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Emine K. Sögütcü, Neşet Aydin and Öznur Gölbaşı

REMARKS ON $*-(\sigma,\tau)-$ LIE IDEALS OF $*-$PRIME RINGS WITH DERIVATION

Abstract. Let R be a $*-$prime ring with characteristic not 2, U a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d a nonzero derivation of R. Suppose σ, τ be two automorphisms of R such that $\sigma d = d\sigma$, $\tau d = d\tau$ and $*$ commutes with σ, τ, d. In the present paper it is shown that if $d(U) \subseteq Z$ or $d^2(U) \subseteq Z$, then $U \subseteq Z$.

Key words: prime ring, derivation, $(\sigma,\tau)-$Lie ideal, involution.

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1. Introduction

Let R will be an associative ring with center Z. Let σ and τ two mappings from R into itself. For any $x, y \in R$, we write $[x, y]$ and $[x, y]_{\sigma,\tau}$, for $xy - yx$ and $x\sigma(y) - \tau(y)x$ respectively and make extensive use of basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$
$$[xy, z] = [x, z]y + x[y, z]$$
$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$$
$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z).$$

We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c \quad \text{for all} \quad x \in R\}$ and call it $(\sigma,\tau)-$center of R. Note that $C_{1,1} = Z(R)$, where $1 : R \longrightarrow R$ is the identity map. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. Kaya [4] first introduced the $(\sigma,\tau)-$Lie ideal as following: Let U be an additive subgroup of R, $\sigma,\tau : R \longrightarrow R$ be two mappings. Then (i) U is a $(\sigma,\tau)-$right Lie ideal of R if $[U, R]_{\sigma,\tau} \subseteq U$. (ii) U is a $(\sigma,\tau)-$left Lie ideal of R if $[R, U]_{\sigma,\tau} \subseteq U$. (iii) U is a $(\sigma,\tau)-$Lie ideal of R if U is both

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a $(\sigma,\tau)-$right Lie ideal and $(\sigma,\tau)-$left Lie ideal of R. Every Lie ideal of R is a $(1,1)-$left (and right) Lie ideal of R, where $1: R \longrightarrow R$ is the identity map of R. But there exist $(\sigma,\tau)-$Lie ideals which are not Lie ideals (Such an example due to [4]).

Recall that a ring R is prime if $xRy = 0$ for $x, y \in R$ implies $x = 0$ or $y = 0$. An additive mapping $*: R \to R$ is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*-$ring. A ring with an involution is said to $*-$prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ for $x, y \in R$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $*-$prime but the converse need not hold general. An example due to Oukhtite [9] justifies the above statement that is, R be a prime ring, $S = R \times R^o$ where R^o is the opposite ring of R. Define involution $*$ on S as $(x,y)^* = (y,x)$. S is $*-$prime, but not prime. This example shows that $*-$prime rings constitute a more general class of prime rings. In all that follows the symbol $S_*(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R, i.e. $S_*(R) = \{x \in R \mid x^* = \pm x\}$. An $(\sigma,\tau)-$Lie ideal of R is said to be a $* - (\sigma,\tau)-$Lie ideal if U is invariant under $*$, i.e. $U^* = U$.

Following Posner [10], an additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Many results in the literature indicate that the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. For example derivations with certain properties investigated in various papers. Bergen et al. proved the following results in [3]: Let R be a prime ring of characteristic different from 2, U a nonzero Lie ideal of R and d a nonzero derivation. If $d(U) \subseteq Z$, then $U \subseteq Z$. In [5], Lee and Lee proved that if R is a prime ring of characteristic different from 2, U a nonzero Lie ideal of R and d a nonzero derivation such that $d^2(U) \subseteq Z$ then $U \subseteq Z$. Further, the above results were extended to $(\sigma,\tau)-$ Lie ideals of R in [1] and [11]. Oukhtite et al. showed that these results are valid for $*-$prime rings in [8]. In this paper our objective is to generalize the above results for a nonzero $* - (\sigma,\tau)-$Lie ideal of a $*-$prime ring with characteristic not two.

2. Results

Lemma 1 ([12], Lemma 2.8)**.** *Let R be a $*-$prime ring, U a nonzero $* - (\sigma,\tau)-$left Lie ideal of R such that τ commutes with $*$. If $U \subseteq C_{\sigma,\tau}$, then $U \subseteq Z$.*

Lemma 2 ([12], Theorem 2.11)**.** *Let R be a $*-$prime ring with characteristic not 2, U a nonzero $* - (\sigma,\tau)-$Lie ideal of R such that τ commutes with $*$. If $a \in S_*(R)$ and $[U,a] = 0$ then $a \in Z$ or $U \subseteq Z$.*

Lemma 3 ([2], Theorem 2.10). *Let R be a $*-$prime ring with characteristic not 2, U a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d a nonzero derivation of R such that $d\tau=\tau d, \sigma d=d\sigma$ and $*$ commutes with σ,τ and d. If $d^2(U)=(0)$, then $U\subseteq Z$.*

Lemma 4. *Let R be a $*-$prime ring, U a nonzero $*-(\sigma,\tau)-$left Lie ideal of R such that σ and τ commutes with $*$. If $[R,U]_{\sigma,\tau}\subseteq Z$, then $U\subseteq Z$.*

Proof. For any $x\in R$, $u\in U$, we get $[x,u]_{\sigma,\tau}\in Z$. Replacing x by $x\sigma(u), u\in U$ in the this equation, we obtain

$$[x,u]_{\sigma,\tau}\sigma(u)\in Z,\ \text{for all}\ x\in R,\ u\in U$$

and so

$$[x,u]_{\sigma,\tau}\sigma(u)r=r[x,u]_{\sigma,\tau}\sigma(u),\ \text{for all}\ x,r\in R,\ u\in U.$$

By the hypothesis, we have

$$[x,u]_{\sigma,\tau}\,[\sigma(u),r]=0,\ \text{for all}\ \ x,r\in R,\ u\in U.$$

Again using the hypothesis, we obtain

$$(1)\qquad [x,u]_{\sigma,\tau}R\,[\sigma(u),r]=0,\ \text{for all}\ x,r\in R,\ u\in U.$$

Assume that $u\in U\cap S_*(R)$. In (1), replacing r^*,u^* instead of r,u respectively, and using $*\sigma=\sigma*$, we get

$$[x,u]_{\sigma,\tau}R\,([\sigma(u),r])^*=0,\ \text{for all}\ x,r\in R,\ u\in U\cap S_*(R).$$

Thus,

$$[x,u]_{\sigma,\tau}R\,[\sigma(u),r]=[x,u]_{\sigma,\tau}R\,([\sigma(u),r])^*=0,$$

for all $x,r\in R$, $u\in U\cap S_*(R)$

By the $*-$primeness of R, we have

$$[x,u]_{\sigma,\tau}=0\ \text{ or }\ [\sigma(u),r]=0,\ \text{for all}\ x\in R,\ u\in U\cap S_*(R).$$

Now, let $[x,u]_{\sigma,\tau}=0$, for all $u\in U\cap S_*(R)$. For any $u\in U$, we find that $u-u^*\in U\cap S_*(R)$, and so $[x,u]_{\sigma,\tau}=[x,u^*]_{\sigma,\tau}$, for all $u\in U$, $x\in R$. In (1), taking r^*,u^* instead of r,u respectively and using $*\sigma=\sigma*$, we get

$$[x,u]_{\sigma,\tau}R\,([\sigma(u),r])^*=0,\ \text{for all}\ x,r\in R,\ u\in U.$$

On the other hand, we get $[\sigma(u),r]=0$, for all $u\in U\cap S_*(R)$. For any $u\in U$, again taking $u-u^*\in U\cap S_*(R)$,and so, $[\sigma(u),r]=[\sigma(u^*),r]$ for all

$r \in R, u \in U$. Replacing r by r^* in (1) and using this equation, $\sigma * = * \sigma$, we have

$$[x,u]_{\sigma,\tau} R \left([\sigma(u), r]\right)^* = 0, \text{ for all } x, r \in R, \ u \in U.$$

Hence we find that

$$(2) \qquad [x,u]_{\sigma,\tau} R \left([\sigma(u), r]\right)^* = 0, \text{ for all } x, r \in R, \ u \in U.$$

for any cases. By (1) and (2), we get

$$[x,u]_{\sigma,\tau} R \left[\sigma(u), r\right] = [x,u]_{\sigma,\tau} R \left([\sigma(u), r]\right)^* = 0, \text{ for all } x, r \in R, \ u \in U.$$

Since R is $*-$prime ring and σ is automorphism, we obtain

$$[x,u]_{\sigma,\tau} = 0 \text{ or } u \in Z \text{ for all } x \in R, \ u \in U.$$

We set $K = \{u \in U \mid [x,u]_{\sigma,\tau} = 0 \}$ and $L = \{u \in U \mid u \in Z\}$. Clearly each of K and L is additive subgroup of U. Morever, U is the set-theoretic union of K and L. But a group can not be the set-theoretic union of its two proper subgroups, hence $K = U$ or $L = U$. In the former case, $U \subseteq C_{\sigma,\tau}$. By Lemma 1, we have $U \subseteq Z$. In the latter case, $U \subseteq Z$. This completes the proof. ■

Theorem 1. *Let R be a $*-$prime ring with characteristic not 2, U a nonzero $* - (\sigma,\tau)-$Lie ideal of R, d a nonzero derivation of R such that $d\tau = \tau d, \sigma d = d\sigma$ and $*$ commutes with σ, τ and d. If $d(U) \subseteq Z$, then $U \subseteq Z$.*

Proof. For any $x \in R$, $u, v \in U$ and $[d(v)x, u]_{\sigma,\tau} \in U$. Thus we have

$$d([d(v)x,u]_{\sigma,\tau}) = d(d(v)[x,u]_{\sigma,\tau} + [d(v), \tau(u)]x) = d(d(v)[x,u]_{\sigma,\tau}) \in Z$$

and so

$$d^2(v)[x,u]_{\sigma,\tau} + d(v)d([x,u]_{\sigma,\tau}) \in Z, \text{ for all } x \in R, u, v \in U.$$

Using the hypothesis, we get

$$d^2(v)[x,u]_{\sigma,\tau} \in Z, \text{ for all } x \in R, \ u, v \in U.$$

Since $d^2(v)[x,u]_{\sigma,\tau} \in Z$, we have

$$d^2(v)[x,u]_{\sigma,\tau} r = r d^2(v)[x,u]_{\sigma,\tau}, \text{ for all } x, r \in R, \ u, v \in U.$$

Again using hypothesis, we obtain that

$$d^2(v) \left[[x,u]_{\sigma,\tau}, r\right] = 0, \text{ for all } x, r \in R, \ u, v \in U,$$

and so

$$d^2(v)R\,[[x,u]_{\sigma,\tau},r]=0,\ \text{ for all } x,r\in R,\ u,v\in U.$$

Replacing v by v^* in last equation and using $d*=*d$, we have

$$\left(d^2(v)\right)^*R\,[[x,u]_{\sigma,\tau},r]=0,\ \text{ for all } x,r\in R,\ u,v\in U.$$

Combining the last two equations and using the $*-$primeness of R, we arrive at

$$d^2(v)=0 \text{ or } [x,u]_{\sigma,\tau}\subseteq Z,\ \text{ for all } x\in R,\ u,v\in U.$$

In the former case, we get $U\subseteq Z$ by Lemma 3. In the latter case, $[R,U]_{\sigma,\tau}\subseteq Z$, and so, $U\subseteq Z$ by Lemma 4. This completes the proof. ■

Theorem 2. *Let R be a $*-$prime ring with characteristic not 2, U a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d a nonzero derivation of R such that $*$ commutes with σ,τ,d. If $a\in S_*(R)$, $d(Z)\neq 0$ and $[d(U),a]_{\sigma,\tau}=0$, then $a\in Z$ or $U\subseteq Z$.*

Proof. Choose $\alpha\in Z$ such that $d(\alpha)\neq 0$. It is easily seen that $\alpha,d(\alpha),d(\alpha^*)\in Z$ and $0\neq d(\alpha)^*=d(\alpha^*)$. For all $x\in R$, $u\in U$, we get

$$\begin{aligned}0 &= [d([x,u]_{\sigma,\tau}\alpha),a]_{\sigma,\tau}=[d([x,u]_{\sigma,\tau})\alpha+[x,u]_{\sigma,\tau}d(\alpha),a]_{\sigma,\tau}\\ &= [d([x,u]_{\sigma,\tau}),a]_{\sigma,\tau}\alpha+d([x,u]_{\sigma,\tau})[\alpha,\sigma(a)]\\ &\quad+[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}d(\alpha)+[x,u]_{\sigma,\tau}[d(\alpha),\sigma(a)].\end{aligned}$$

Using the hypothesis and $\alpha,d(\alpha)\in Z$, we obtain

$$[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}d(\alpha)=0,\ \text{ for all } x\in R,\ u\in U$$

and so for all $\alpha\in Z$ such that $d(\alpha)\neq 0$, we get

$$[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}Rd(\alpha)=0,\ \text{ for all } x\in R,\ u\in U.$$

Arguing the same ways above and using $*$ commutes with d and $\alpha^*\in Z$ such that $d(\alpha^*)\neq 0$, we obtain that

$$[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}Rd(\alpha)^*=0,\ \text{ for all } x\in R,\ u\in U$$

Hence we get

$$[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}Rd(\alpha)=[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}Rd(\alpha)^*=0,\ \text{ for all } x\in R,\ u\in U.$$

Since R is $*-$prime ring and $0\neq d(\alpha)\in Z$, we see that

$$[[x,u]_{\sigma,\tau},a]_{\sigma,\tau}=0,\ \text{ for all } x\in R,\ u\in U. \tag{3}$$

Substituting $x\sigma(u)$ for x in (3) and using this equation, we obtain

$$[x,u]_{\sigma,\tau}\sigma([u,a])=0, \text{ for all } x\in R,\ u\in U.$$

Replacing x by $\tau(y)x$, $y\in R$ in the last equation and using this equation, we get

$$\tau\left([y,u]\right)R\sigma([u,a])=0, \text{ for all } y\in R,\ u\in U. \tag{4}$$

Suppose that $u\in U\cap S_*(R)$. Taking y^* instead of y in (4) and using $\tau *=*\tau$, we have

$$\tau^*\left([y,u]\right)R\sigma([u,a])=0, \text{ for all } x\in R,\ u\in U\cap S_*(R).$$

That is,

$$\tau\left([y,u]\right)R\sigma([u,a])=\tau^*\left([y,u]\right)R\sigma([u,a])=0, \text{ for all } y\in R,\ u\in U\cap S_*(R).$$

Since R is a $*-$prime ring and σ,τ are automorphisms, we get

$$u\in Z \text{ or } [u,a]=0, \text{ for all } u\in U\cap S_*(R).$$

This implies that $[u,a]=0$, for all $u\in U\cap S_*(R)$.

Assume that $u\in U$. We know that $u-u^*\in U\cap S_*(R)$. The last equation gives that $[u,a]=[u^*,a]$, for all $u\in U$. Replacing y, u by y^*, u^* respectively in (4) and using $\tau *=*\tau$, we get

$$\tau^*\left([y,u]\right)R\sigma([u,a])=0, \text{ for all } y\in R,\ u\in U. \tag{5}$$

By (4) and (5), we get

$$\tau^*\left([y,u]\right)R\sigma([u,a])=\tau\left([y,u]\right)R\sigma([u,a])=0, \text{ for all } y\in R,\ u\in U.$$

Since R is a $*-$prime ring and σ,τ are automorphisms, we get

$$u\in Z \text{ or } [u,a]=0, \text{ for all } u\in U.$$

We have $[U,a]=0$ for any cases. Hence we arrive at $a\in Z$ or $U\subseteq Z$ by Lemma 2. This the proof is completed. ■

Theorem 3. *Let R be a $*-$prime ring with characteristic not 2 and 3, U a nonzero $*-(\sigma,\tau)-$Lie ideal of R, d a nonzero derivation of R such that $d\tau=\tau d,\sigma d=d\sigma$ and $*$ commutes with σ, τ and d. If $d(U)\subseteq U$ and $d^2(U)\subseteq Z$, then $U\subseteq Z$.*

Proof. Assume that $d(Z) = (0)$. This implies that

$$d^3(U) = d(d^2(U)) \subseteq d(Z) = (0).$$

For any $x \in R$, $u \in U$ and $\tau(u)[x,u]_{\sigma,\tau} \in U$, we get

$$d^3(\tau(u)[x,u]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U.$$

Expanding this equation by using $d\tau = \tau d$ and $d^3(U) = (0)$, we arrive at

$$0 = 3(d^2(\tau(u))d([x,u]_{\sigma,\tau}) + d(\tau(u))d^2([x,u]_{\sigma,\tau}).$$

Since $char R \neq 3$, we obtain

$$d^2(\tau(u))d([x,u]_{\sigma,\tau}) + d(\tau(u))d^2([x,u]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U.$$

Replacing u by $d(u)$ in the last equation and using $\tau d = d\tau$, $d^3(U) = 0$, we have

$$d^2(\tau(u))d^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U.$$

By the hypothesis, we have

$$d^2(\tau(u))Rd^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U. \tag{6}$$

Assume that $u \in U \cap S_*(R)$. In (6), replacing u by u^* and using $*$ commutes with τ and d, we get

$$d^2(\tau(u))^* Rd^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U \cap S_*(R).$$

This yields that

$$d^2(\tau(u))Rd^2([x,d(u)]_{\sigma,\tau}) = \left(d^2(\tau(u))\right)^* Rd^2([x,d(u)]_{\sigma,\tau}) = 0,$$

for all $x, r \in R$, $u \in U \cap S_*(R)$. The $*-$primeness of R gives

$$d^2(\tau(u)) = 0 \text{ or } d^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U \cap S_*(R).$$

Now, let $d^2(\tau(u)) = 0$ for all $u \in U \cap S_*(R)$. For any $u \in U$, we know that $u - u^* \in U \cap S_*(R)$, and so $d^2(\tau(u)) = d^2(\tau(u^*))$ for all $u \in U$. By using the last equation in (6) and using $*$ commutes with τ and d, we get

$$\left(d^2(\tau(u))\right)^* Rd^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R, \ u \in U.$$

On the other hand, we get $d^2([x,d(u)]_{\sigma,\tau}) = 0$, for all $u \in U \cap S_*(R)$. For any $u \in U$, again taking $u - u^* \in U \cap S_*(R)$, and so, $d^2([x,d(u)]_{\sigma,\tau}) =$

$d^2([x,d(u^*)]_{\sigma,\tau})$, for all $x \in R, u \in U$. Replacing u by u^* in (6) and using this equation, $*$ commutes with τ and d, we arrive at

$$\left(d^2(\tau(u))\right)^* Rd^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R, u \in U.$$

Hence we find that

$$\left(d^2(\tau(u))\right)^* Rd^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R,\ u \in U. \tag{7}$$

for any cases. By (6) and (7), we get

$$d^2(\tau(u))Rd^2([x,d(u)]_{\sigma,\tau}) = \left(d^2(\tau(u))\right)^* Rd^2([x,d(u)]_{\sigma,\tau}) = 0,$$

for all $x \in R$, $u \in U$. Since R is $*-$prime ring, τ is automorphism and $d\tau = \tau d$, we obtain

$$d^2(u) = \text{ or } d^2([x,d(u)]_{\sigma,\tau}) = 0 \text{ for all } x \in R,\ u \in U.$$

Let us define $K = \{u \in U | d^2(u) = 0\}$ and $L = \{u \in U | d^2([x,d(u)]_{\sigma,\tau}) = 0$, for all $x \in R\}$. Clearly, both K and L are additive subgroups of U. Moreover, U is the set-theoretic union of K and L. But a group cannot be the set-theoretic union of two proper subgroups. Hence $K = U$ or $L = U$. If $K = U$ then $U \subseteq Z$ by Lemma 3. So, we have $L = U$. That is,

$$d^2([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R,\ u \in U. \tag{8}$$

Replacing x by $\tau(d(u))x$ in (8) and using $\tau d = \tau d$, we get

$$\begin{aligned} 0 &= d^2([\tau(d(u))x,d(u)]_{\sigma,\tau}) = d^2(\tau(d(u))[x,d(u)]_{\sigma,\tau}) \\ &= \tau(d^3(u))[x,d(u)]_{\sigma,\tau} + 2\tau(d^2(u))d([x,d(u)]_{\sigma,\tau}) \\ &\quad + \tau(d(u))d^2([x,d(u)]_{\sigma,\tau}). \end{aligned}$$

By equation (8) and $d^3(U) = (0)$, $char R \neq 2$, we get

$$\tau(d^2(u))d([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R,\ u \in U.$$

Using the same arguments after equation (6), we have

$$d^2(u) = 0 \text{ or } d([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all } x \in R,\ u \in U.$$

Let $M = \{u \in U | d^2(u) = 0\}$ and $N = \{u \in U | d([x,d(u)]_{\sigma,\tau}) = 0, \forall x \in R\}$. Each of M and N is an additive subgroup of U such that $U = M \cup N$. The above trick gives us $U = M$ or $U = N$. In the former case, $d^2(U) = 0$, which forces $U \subseteq Z$ by Lemma 3. If $U = N$, then $d([x,d(u)]_{\sigma,\tau}) = 0$ for all $u \in U$. Replacing x by $\tau(d(u))x$ in this equation, we have

$$\tau(d^2(u))[x,d(u)]_{\sigma,\tau} = 0, \text{ for all } x \in R,\ u \in U.$$

Using $d^2(u) \in Z$ and again applying the above trick, we obtain that $[x, d(u)]_{\sigma,\tau} = 0$. Writing x by $xy, y \in R$ in this equation and using the last equation, we have

$$0 = [xy, d(u)]_{\sigma,\tau} = x[y, d(u)]_{\sigma,\tau} + [x, \sigma(d(u))]y = [x, \sigma(d(u))]y,$$

and so

$$[x, \sigma(d(u))]R = 0, \text{ for all } x \in R, u \in U.$$

Replacing x by x^*, u by u^* and using $*$ commutes with σ and d, we get

$$([x, \sigma(d(u))])^* R = 0, \text{ for all } x \in R, \ u \in U.$$

Thus we have

$$[x, \sigma(d(u))]R = ([x, \sigma(d(u))])^* R = 0, \text{ for all } x \in R, \ u \in U.$$

Since R is a $*-$prime ring and σ is an automorphism, we obtain $d(U) \subseteq Z$. Theorem 1 gives that $U \subseteq Z$. Hence the proof is completed in the case of $d(Z) = (0)$.

Now, we suppose that $d(Z) \neq (0)$. Choose $\alpha \in Z$ such that $d(\alpha) \neq 0$. For any $x \in R$, $u \in U$ and $[\alpha x, u]_{\sigma,\tau} = \alpha[x, u]_{\sigma,\tau} \in U$. By the hypothesis, we have

$$d^2(\alpha[x, u]_{\sigma,\tau}) = d^2(\alpha)[x, u]_{\sigma,\tau} + 2d(\alpha)d([x, u]_{\sigma,\tau}) + \alpha d^2([x, u]_{\sigma,\tau}) \in Z$$

and so

$$d^2(\alpha)[x, u]_{\sigma,\tau} + 2d(\alpha)d([x, u]_{\sigma,\tau}) \in Z, \text{ for all } x \in R, u \in U. \tag{9}$$

Replacing x by $x\alpha$ in (9), we get

$$(d^2(\alpha)[x, u]_{\sigma,\tau} + 2d(\alpha)d([x, u]_{\sigma,\tau}))\alpha + 2d(\alpha)[x, u]_{\sigma,\tau} d(\alpha) \in Z.$$

Using equation (9), we obtain

$$d(\alpha)[x, u]_{\sigma,\tau} d(\alpha) \in Z.$$

That is $(d(\alpha))^2 [x, u]_{\sigma,\tau} \in Z$ and so

$$(d(\alpha))^2 [x, u]_{\sigma,\tau} r = r (d(\alpha))^2 [x, u]_{\sigma,\tau} \text{ for all } x, r \in R, \ u \in U.$$

Again using hypothesis, we arrive at

$$(d(\alpha))^2 [[x, u]_{\sigma,\tau}, r] = 0, \text{ for all } x, r \in R, \ u \in U,$$

and so for all $\alpha \in Z$ such that $d(\alpha) \neq 0$, we have

$$(d(\alpha))^2 R [[x,u]_{\sigma,\tau}, r] = 0, \text{ for all } x, r \in R, \ u \in U.$$

Arguing the same ways above and using $*$ commutes with d and $\alpha^* \in Z$ such that $d(\alpha^*) \neq 0$, we have

$$\left((d(\alpha))^2\right)^* R [[x,u]_{\sigma,\tau}, r] = 0, \text{ for all } x, r \in R, \ u \in U.$$

Combining the last two equations and using the $*-$primeness of R, we arrive at

$$(d(\alpha))^2 = 0 \text{ or } [x,u]_{\sigma,\tau} \subseteq Z, \text{ for all } x \in R, \ u, v \in U.$$

Since $0 \neq d(\alpha) \in Z$, we must have $[R,U]_{\sigma,\tau} \subseteq Z$. Lemma 4 yields that $U \subseteq Z$. This completes the proof. ■

Dedication: This study is dedicated to our pioneer in this area, Prof. Dr. Hatice Kandamar.

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EMINE KOÇ SÖGÜTCÜ
CUMHURIYET UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
SIVAS, TURKEY
e-mail: eminekoc@cumhuriyet.edu.tr

NEŞET AYDIN
ÇANAKKALE 18 MART UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF MATHEMATICS
ÇANAKKALE, TURKEY
e-mail: neseta@comu.edu.tr

ÖZNUR GÖLBAŞI
CUMHURIYET UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
SIVAS, TURKEY
e-mail: ogolbasi@cumhuriyet.edu.tr

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