\}, \{1,2,6,11,10,9,8\}, \{1,2,6,11,10,9,7\},$
$\{1,3,4,11,12,7,8\}, \{1,3,4,11,10,9,8\}, \{1,3,4,10,9,8,7\},$
$\{1,3,6,7,12,11,10\}, \{1,3,6,10,9,8,7\}, \{1,3,6,10,9,8,11\},$
$\{1,4,5,8,7,12,11\}, \{1,4,5,9,8,7,12\}, \{1,4,5,11,10,9,8\},$
$\{1,4,6,8,7,12,11\}, \{1,4,6,11,12,7,10\}, \{1,4,6,11,10,9,8\},$
$\{1,5,6,10,9,8,7\}, \{1,5,6,10,9,8,12\}, \{1,5,6,10,9,8,11\},$
$\{1,5,6,10,11,12,7\}, \{1,5,6,10,11,12,8\}, \{1,5,6,10,11,12,9\},$
$\{1,2,8,7,12,11,10\}, \{1,2,8,9,10,11,12\}, \{1,2,9,10,11,12,7\},$
$\{1,2,9,8,7,12,11\}, \{1,3,12,8,7,11,10\}, \{1,3,8,9,10,11,12\},$
$\{1,3,10,9,8,7,12\}, \{1,4,8,7,12,11,10\}, \{1,4,8,9,10,11,12\},$
$\{1,4,11,10,9,8,7\}, \{1,4,11,12,7,8,9\}, \{1,5,10,8,7,12,11\},$
$\{1,5,10,8,9,12,11\}, \{1,5,12,7,8,9,10\}, \{1,6,10,11,7,12,9\},$
$\{1,6,10,11,7,8,9\}, \{1,6,10,11,8,9,12\}, \{1,6,10,11,8,7,12\},$
$\{1,8,7,9,10,11,12\}\}.$

Therefore $d_c(G,7) = \frac{154\times 12}{7} = 264$. ■

Lemma 10. *For the $G = GP(6,1)$, $d_c(G,8) = \binom{12}{8} - 153$.*

Proof. It suffices to determine the number of 8-subsets of vertices which are not connected dominating set. We consider 3 cases.

Case 1. We select 5 vertices on the outer 6-cycle and 3 vertices on the inner 6-cycle. If 5 vertices are selected, then one of vertices on the inner 6-cycle is not dominated. For example if a_1, a_2, a_3, a_4, a_5 are be selected, then vertex b_6 is not dominated. We consider 2 cases for selecting three vertices on the inner 6-cycle such that the vertex b_6 is not dominated yet.

(i) By selecting b_6 and 2 its nonadjacent vertices, that is possible in $\binom{3}{2} = 3$ ways.

(ii) By selecting 3 nonadjacent vertices of b_6. It is possible in 1 ways.

Therefore the number of selection of 5 vertices on the outer 6-cycle including $a_1 = 1$ which are not connected dominating sets is 20.

Case 2. We select 4 vertices on the outer 6-cycle and 4 vertices on the inner 6-cycle. If 4 vertices are adjacent, then 2 adjacent vertices on the inner 6-cycle are not dominated. So we can select one of them and its three nonadjacent vertices or we can select two adjacent vertices and two other vertices that are not adjacent to them. It is possible in 3 ways.
For example if a_1, a_2, a_3, a_4 are selected, then vertices b_5 and b_6 are not dominated. So we can select b_6, b_2, b_3, b_4; b_5, b_1, b_2, b_3 or b_5, b_6, b_2, b_3.
If 3 vertices are adjacent and 1 vertex is not their adjacent, then two non-adjacent vertices on the inner 6-cycle are not dominated. For example if a_1, a_2, a_3, a_5 are selected, then vertices b_4 and b_6 are not dominated. We can select four vertices in 9 ways, b_6, b_2, b_3, b_4; b_4, b_1, b_6, b_2; b_6, b_1, b_2, b_3; b_6, b_1, b_3, b_4; b_1, b_2, b_3, b_4; b_6, b_2, b_4, b_5; b_1, b_2, b_3, b_5; b_1, b_2, b_4, b_5 or b_6, b_2, b_3, b_5.
If two vertices are adjacent and two other vertices are adjacent, too, then two nonadjacent vertices on the inner 6-cycle are not dominated. For example to select a_1, a_2, a_4, a_5 then vertices b_3, b_6 are not dominated. Since b_3, b_6 are dominated in 4 ways, we can four vertices in $\binom{6}{4} - 4 = 11$ ways the b_3, b_6 are not dominated, b_6, b_1, b_2, b_3; b_6, b_1, b_2, b_4; b_6, b_1, b_3, b_4; b_6, b_1, b_3, b_5; b_6, b_2, b_3, b_4; b_6, b_2, b_3, b_5; b_6, b_2, b_4, b_5; b_6, b_3, b_4, b_5; b_1, b_2, b_3, b_5; b_1, b_2, b_4, b_5 or b_1, b_3, b_4, b_5.
Therefore the number of selection of 4 vertices on the outer 6-cycle including $a_1 = 1$ which are not connected dominating sets is 70.

Case 3. We select 3 vertices on the outer 6-cycle and 5 vertices on the inner 6-cycle. If 3 vertices are adjacent, then one of vertices for example a_i on the outer 6-cycle is not dominated, so we can select 5 vertices on the inner 6-cycle unless b_i in 1 way. For example to select a_1, a_2, a_3; vertex a_5

is not dominated. So we should be selected b_1, b_2, b_3, b_4, b_6. If 2 vertices are adjacent and 1 vertex for example a_i is not adjacent to them on the outer 6-cycle, it suffices we select 5 vertices unless b_i. For example if we select a_1, a_2, a_4, then we should select b_1, b_2, b_3, b_5, b_6. If 3 vertices are not adjacent, we can not select one of adjacent vertices to them. For example to select a_1, a_3, a_5, we should be selected b_2, b_3, b_4, b_5, b_6; b_1, b_2, b_4, b_5, b_6 or b_1, b_2, b_3, b_4, b_6. Therefore the number of selection of 3 vertices on the outer 6-cycle containing $a_1 = 1$ which are not connected dominating sets is 12. Since the number of 8-subsets of vertices containing $a_1 = 1$ which are not connected dominating set is 102. The number of 8-subsets which are not connected dominating set is $\frac{102\times 12}{8} = 153$. So the number of 8-subsets of vertices connected dominating sets is $\binom{12}{8} - 153 = 342$.

These disconnected dominating sets are listed below.

$$
\begin{aligned}
\bar{D}_{c_1}(G,8) \;=\; & \{\{1,2,3,4,5,7,9,10\}, \{1,2,3,4,5,7,9,11\}, \{1,2,3,4,5,7,10,11\},\\
& \{1,2,3,5,6,11,7,8\}, \{1,2,3,5,6,11,7,9\}, \{1,2,3,5,6,11,8,9\},\\
& \{1,2,3,5,6,7,8,9\}, \{1,2,3,4,6,12,8,9\}, \{1,2,3,4,6,12,8,10\},\\
& \{1,2,3,4,6,12,9,10\}, \{1,2,3,4,6,8,9,10\}, \{1,2,4,5,6,10,12,7\},\\
& \{1,2,4,5,6,10,12,8\}, \{1,2,4,5,6,10,7,8\}, \{1,2,4,5,6,7,8,12\},\\
& \{1,3,4,5,6,9,7,12\}, \{1,3,4,5,6,9,7,11\}, \{1,3,4,5,6,9,11,12\},\\
& \{1,3,4,5,6,7,11,12\}, \{1,2,3,4,7,9,10,11\}, \{1,2,3,4,7,12,9,10\},\\
& \{1,2,3,4,12,8,9,10\}, \{1,2,3,5,7,9,10,11\}, \{1,2,3,5,11,8,7,9\},\\
& \{1,2,3,5,7,8,9,10\}, \{1,2,3,5,7,8,10,11\}, \{1,2,3,5,8,9,10,11\},\\
& \{1,2,3,5,7,9,11,12\}, \{1,2,3,5,8,9,10,12\}, \{1,2,3,5,8,9,11,12\},\\
& \{1,2,3,5,7,9,10,12\}, \{1,2,3,6,11,7,8,9\}, \{1,2,3,6,12,8,9,10\},\\
& \{1,2,3,6,11,12,8,9\}, \{1,2,4,5,7,8,9,10\}, \{1,2,4,5,7,8,9,11\},\\
& \{1,2,4,5,7,8,10,11\}, \{1,2,4,5,7,8,10,12\}, \{1,2,4,5,7,9,10,11\},\\
& \{1,2,4,5,7,9,10,12\}, \{1,2,4,5,7,9,11,12\}, \{1,2,4,5,7,10,11,12\},\\
& \{1,2,4,5,8,9,10,12\}, \{1,2,4,5,8,9,11,12\}, \{1,2,4,5,8,10,11,12\},\\
& \{1,2,4,6,7,8,9,10\}, \{1,2,4,6,7,8,9,11\}, \{1,2,4,6,7,8,10,11\},\\
& \{1,2,4,6,7,8,10,12\}, \{1,2,4,6,7,8,9,12\}\{1,2,4,6,7,9,10,12\},\\
& \{1,2,4,6,8,9,10,12\}, \{1,2,4,6,8,9,11,12\}, \{1,2,4,6,8,10,11,12\},\\
& \{1,2,5,6,7,8,9,11\}, \{1,2,5,6,7,8,10,11\}, \{1,2,5,6,7,8,10,12\},\\
& \{1,3,4,5,7,8,9,11\}, \{1,3,4,5,7,8,10,11\}, \{1,3,4,5,7,9,10,11\},\\
& \{1,3,4,5,7,9,10,12\}, \{1,3,4,5,7,9,11,12\}, \{1,3,4,5,7,10,11,12\},\\
& \{1,3,4,5,8,9,11,12\}, \{1,3,4,5,9,10,11,12\}, \{1,3,4,5,8,10,11,12\},\\
& \{1,3,5,6,7,8,9,11\}, \{1,3,5,6,7,8,10,11\}, \{1,3,5,6,7,8,10,12\},
\end{aligned}
$$

$\{1,3,5,6,7,8,11,12\},\{1,3,5,6,7,8,9,12\},\{1,3,5,6,7,9,10,11\},$
$\{1,3,5,6,7,9,10,12\},\{1,3,5,6,7,9,11,12\},\{1,3,5,6,8,9,11,12\},$
$\{1,4,5,6,7,8,10,12\},\{1,4,5,6,7,9,11,12\},\{1,4,5,6,7,9,10,12\},$
$\{1,3,4,6,7,8,9,11\},\{1,3,4,6,7,8,10,12\},\{1,3,4,6,7,8,9,12\},$
$\{1,3,4,6,7,9,10,11\},\{1,3,4,6,7,9,10,12\},\{1,3,4,6,7,9,11,12\},$
$\{1,3,4,6,8,9,10,12\},\{1,3,4,6,8,9,11,12\},\{1,3,4,6,9,10,11,12\},$
$\{1,3,4,6,8,10,11,12\},\{1,3,4,6,7,8,10,11\},\{1,2,3,7,8,9,10,11\},$
$\{1,2,4,7,8,9,10,12\},\{1,2,5,7,8,9,10,11\},\{1,2,6,7,8,9,10,12\},$
$\{1,3,4,7,9,10,11,12\},\{1,3,5,7,9,10,11,12\},\{1,3,5,7,8,9,11,12\},$
$\{1,3,5,7,8,9,10,11\},\{1,3,6,7,8,9,11,12\},\{1,4,5,7,9,10,11,12\},$
$\{1,4,6,7,9,10,8,12\},\{1,5,6,7,8,9,11,12\},\{1,2,3,4,5,9,10,11\}\}$

Therefore $d_c(G,8)=\binom{12}{8}-153$. ■

Lemma 11. *For the $G=GP(6,1)$, $d_c(G,9)=\binom{12}{9}-12$.*

Proof. It suffices to determine the number of 9-subsets which are not connected dominating set. Suppose that $S\subseteq V(G)$, $|S|=9$, and S is not a connected dominating set for G. Thus there exists $v\in V(G)$ such that $N[v]\cap S=\emptyset$ or S is not connected. By Lemma 3, S is a dominating set. Now, note that for every $x\in V(G)$, $V(G)\setminus N(x)$ is not connected. The number of 9-subsets of $V(G)$ which are not dominating sets is 12. So we have $d_c(G,9)=\binom{12}{9}-12$. ■

Lemma 12. *For the $G=GP(6,1)$, $d_c(G,i)=\binom{12}{i}$ for $i=10,11,12$.*

Proof. Note that every i selected vertices for $i=10,11,12$ is connected. So the result follows by Lemma 3. ■

Now, we have the main theorem that follows from lemmas.

Theorem 6. *The connected domination polynomial of the $G=GP(6,1)$ is:*

$$\begin{aligned} D_c(G,X) &= X^{12}+12X^{11}+\binom{12}{10}X^{10}+\left(\binom{12}{9}-12\right)X^9 \\ &\quad+\left(\binom{12}{8}-153\right)X^8+264X^7+74X^6. \end{aligned}$$

Proof. That follows from Lemmas 8 - 12. ■

References

[1] Akbari S., Alikhani S., Peng Y.H., Characterization of graphs using domination Polynomials, *European J. Combin.,* 31(2010), 1714-1724.

[2] Alikhani S., Peng Y.H., Dominating sets and domination polynomials of cycles, *Global Journal of Pure and Applied Mathematics,* 4(2)(2008), 151-162.

[3] Alikhani S., Peng Y.H., Domination polynomials of cubic graphs of order 10, *Turkish Journal of Mathematics,* 35(2011), 355-366.

[4] Alikhani S., Peng Y.H., Dominating sets and domination polynomials of Paths, *International Journal of Mathematics and Mathematical Science,* Vol. 2009, Article ID 542040 (2009).

[5] Biggs N.L., *Algebraic Graph Theory,* 2nd ed. Cambridge, Cambridge University press., England, 1993.

[6] Bondy J.A., Murty U.S.R., *Graph Theory,* Graduate Texts Mathematics 244, 2008.

[7] Haynes T.W., Hedetniemi S.T., Slater P.J., *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

D. A. Mojdeh
Department of Mathematics
University of Mazandaran
Babolsar, Iran
e-mail: damojdeh@umz.ac.ir

A. S. Emadi
Department of Mathematics
University of Mazandaran
Babolsar, Iran
e-mail: math_emadi2000@yahoo.com

Received on 13.11.2017 and, in revised form, on 01.03.2018.