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NEW MAXIMAL AND MINIMAL SETS VIA $\beta\theta$-OPEN SETS

Abstract. Nakaoka and Oda ([9] and [10]) introduced the notion of maximal open sets and minimal closed sets. In this paper, we introduce new classes of sets called maximal $\beta\theta$-open sets, minimal $\beta\theta$-closed sets, $\beta\theta$-semi maximal open sets and $\beta\theta$-semi minimal closed sets and investigate some of their fundamental properties.

Key words: topological space, maximal open set, minimal closed set, $\beta\theta$-open set, $\beta\theta$-semi maximal open set.

AMS Mathematics Subject Classification: 54B05, 54C08.

1. Introduction and preliminaries

It is the common viewpoint of many topologists that generalized open sets are important ingredients in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of β-open sets or semipreopen sets introduced by Abd. El-Monsef et al. [1] and Andrijević [2], respectively. In 2003, Noiri [11] used this notion and the β-closure [1] of a set to introduce the concepts of $\beta\theta$-open and $\beta\theta$-closed sets which provide a formulation of the $\beta\theta$-closure of a set in a topological space. Caldas, jafari and Ekici [3, 4, 5, 6, 8] continued the work of Noiri and defined other concepts utilizing $\beta\theta$-closed sets.

F. Nakaoka and N. Oda in [9] and [10] introduced the notion of maximal open sets and minimal closed sets. The purpose of the present paper is to introduce the concept of a new class of open sets called maximal $\beta\theta$-open sets, minimal $\beta\theta$-closed sets, $\beta\theta$-semi maximal open sets and $\beta\theta$-semi minimal closed sets. We also investigate some of their fundamental properties.

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset A of a space (X,τ) is said to be β-open [1] if $A \subset Cl(Int(Cl(A)))$, where $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. The complement of a β-open set is said to be β-closed. The intersection of all β-closed sets containing A is said to be the β-closure of A and is denoted by $\beta Cl(A)$. The β-interior $\beta Int(A)$ of a subset $A \subset X$ is the union of all β-open sets contained in A. The $\beta\theta$-closure of A [11], denoted by $\beta Cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $\beta Cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X,\tau)$ with $x \in O$. The set $\{x \in X : \beta Cl_\theta(O) \subset A \ for \ some \ O \in \beta(X,x)\}$ is called the $\beta\theta$-interior of A and is denoted by $\beta Int_\theta(A)$. A subset A is said to be $\beta\theta$-closed [11] if $A = \beta Cl_\theta(A)$. The complement of a $\beta\theta$-closed set is said to be $\beta\theta$-open. The family of all $\beta\theta$-open (resp. $\beta\theta$-closed) subsets of X is denoted by $\beta\theta O(X,\tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X,\tau)$). We set $\beta\theta O(X,x) = \{U : x \in U \in \beta\theta O(X,\tau)\}$ and $\beta\theta C(X,x) = \{U : x \in U \in \beta\theta C(X,\tau)\}$.

A proper nonempty open set (resp. closed set) U of X (resp. V of X) is called a maximal open set [9] (resp. minimal closed set [10]) if any open (resp. closed) set which contains U is either X or U (resp. contained in V is either $\emptyset$ or V). The family of all maximal open (resp. minimal closed) sets will be denoted by $M_aO(X)$ (resp. $M_iC(X)$). We set $M_aO(X,x) = \{U \mid x \in U \in M_aO(X)\}$ and $M_iC(X,x) = \{V \mid x \in V \in M_iC(X)\}$.

2. Minimal $\beta\theta$-closed sets

Definition 1. *A proper nonempty $\beta\theta$-closed set B of X is called a minimal $\beta\theta$-closed set if any $\beta\theta$-closed set which is contained in B is either $\emptyset$ or B.*

The family of all minimal $\beta\theta$-closed sets will be denoted by $M_i\beta\theta C(X)$. We set $M_i\beta\theta C(X,x) = \{F \mid x \in F \in M_i\beta\theta C(X)\}$.

The following example shows that minimal-closed sets and minimal $\beta\theta$-closed sets are in general independent.

Example 1. Let (X,τ) be a topological space such that $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$. Clearly $\beta\theta O(X,\tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$. Then $\{b\}$ is a minimal $\beta\theta$-closed set which is not minimal-closed, and $\{b,c\}$ is a minimal-closed set which is not minimal $\beta\theta$-closed.

Theorem 1. *For the subsets F and G of a topological space X, we have the following:*

(1) *Let F be a minimal $\beta\theta$-closed set and G a $\beta\theta$-closed set. Then $F \frown G = \emptyset$ or $F \subset G$.*

(2) *Let F and G be minimal $\beta\theta$-closed sets. Then $F \frown G = \emptyset$ or $F = G$.*

Proof. (1) Let F be a minimal $\beta\theta$-closed set and G be $\beta\theta$-closed set. If $F \frown G = \emptyset$, then there is nothing to prove. But if $F \frown G \neq \emptyset$, then we have to prove that $F \subset G$. Now if $F \frown G \neq \emptyset$ then $F \frown G \subset F$ and $F \frown G \subset G$. Since $F \frown G \subset F$ and given that F is minimal $\beta\theta$-closed, then by definition $F \frown G = F$ or $F \frown G = \emptyset$. But $F \frown G \neq \emptyset$, then $F \frown G = F$ which implies $F \subset G$.

(2) Let F and G be two minimal $\beta\theta$-closed sets. If $F \frown G = \emptyset$, then there is nothing to prove. But if $F \frown G \neq \emptyset$, then we have to prove that $F = G$. Now if $F \frown G \neq \emptyset$ then $F \frown G \subset F$ and $F \frown G \subset G$. Since F and G are minimal $\beta\theta$-closed sets, by (1) $F \subset G$ and $G \subset F$. Therefore, $F = G$. ■

Theorem 2. *If F is a nonempty finite $\beta\theta$-closed set of X, then there exists at least one (finite) minimal $\beta\theta$- closed set G such that $G \subset F$.*

Proof. If F is a minimal $\beta\theta$-closed set, we may set $G = F$. If F is not a minimal $\beta\theta$-closed, then there exists a (finite) $\beta\theta$-closed set F_1 such that $\emptyset \neq F_1 \subset F$. If F_1 is a minimal $\beta\theta$-closed set, we may set $G = F_1$. If F_1 is not a minimal $\beta\theta$-closed, then there exists a (finite) $\beta\theta$-closed set F_2 such that $\emptyset \neq F_2 \subset F_1$. Continuing this process, we have a sequence of $\beta\theta$-closed sets $..... \subset F_k \subset ....F_3 \subset F_2 \subset F_1 \subset F$. Since F is a finite set, this process repeats only finitely many times and finally we get a minimal $\beta\theta$-closed set $G = F_n$ for some positive integer n. ■

Theorem 3. (1) *Let F be a minimal $\beta\theta$-closed set of X. If $x \in F$ then $F \subset G$ for any $\beta\theta$-closed set G containing x.*

(2) *Let F be a minimal $\beta\theta$-closed set of X. Then $F = \frown \{G \mid x \in G \in \beta\theta C(X)\}$ for any element x of F.*

Proof. (1) Let $F \in M_i\beta\theta C(X, x)$ and $G \in \beta\theta C(X, x)$ such that $F \not\subset G$. This implies that $F \frown G \subset F$ and $F \frown G \neq \emptyset$. But since F is minimal $\beta\theta$-closed, by Definition 2 $F \frown G = F$ which contradicts $F \frown G \subset F$. Therefore $F \subset G$.

(2) By (1) and the fact that F is $\beta\theta$-closed containing x, we have $F \subset \frown \{G \mid G \in \beta\theta C(X, x)\} \subset F$. Hence the result. ■

Theorem 4. (1) *Let F and F_λ $(\lambda \in \Lambda)$ be minimal $\beta\theta$-closed sets. If $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda$ then there exists $\lambda \in \Lambda$ such that $F = F_\lambda$.*

(2) *Let F and F_λ, $(\lambda \in \Lambda)$ be minimal $\beta\theta$-closed sets. If $F \neq F_\lambda$ for any $\lambda \in \Lambda$, then $(\bigcup_{\lambda\in\Lambda} F_\lambda) \frown F = \emptyset$.*

Proof. (1) Let F and F_λ ($\lambda \in \Lambda$) be minimal $\beta\theta$-closed sets with $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda$. We have to prove that $F \frown F_\lambda \neq \emptyset$. If $F \frown F_\lambda = \emptyset$ then $F_\lambda \subset X\backslash F$ and hence $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda \subset X\backslash F$ which is a contradiction. Now as $F \frown F_\lambda \neq \emptyset$, then $F \frown F_\lambda \subset F$ and $F \frown F_\lambda \subset F_\lambda$. Since $F \frown F_\lambda \subset F$ and given that F is minimal $\beta\theta$-closed, then by definition $F \frown F_\lambda = F$ or $F \frown F_\lambda = \emptyset$. But $F \frown F_\lambda \neq \emptyset$ Then $F \frown F_\lambda = F$ which implies $F \subset F_\lambda$. Similarly if $F \frown F_\lambda \subset F_\lambda$ and given that F_λ is minimal $\beta\theta$-closed, then by definition $F \frown F_\lambda = F_\lambda$ or $F \frown F_\lambda = \emptyset$. But $F \frown F_\lambda \neq \emptyset$ then $F \frown F_\lambda = F_\lambda$ which implies $F_\lambda \subset F$. Then $F = F_\lambda$.

(2) Suppose that $(\bigcup_{\lambda\in\Lambda} F_\lambda) \frown F \neq \emptyset$. Then there exists $\lambda \in \Lambda$ such that $F_\lambda \frown F \neq \emptyset$. By Theorem 2.2(2), we have $F = F_\lambda$ which contradicts the fact that $F \notin F_\lambda$ for any $\lambda \in \Lambda$. Hence $(\bigcup_{\lambda\in\Lambda} F_\lambda) \frown F = \emptyset$. ■

3. Maximal $\beta\theta$-open sets

Definition 2. *A proper nonempty $\beta\theta$-open set A of X is called a maximal $\beta\theta$-open set if any $\beta\theta$-open set which contains A is either X or A.*

The family of all maximal $\beta\theta$-open sets will be denoted by $M_a\beta\theta O(X)$. We set $M_a\beta\theta O(X, x) = \{A \mid x \in A \in M_a\beta\theta O(X)\}$.

Example 2. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Clearly $\beta\theta O(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\{a, c\}$ is a maximal $\beta\theta$-open set which is not maximal-open, and $\{a\}$ is a maximal-open set which is not maximal $\beta\theta$-open.

Theorem 5. *Let A a proper nonempty subset A of X. Then A is maximal $\beta\theta$-open if and only if $X\backslash A$ is a minimal $\beta\theta$-closed set.*

Proof. *Necessity.* Let A be a maximal $\beta\theta$-open set. Then $A \subset X$ or $A \subset A$. Hence $\emptyset \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Therefore by Definition 2, $X\backslash A$ is a minimal $\beta\theta$-closed set.

Sufficiency. Let $X\backslash A$ is a minimal $\beta\theta$-closed set. Then $\emptyset \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Hence $A \subset X$ or $A \subset A$ which implies that A is a maximal $\beta\theta$-open set. ■

Theorem 6. *The following statements are true for any topological space X*

(1) *Let A be a maximal $\beta\theta$-open set and B be a $\beta\theta$-open set. Then $A \smile B = X$ or $B \subset A$.*

(2) *Let A and B be maximal $\beta\theta$-open sets. Then $A \smile B = X$ or $B = A$.*

Proof. (1) Let A be a maximal $\beta\theta$-open set and B a $\beta\theta$-open set. If $A \smile B = X$ then we are done. But if $A \smile B \neq X$, then we have to prove that $B \subset A$. Now $A \smile B \neq X$ means $B \subset A \smile B$ and $A \subset A \smile B$. Therefore we have $A \subset A \smile B$ and A is maximal $\beta\theta$-open, then by definition $A \smile B = X$ or $A \smile B = A$ but $A \smile B \neq X$, then $A \smile B = A$ which implies $B \subset A$.

(2) Let A and B be maximal $\beta\theta$-open sets. If $A \smile B = X$ then we are done. But if $A \smile B \neq X$, then we have to prove that $B = A$. Now $A \smile B \neq X$ means $A \subset A \smile B$ and $B \subset A \smile B$. Now $A \subset A \smile B$ and A is maximal $\beta\theta$-open set, then by definition $A \smile B = X$ or $A \smile B = A$ but $A \smile B \neq X$. Therefore $A \smile B = A$ which implies $B \subset A$. Similarly if $B \subset A \smile B$, we obtain $A \subset B$. Therefore $A = B$. ■

Theorem 7. (1) *Let A be a maximal $\beta\theta$-open and x an element of $X \backslash A$. Then $X \backslash A \subset B$ for any $\beta\theta$-open B containing x.*

(2) *Let A be a maximal $\beta\theta$-open set. Then, either of the following (i) and (ii) holds:*

(i) *For each $x \in X \backslash A$ and each $\beta\theta$-open set B containing x, $B = X$.*

(ii) *There exists a $\beta\theta$-open set B such that $X \backslash A \subset B$ and $B \subset X$.*

(3) *Let A be a maximal $\beta\theta$-open set. Then, either of the following (i) and (ii) holds:*

(i) *For each $x \in X \backslash A$ and each $\beta\theta$-open B containing x, we have $X \backslash A \subset B$.*

(ii) *There exists a $\beta\theta$-open set B such that $X \backslash A = B \neq X$.*

Proof. (1) Since $x \in X \backslash A$, we have $B \not\subset A$ for any $\beta\theta$-open B containing x. Then $A \smile B = X$ by Theorem 3.3(1). Therefore $X \backslash A \subset B$.

(2) If (i) does not hold, then there exist an element x of $X \backslash A$ and a $\beta\theta$-open B containing x such that $B \subset X$. By (1) we have $X \backslash A \subset B$.

(3) If (ii) does not hold, then, by (1), we have $X \backslash A \subset B$ for each $x \in X \backslash A$ and each $\beta\theta$-open B containing x. Hence $X \backslash A \subset B$. ■

Remark 1. If we consider the case which A is a maximal $\beta\theta$-open set. Then, one can claim that either of the following (i) and (ii) holds:

(i) For each $x \in X \backslash A$ and each $\beta\theta$-open B containing x, we have $X \backslash A \subset B$.

(ii) There exists a $\beta\theta$-open set B such that $X \backslash A = B \neq X$.

But By (1), (i) is always true and therefore only (i) holds. This implies that such a case is contained in (1).

Theorem 8. *Let A, B, C be maximal $\beta\theta$-open sets such that $A \neq B$. If $A \frown B \subset C$, then either $A = C$ or $B = C$.*

Proof. Given that $A \frown B \subset C$. If $A = C$ then there is nothing to prove. But if $A \neq C$, then we have to prove $B = C$. By using Theorem 3.3(2) we have :

$$\begin{aligned} B \frown C &= B \frown [C \frown X] = B \frown [C \frown (A \smile B)] \\ &= B \frown [(C \frown A) \smile (C \frown B)] \\ &= (B \frown C \frown A) \smile (B \frown C \frown B) \\ &= (A \frown B) \smile (C \frown B) \text{ since } A \frown B \subset C. \\ &= (A \smile C) \frown B \\ &= X \frown B = B, \text{ since } A \smile C = X. \end{aligned}$$

This implies $B \subset C$. It follows from the definition of maximal $\beta\theta$-open set that $B = C$. ■

Theorem 9. *Let A, B, C be maximal $\beta\theta$-open sets which are different from each other. Then $(A \frown B) \not\subset (A \frown C)$.*

Proof. Let $(A \frown B) \subset (A \frown C)$. Then, $(A \frown B) \smile (C \frown B) \subset (A \frown C) \smile (C \frown B)$. Hence $(A \smile C) \frown B \subset C \frown (A \smile B)$. Since by Theorem 3.3(2) $A \cup C = X$ and $A \cup B = X$, we have $X \frown B \subset C \frown X$ which implies $B \subset C$. From definition of maximal $\beta\theta$-open set it follows that $B = C$. This contradicts the fact that A, B and C are different from each other. Therefore $(A \frown B) \not\subset (A \frown C)$. ■

We call a set cofinite if its complement is finite.

Theorem 10. *If A is a proper nonempty cofinite $\beta\theta$-open set of X, then there exists (cofinite) maximal $\beta\theta$- open set B such that $A \subset B$.*

Proof. If A is a maximal $\beta\theta$-open set, we may set $A = B$. If A is not a maximal $\beta\theta$-open, then there exists a (cofinite) $\beta\theta$-open set A_1 such that $A \subset A_1 \neq X$. If A_1 is a maximal $\beta\theta$-open set, we may set $B = A_1$. If A_1 is not a maximal $\beta\theta$-open set, then there exists a (cofinite) $\beta\theta$-open set $A_2 \neq X$ such that $A \subset A_1 \subset A_2 (\neq X)$. Continuing this process, we have a sequence of $\beta\theta$-open sets such that $A \subset A_1 \subset A_2 \subset \ldots A_k \subset \ldots$. Since A is a cofinite set, this process repeats only finitely many times and finally we get a maximal $\beta\theta$-open set $B = A_n$ for some positive integer n. ■

Theorem 11. *The following statements are true for any topological space X.*

(1) *Let A be a maximal $\beta\theta$-open set of X. Then either $\beta Cl_\theta(A) = X$ or $\beta Cl_\theta(A) = A$.*

(2) *Let A be a maximal $\beta\theta$-open set of X. Then either $\beta Int_\theta(X \backslash A) = X \backslash A$ or $\beta Int_\theta(X \backslash A) = \emptyset$*

Proof. (1) Since A is a maximal $\beta\theta$-open set, only the following cases (i) and (ii) are possible by Theorem 3.4(3). (i) Let $\beta Cl_\theta(A) \neq X$. Then there exists $x \in X \setminus \beta Cl_\theta(A)$. Hence there exists a $\beta\theta$-open set B such that $x \in B$ and $B \cap A = \emptyset$. Therefore, $B \subset X \setminus A$. On the other hand, by Theorem 3.4(1), $X \setminus A \subset B$ and $B = X \setminus A$. Hence A is $\beta\theta$-closed and $\beta Cl_\theta(A) = A$.

(2) By (1), we have $\beta Cl_\theta(A) = A$ or $\beta Cl_\theta(A) = X$. Hence $\beta Int_\theta(X\backslash A) = X\backslash A$ or $\beta Int_\theta(X\backslash A) = \emptyset$. ■

Theorem 12. (1) *Let A be a maximal $\beta\theta$-open set of X and B a nonempty subset of $X\backslash A$. Then $\beta Cl_\theta(B) = X\backslash A$.*

(2) *Let A be a maximal $\beta\theta$-open set of X. and G a proper subset of X with $A \subset G$. Then $\beta Int_\theta(G) = A$.*

Proof. (1) Since $\emptyset \neq B \subset X\backslash A$, by Theorem 3.4(1) we have that $W \frown A \neq \emptyset$ for $x \in X \setminus A$ and $\beta\theta$-open set W containing x. Hence $W \frown B \neq \emptyset$ for any $\beta\theta$-open set W containing x. Thus, $X\backslash A \subset \beta Cl_\theta(B)$. Since $X\backslash A$ is $\beta\theta$-closed and $B \subset X\backslash A$, we have $\beta Cl_\theta(B) \subset X\backslash A$. Therefore $\beta Cl_\theta(B) = X\backslash A$.

(2) If $G = A$, then $\beta Int_\theta(G) = \beta Int_\theta(A) = A$. If $G \neq A$, then we have $A \subset G$. Thus $A \subset \beta Int_\theta(G)$. Since A is maximal $\beta\theta$- open and $\beta Int_\theta(G)$ is $\beta\theta$-open containing A, then, $\beta Int_\theta(G) = X$ or $\beta Int_\theta(G) = A$. Since G is proper, $\beta Int_\theta(G) \neq X$. Therefore $\beta Int_\theta(G) = A$. ■

4. $\beta\theta$-semi-maximal open sets and $\beta\theta$-semi-minimal closed sets

Definition 3. *A set A in a topological space X is said to be $\beta\theta$-semi-maximal open if there exists a maximal $\beta\theta$-open set U such that $U \subset A \subset Cl(U)$. The complement of a $\beta\theta$-semi-maximal open set is called a $\beta\theta$-semi-minimal closed set.*

Example 3. Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c\}, \{c, d\}, \{a, b\}, \{a, b, c\}, X\}$. Clearly $\theta O(X, \tau) = \{\emptyset, \{a, b\}, \{c, d\}, X\}$. Then $\{a, b\}$ is a θ-semi-maximal open set.

The collection of all $\beta\theta$-semi-maximal open (resp. $\beta\theta$-semi-minimal closed) sets of X is denoted by $\beta\theta SM_aO(X)$ (resp. $\beta\theta SM_aC(X)$).

Remark 2. Every maximal $\beta\theta$-open (resp. minimal $\beta\theta$-closed) set is $\beta\theta$-semi-maximal open (resp. $\beta\theta$-semi-minimal closed).

Theorem 13. *If A is a $\beta\theta$-semi-maximal open set of X and $A \subset B \subset Cl(A)$. Then B is a $\beta\theta$-semi-maximal open set of X.*

Proof. Since A is a $\beta\theta$-semi-maximal open, there exists a maximal $\beta\theta$-open set U such that $U \subset A \subset Cl(U)$. Then $U \subset A \subset B \subset Cl(A) \subset Cl(U)$. Hence $U \subset B \subset Cl(U)$. Thus B is $\beta\theta$-semi-maximal open. ■

Theorem 14. *A subset F of X is $\beta\theta$-semi-minimal closed if and only if there exists a minimal $\beta\theta$-closed set G in X such that $Int(G) \subset F \subset G$.*

Proof. Suppose F is $\beta\theta$-semi-minimal closed in X. By Definition 3, $X\backslash F$ is $\beta\theta$-semi-maximal open in X. Therefore, there exists a maximal $\beta\theta$-open set U such that $U \subset X\backslash F \subset Cl(U)$ which implies $Int(X\backslash U) = X\backslash Cl(U) \subset F \subset X\backslash U$. Take $G = X\backslash U$ so that G is a minimal $\beta\theta$-closed set and $Int(G) \subset F \subset G$.

Conversely, Suppose that there exists a minimal $\beta\theta$-closed set G in X such that $Int(G) \subset F \subset G$. Hence $X\backslash G \subset X\backslash F \subset X\backslash Int(G) = Cl(X\backslash G)$. Therefore there exists a maximal $\beta\theta$-open set $U = X\backslash G$ such that $U \subset X\backslash F \subset Cl(U)$, i.e., $X\backslash F$ is $\beta\theta$-semi-maximal open in X. It follows that F is $\beta\theta$-semi-minimal closed. ■

Theorem 15. *If G is $\beta\theta$-semi-minimal closed in X and $Int(G) \subset F \subset G$, then F is also $\beta\theta$-semi-minimal closed in X.*

Proof. Let G be a $\beta\theta$-semi-minimal closed set of X. Then, there exists a minimal $\beta\theta$-closed set H in X such that $Int(H) \subset G \subset H$. Hence $Int(H) \subset Int(G) \subset F \subset G \subset H$. It follows $Int(H) \subset F \subset H$. Therefore F is a $\beta\theta$-semi-minimal closed set of X. ■

Theorem 16. *Let Y be an open subspace of X and $A \subset Y$. If A is a $\beta\theta$-semi-maximal open set of X, then A is also a $\beta\theta$-semi-maximal open set of Y.*

Proof. Since A is a $\beta\theta$-semi-maximal open set of X, there exists a maximal $\beta\theta$-open set U such that $U \subset A \subset Cl(U)$. Hence U is a subset of Y. Since U is maximal $\beta\theta$-open in X, $Y \frown U = U$ is maximal $\beta\theta$-open in Y and $U = Y \frown U \subset Y \frown A \subset Y \frown Cl(U)$, i.e., $U \subset A \subset Cl_Y(U)$. Hence A is $\beta\theta$-semi-maximal open in Y. ■

Theorem 17. *If A_i is a $\beta\theta$-semi-maximal open set of X_i $(i = 1, 2)$ then $A_1 \times A_2$ is a $\beta\theta$-semi-maximal open set of $X_1 \times X_2$.*

Proof. For $i = 1, 2$, there exists a maximal $\beta\theta$-open set U_i such that $U_i \subset A_i \subset Cl_{X_i}(U_i)$. Therefore $U_1 \times U_2 \subset A_1 \times A_2 \subset Cl_{X_1}(U_1) \times Cl_{X_2}(U_2) = Cl_{X_i \times X_2}(U_1 \times U_2)$. Hence $A_1 \times A_2$ is $\beta\theta$-semi-maximal open in $X_1 \times X_2$. ■

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